Fractal Spectrum and Anomalous Diffusion in the Kicked Harper Model

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We consider a kicked system on the cylinder obtained upon quantization of a chaotic area-preserving map. We use the thermodynamic formalism to investigate the scaling properties of the fractal spectrum. In time evolution we observe anomalous diffusion with an exponent closely related to the Hausdorff dimension of the spectrum, and dependent upon the parameters of the system.

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Periodically driven systems have been intensely studied in the framework of quantum chaos: The kicked rotator (see [1] for a review) is a typical system of this class (its classical analog being the standard map), and the quantum behavior is characterized by the suppression [2] of classical deterministic diffusion which the standard map exhibits in the chaotic regime [3]. This remarkable phenomenon has also been invoked to explain experimental results, such as, for instance, peculiar behavior of highly excited hydrogen atoms in a microwave field (see [4,5] for a review).

Some light has been shed on this model by showing that it bears significant analogies with Anderson localization in 1D lattices [6]. Besides the classical perturbation parameter, the dynamics of the quantum system is governed by the ratio of the unperturbed and the external frequencies: In particular dynamical localization appears for almost all irrational values of this ratio, while for rational values the energy grows quadratically [2]. The pure point character of the spectrum for irrational values induces the absence of scaling in the hierarchy of finer and finer band structures obtained by approaching the frequency ratio through a sequence of rational approximants [7,8], as bandwidths shrink exponentially with respect to the number of bands and this excludes the possibility of self-similarity.

On the other hand, it is known that fractal features of the spectrum can appear in systems with quasiperiodic potentials, as, for example, in the Harper model [9]. Recently some numerical investigations [10] have suggested the existence of a quantum system, the so-called kicked Harper model (KHM), displaying a rich variety of temporal behavior, from dynamical localization to quadratic growth of energy or to diffusion. An important property of this model is that the corresponding classical system is chaotic, while the usual Harper model is integrable in the classical limit. Interesting features of these models have been analyzed [11,12], in particular as regards the level spacing distribution and the decay of autocorrelation functions. The investigation of such models is closely connected to the motion of electrons in 2D lattices in the presence of a magnetic field [13] and can lead to a better understanding of the physics of such systems.

In this Letter we present results concerning time evolution and scaling properties of the spectrum. Dynamically we find that the KHM system (in contrast to the Harper model [11]) exhibits anomalous diffusion, with an exponent depending upon the parameters of the map. The scaling properties of the spectrum are studied within the thermodynamic formalism for multifractals [14–18]. The most important observation concerns the occurrence of a phase transition, similar to the one observed in the Hénon attractor [19]. To our knowledge this is the first instance in which this kind of transition has been observed in a quantum-mechanical framework.

The system we consider is obtained by quantizing the area-preserving Harper map (see [10] for some discussion on its behavior upon variation of the parameters):

$$p_{n+1}=p_n+K\sin(x_n),\quad x_{n+1}=x_n-L\sin(p_{n+1})\,.$$

Following the usual procedure for quantum kicked systems we are led to the following one-period evolution operator:

$$\hat{U}=\exp[-i(L/\hbar)\cos(\hbar\hat{n})]\exp[-i(K/\hbar)\cos(x)]\,, \quad (1)$$

where $\hat{n}=-id/dx$, and no quasimomentum appears as we consider x on a circle (and thus the global model acting on a cylinder); $\hbar$ plays the role of an effective Planck constant, which includes the frequency ratio of the unperturbed system and the external driving. The quasienergies and the corresponding eigenfunctions are determined by

$$\hat{U}\psi_\omega=e^{-i\omega}\psi_\omega\,. \quad (2)$$

There are rigorous mathematical results ([20], and references therein) that allow one to conclude that in the symmetric case ($K=L$ self-dual case [21]), the quasienergy eigenstates are delocalized [22].

We now focus our attention on strongly irrational values of $\hbar/2\pi$: In particular we consider $\hbar=2\pi/(m+\rho_{\rm GM})$ with $\rho_{\rm GM}=(\sqrt{5}+1)/2$, approaching this value through a sequence of rational approximants $\{p_n/q_n\}$ determined by successive truncations of the continued-fraction expansion of $\hbar/2\pi$. Each of these approximants introduces periodicity in p, leading to a band spectrum with Bloch eigenfunctions. On the unperturbed basis we thus have eigenfunctions of the form $\phi_{s+q_nl}=e^{-ial}\phi_s$, $s=1,\ldots,q_n$, $l\in\mathbb{Z}$, $a\in[0,1)$, and (2) reduces to the matrix equation [23]

$$[U(a)]_{s,s'}\phi(a)_{s'}=\phi(a)_s e^{-i\omega(a)}.$$

By varying a, and diagonalizing $U(a)$, we thus get q_n bands, whose widths are denoted by $\{w_{i(n)}\}$. Note that order by order the number of the bands increases exponentially, with a geometric factor given by $\rho_{\rm GM}$, as each $\{p_n\},\{q_n\}$ sequence obeys Fibonacci recursion relations. The fractal features of the limiting set are thus analyzed, in the framework of the thermodynamic formalism [14–18], by introducing the free energy $g(\tau)$,

$$q_n^{g_n(\tau)}=\sum_i^{(n)} w_{i(n)}^{-\tau},\quad g(\tau)=\lim_{n\to\infty} g_n(\tau). \tag{3}$$

This is the correct thermodynamic function if we assign equal probability to each band. This is physically motivated by considering that each band contains the same number of states [9], but other choices are in principle possible [24,25] (like use of the spectral measure as in [12]; we will discuss the different possible thermodynamics in a separate publication [26]). The set of scaling exponents μ [24], giving the geometric rate of shrinking for bandwidths, are determined by $\mu_{i(n)}=-\log w_{i(n)}/\log q_n$; their relative relevance is exhibited via the scaling spectrum $s(\mu)$, which is introduced by reordering the partition sum appearing in (3) by increasing bandwidths,

$$\sum_i^{(n)} w_{i(n)}^{-\tau}=\int_{\mu_{\min}}^{\mu_{\max}} d\mu\, q_n^{s_n(\mu)+\mu\tau},\quad s(\mu)=\lim_{n\to\infty} s_n(\mu). \tag{4}$$

These functions [24,25] are related by the usual thermodynamical Legendre transform

$$g(\tau)=S(\mu)+\tau\mu,$$

$$\mu=\frac{dg(\tau)}{d\tau},\quad \tau=-\frac{dS(\mu)}{d\mu},$$

where $S(\mu)$ [obtained by a stationary phase evaluation of the integral in (4)] is the convex envelope of the scaling spectrum $s(\mu)$. We remark that distinguishing between $s(\mu)$ and its convex envelope $S(\mu)$ is not a mathematical subtlety: It is a possible way of diagnosing the occurrence of phase transitions [24,25]. In the same fashion we have to pay attention to the way the limit procedure on (3) and (4) is attained, since relevant, and sometimes subtle, finite-order effects [24,27] may appear. We notice that the Hausdorff dimension plays a particularly relevant role in this context, as it does not depend on the probability measure upon the set, being uniquely determined by metric properties of the asymptotic set.

Our investigations focus on the symmetric case, $K=L$ [28]. The thermodynamic analysis of the spectrum along this critical line shows how well scaling behavior coexists with abnormal scaling: In contrast with the critical case

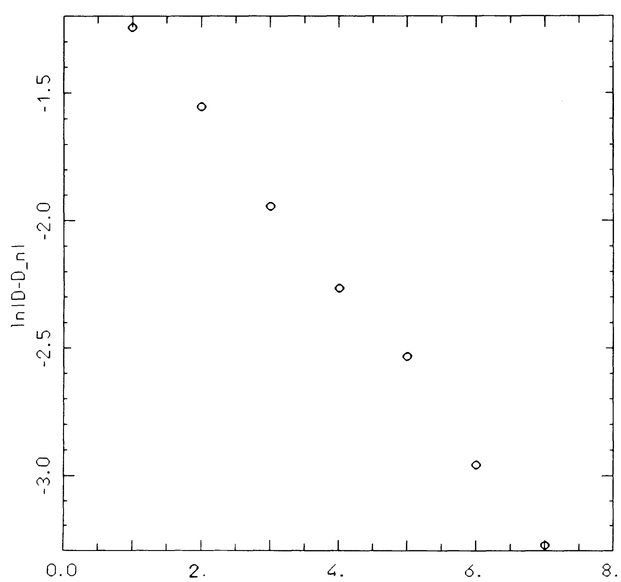

FIG. 1. $\ln|D_n-D_H|$ vs approximant order n (labeling p_n/q_n rationals) for $K=L=5$, $\hbar=2\pi/(18+\rho_{\rm GM})$. Geometric character of convergence is manifest.

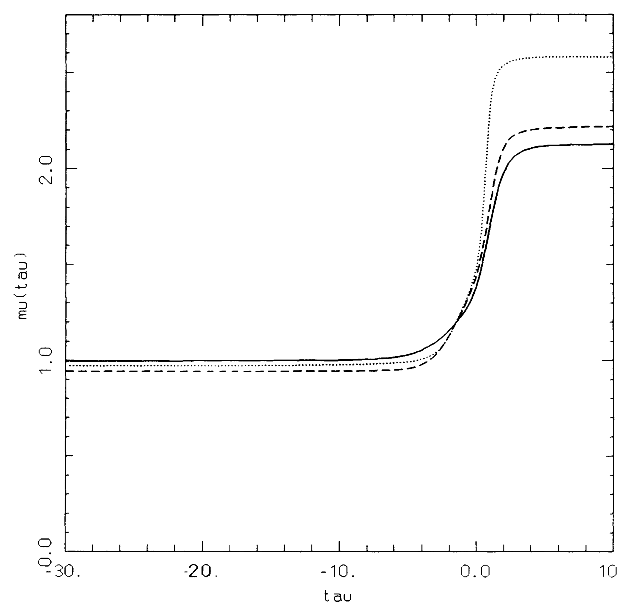

FIG. 2. Mean scaling index $g'(\tau)$ for $K=L=5$, $\hbar=2\pi/(18+\rho_{\rm GM})$ for different rational approximants (solid line: $p_n/q_n=8/157$; dashed line: $p_n/q_n=13/255$; dotted line: $p_n/q_n=21/412$).

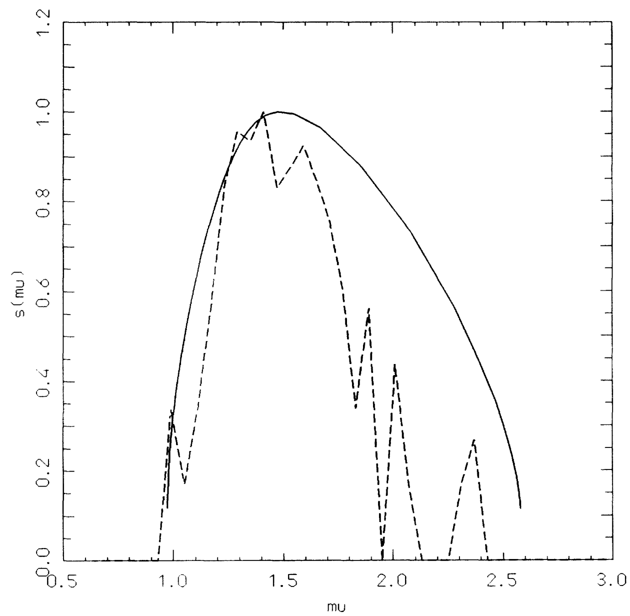

FIG. 3. Scaling spectrum (dashed line) and its convex envelope (solid line) for $K=L=5$, $\hbar=2\pi(21/412)$: The difference between the two curves signals a Hénon-like phase transition.

of the Harper model [9] maximal scaling indices $\mu_{\max}$ increase by taking successive rational approximations (this corresponds to bandwidths shrinking faster than geometrically). In analogy with solid-state models [9] and kicked rotator analysis [7] this may be interpreted as a remnant of the discrete spectrum. Along with these irregular features signatures of regular scaling coexist: Computation of the Hausdorff and information dimensions shows an overall geometric convergence (see Fig. 1). Mean scaling plots (see Fig. 2) display a rather sharp distinction between the two phases, respectively dominated by $\mu_{\min}$ and $\mu_{\max}$. However, there is no quantitative indication of a conventional phase transition (like scaling of the maximum of the second derivative of the free energy [29]), but the situation resembles what happens for the Hénon attractor [19]: There, nonconvergence of $\mu_{\max}$ is also present (being induced by the existence of almost stable periodic orbits). Another confirmation of irregular behavior for high values of the scaling index comes from the analysis of the scaling spectrum $s(\mu)$ (built up by just making a statistics of bandwidths $w_{i(n)}$) as compared with

TABLE I. Hausdorff-dimension estimates (by geometric extrapolation of finite order data) for a set of parameter choices, together with the diffusion exponent divided by 2.

Parameters	D_H	$\alpha/2$
$K=L=5$, $\hbar=2\pi/(18+\rho_{GM})$	0.68 ± 0.01	0.71 ± 0.02
$K=L=5$, $\hbar=2\pi/(6+\rho_{GM})$	0.56 ± 0.02	0.57 ± 0.05
$K=4$, $L=2$, $\hbar=2\pi/(6+\rho_{GM})$	0.95 ± 0.01	1.00 ± 0.01

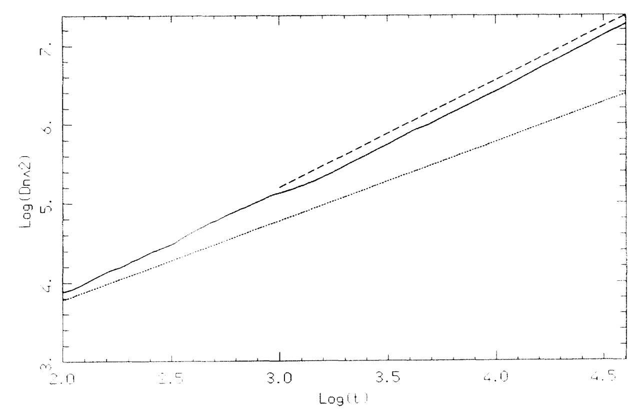

FIG. 4. Asymptotic dependence of $\log\langle\Delta n_t^2\rangle$ on $\log t$ for $K=L=5$, $\hbar=2\pi/(18+\rho_{GM})$ (solid line): The dashed line has a slope $2D_H=1.36$, while the dotted line has slope 1 (case of normal diffusion).

its convex envelope (obtained via a Legendre transform) (see Fig. 3). The lack of convergence for $\tau>\tau_c$ (which is close to zero) leads to the conclusion that only scaling indices pertaining to this $\tau\le\tau_c$ stable phase should have any meaning: This is the case for the Hausdorff dimension which is determined by $g(-D_H)=0$.

So far we have considered scaling features of the spectrum: We would like to connect these features to dynamical behavior. A claim has been made that $\langle\Delta n_t^2\rangle\sim t^\alpha$ and $\alpha=2D_H$ [30,31], where universal features of small-scale level separation should yield $D_H=0.5$ [11,30] in the sym-

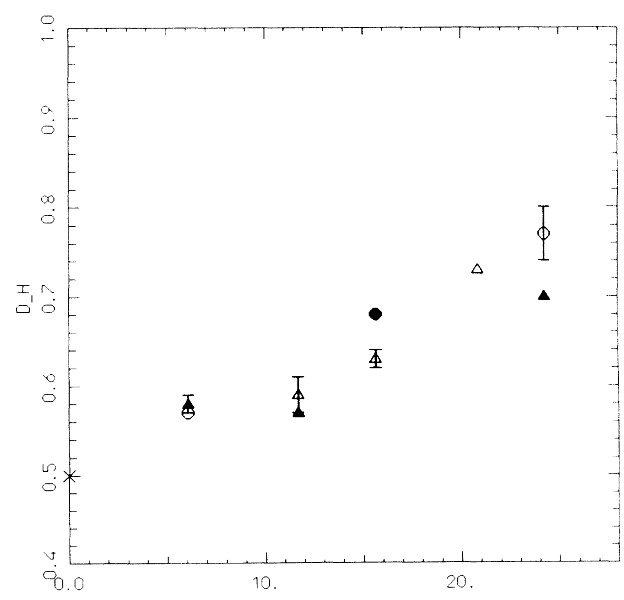

FIG. 5. Hausdorff-dimension estimates [(○) $\hbar=2\pi/(6+\rho_{GM})$, (●) $\hbar=2\pi/(18+\rho_{GM})$, (△) $\hbar=2\pi/(1+\rho_{GM})$, (▲) $\hbar=2\pi/(3+\rho_{GM})$] vs $K/\hbar$ for a few critical states; × denotes the limit point corresponding to the integrable Harper model. For a few points error bars are also reported.

metric case. A heuristic argument to explain the relation $\alpha = 2D_H$ is as follows [32]: Suppose we consider a time evolution of an initial wave packet up to time T. This amounts to exploring the quasienergy spectrum with a resolution bound by the uncertainty principle $\delta\omega \sim T^{-1}$ and this means that the quasienergy spectrum is resolved in $N_\omega(\delta\omega) \sim \delta\omega^{-D_H}$ components, that is $N_\omega(T) \sim T^{D_H}$. The corresponding spreading on an unperturbed basis is given by $\langle \Delta n_T^2 \rangle^{1/2} \sim T^{\alpha/2}$, so $\alpha = 2D_H$. We analyzed symmetric and asymmetric parameter pairs and found that the relation $\alpha = 2D_H$ is approximately satisfied (see Table I). The most remarkable feature is that the Hausdorff dimension depends on the parameters of the model even for the symmetric case; dynamically this means that the diffusion is anomalous [see Fig. 4]. Anomalous diffusion has also been observed for the Fibonacci model [33] and for random walks on fractal structures [34]; however, it was believed [11,12] that D_H in KHM should be the same as for Harper's model. In Fig. 5 we present numerical data showing the dependence of D_H on the kick amplitude $K/\hbar$ in the classically chaotic case; these results demonstrate an increase of D_H with $K/\hbar$. For moderate values of $K/\hbar$ the Hausdorff dimension D_H is close to the Harper limit $D_H = \frac{1}{2}$: This explains why previous numerical studies reported normal diffusion [10-12,31].

In conclusion we have analyzed spectrum scaling and dynamical evolution of the kicked Harper model. The multifractal analysis of its spectrum in the symmetric case reveals the coexistence of robust geometric scaling and irregular scaling, originating in a phase transition very similar to the one observed for the Hénon attractor. In contrast to the integrable Harper model we observe anomalous diffusion ruling dynamical behavior. The exponent depends on the parameters and is approximately equal to $2D_H$. As the Hausdorff dimension has been related to universal properties of spectra [30], our results should call for high-precision tests on level statistics, which should depend upon the choice of parameters.

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