Delocalization of Quantum Chaos by Weak Nonlinearity

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The destruction of quantum localization of chaos by weak nonlinearity is analyzed on the basis of the Chirikov criterion of overlapping resonances. It is shown that for the nonlinear coupling constant there is a delocalization border above which localization is destroyed. In this delocalized phase, excitation is described by a universal anomalous subdiffusion law. Applications of this phenomenon to nonlinear wave propagation in disordered media and Anderson localization are discussed.

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During the last decade great progress has been achieved in the understanding of quantum dynamics of classically chaotic systems [1]. One of the most interesting phenomena in this field is the quantum localization of dynamical chaos [2]. At first this effect was observed in numerical experiments with the kicked rotator model [3] and later it was explained theoretically [4,5]. In some respect this phenomenon can be considered as the dynamical version of Anderson localization for which quantum interference leads to suppression of classical chaotic diffusion [5,6]. The manifestations of dynamical localization were observed not only in the numerical experiments with simple models but also in real laboratory experiments with hydrogen atoms in a microwave field (see [2] and references therein).

Another domain where the discussed phenomenon can take place is the propagation of linear waves in waveguides (or fibers). For propagating waves the localization suppresses the growth of aperture angle with the waveguide length and leads to effective intensity transmission [7]. Here a new and interesting type of problem arises if the waves propagate in a nonlinear media. This problem puts the question of general interest: How is the localization, appearing as the result of linear wave interference, modified by the introduction of small nonlinear wave interaction? We will see that there is a critical strength of nonlinear coupling below which the localization remains. Above this border a delocalization takes place and the number of excited linear modes grows according to the derived anomalous subdiffusion law. This excitation is much slower than the chaotic diffusion of classical rays so that the suppression of classical chaos by quantum (or linear waves) interference is not completely destroyed. The obtained subdiffusion law is of a universal nature since it always takes place in the limit of weak nonlinearity when the energy of nonlinear four-wave interaction ($|\psi|^4$) is much less than the energy of linear modes.

In such a formulation the problem of destruction of quantum localization of chaos by weak nonlinearity is closely connected with the problem of propagation of nonlinear waves in disordered systems (see a recent interesting review [8]). However, while there the main theoretical [8] and very recent experimental [9] efforts were devoted to the investigation of properties of the stationary transmission via a nonlinear layer and of the properties of stationary solutions, here I will address mainly the time-dependent problem of destruction of localized states.

Recently a few attempts [10,11] have been made to understand how nonlinear interaction manifests itself in the domain of quantum chaos. The appearance of nonlinear terms in the Schrödinger equation can arise as the result of a mean field approximation for many body interactions [10] or for waves as the result of propagation through nonlinear media [11]. In the last case the wave propagation under quite general assumptions can be described by the nonlinear Schrödinger equation (NSE). Sinusoidal modulation of the waveguide boundary then leads, for small aperture angles, to the kicked NSE model of wave propagation [11]:

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial^2 x} - \tilde{\beta}|\psi|^2\psi + k\cos x\,\psi \sum_{m=-\infty}^{+\infty}\delta(t-mT/2)\,. \tag{1}$$

Here $\tilde{\beta}$ and k are two parameters which measure the nonlinearity and the kick strength, respectively, we set $\hbar=1$, and the integrated probability $|\psi|^2$ is the integral of motion equal to 1. Motion is considered on a ring, so that $\psi(x,t)=\psi(x+2\pi,t)$. The kicks occur with the period $T/2$. The time variable plays the role of longitudinal direction z along the waveguide. In the linear case ($\tilde{\beta}=0$) this equation describes the kicked rotator model. The periodic kicks lead to some energy excitation since without them the energy of the NSE is an exact integral of motion. Under excitation we will understand the growth of the number Δn of excited linear modes. For a large value of Δn the contribution from the nonlinear term is relatively small. Let us mention that the nonlinear term in (1) is considered as a given one and we will not discuss the conditions under which it is valid.

Numerical investigations carried out in [11] showed that the nonlinear interaction in the model (1) does not destroy the quantum suppression of classical diffusion in the chaos region $K=kT>1$. The question as to the asymptotic law of excitation remained open, however. For a better understanding of the process of excitation in the kicked NSE model and general properties of the destruction of localization of chaos by nonlinear effects, I introduce a kicked nonlinear rotator (KNR) model. This model is much simpler for numerical simulation than (1) and allows us to understand the way in which nonlinearity destroys localization. The dynamics of the model is given by the following map:

$$\bar{A}_n = \sum_m (-i)^{n-m} J_{n-m}(k) A_m \exp(-i\tfrac{1}{2}Tm^2 + i\Delta\phi_m); \quad (2)$$

$$\Delta\phi_m = \beta |A_m|^2 .$$

The Bessel function J_{n-m} appears as the result of kick which gives $\tilde{\psi}(x) = \exp(-ik\cos x)\psi$. The Fourier harmonics A_n are connected with the wave function by the relation $\psi(x) = (1/\sqrt{2\pi})\sum_n e^{inx} A_n$. In fact (2) is practically the same map as for the kicked rotator but now the change of the phase of Fourier harmonic A_n between two kicks depends on the amplitude of the harmonic. This dependence is the same as for one harmonic in NSE when $\psi(x) = A\exp(inx)/\sqrt{2\pi}$ and $\tilde{\beta}$ is small. Therefore, in this case we have $\beta = T\tilde{\beta}/4\pi$. It is natural to assume that such a change of evolution between kicks will not change the asymptotic law of spreading. Indeed, without kicks the square width of the distribution $\sigma = (\Delta n)^2$ remains approximately constant in both cases. In the presence of kicks, transitions between harmonics due to the nonlinear term in NSE are not of primary importance since such transitions also take place in the KNR model (2) due to the kick. Furthermore the type of nonlinearity is the same in both models.

At first glance it seems that the nonlinear phase shift $\Delta\phi_n$ in (2) is not very important since for many excited harmonics (levels) its value is small. Indeed, from the normalization condition it follows that $|A_n|^2 \sim 1/\Delta n$ and $\Delta\phi_n \approx \beta|A_n|^2 \approx \beta/\Delta n$, where Δn is the width of the distribution over unperturbed levels. Consequently the growth of the width leads to a decrease of the nonlinear shift. However, even a small shift can change the nature of motion leading to slow delocalization. This can be understood on the basis of the Chirikov criterion of overlapping resonances [12]. According to this criterion a chaotic spreading takes place over the domain of overlapped resonances. In the case of classical chaos $K > 1$ the average distance between the resonances is $\Delta\omega \sim 1/\Delta n$ since all quasienergies of linear problem ($\beta = 0$) are homogeneously distributed in the interval $(0, 2\pi)$. On the other side, the width of a resonance $\delta\omega$ is of the order of the nonlinear shift $\beta|A_n|^2 \sim \beta/\Delta n$. Therefore, the overlapping parameter $S = \delta\omega/\Delta\omega$ is of the order of β ($S \sim \beta$) and chaotic spreading over all levels n will take place for β larger than some critical value $\beta_c \sim 1$. Below the critical value the initial distribution is always localized in n. In the limit of very small β one enters in the Kolmogorov-Arnol'd-Moser regime where the motion is integrable almost everywhere [13].

To obtain the law of spreading in the delocalized phase it is convenient to use the basis of localized eigenstates C_m of the linear case with $\beta = 0$. For a larger Δn the nonlinear frequency shift in (2) is small and equations for amplitudes of localized states can be written in the form

$$i\frac{\partial C_m}{\partial t} = \epsilon_m C_m + \beta \sum_{m_1, m_2, m_3} V_{m,m_1,m_2,m_3} C_{m_1} C^*_{m_2} C_{m_3}, \quad (3)$$

where the transformation between the unperturbed basis and linear eigenstates is determined by $A_n = \sum_m R_{n,m} C_m$ and $V_{m,m_1,m_2,m_3} = \sum_n R^{-1}_{n,m} R_{n,m_1} R^*_{n,m_2} R_{n,m_3}$ and ϵ_m are quasienergies of linear problems. Here and below t is measured in the number of kicks.

In the regime of quantum localization of chaos the values of R can be approximately represented as $R_{n,m} \approx \exp(-|n-m|/l - i\chi_{n,m})/\sqrt{l}$, where $l \approx k^2/4$ is the linear localization length and $\chi_{n,m}$ are random phases. This leads to the following estimate of the matrix elements $V \sim 1/l^{3/2}$. The total number of terms contributing to the sum in (3) is of the order of l^3. All quasienergies ϵ_m are distributed in the interval $(0, 2\pi)$ so that the typical density of frequencies in (3) is $\rho \sim l^3$. From that we find the rate of transition from a quasienergy level m to other levels: $\Gamma_c \sim (\beta C^3 V)^2 \rho \sim \beta^2/(\Delta n)^3$, where we used the relation $C^2 \sim 1/\Delta n$ valid for $\Delta n \gg l$. Since during a transition the change of m is of the order of l we find the estimate for the diffusion rate in n (in m it is the same): $(\Delta n)^2/\Delta t = D_\beta \sim l^2 \Gamma_c \sim l^2\beta^2/(\Delta n)^3$. This gives the following law of spreading over unperturbed levels in the delocalized phase with $\beta > \beta_c$:

$$(\Delta n)^2 \approx \gamma \beta^{4/5} l^{4/5} t^{2/5}, \quad (4)$$

where γ is some constant. It is interesting to note that in fact the diffusion rate D_β is given by the same expression

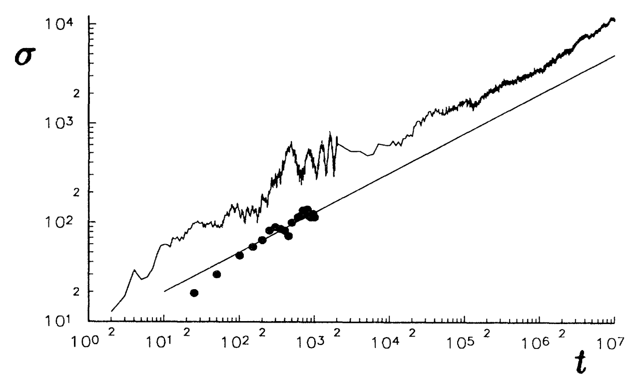

FIG. 1. The square width of the distribution over unperturbed levels $\sigma = \langle n^2 \rangle$ in the KNR model (2) as a function of time t measured in the number of kicks: $k = 5$, $T = 1$, $\beta = 1$ (full line). Initial state is $n = 0$. Dots represent the data (taken from [11], Fig. 4) for the kicked NSE model (1) with $k = 2.5$, $T = 2$, $\tilde{\beta} = 10$, and initial condition in the form of soliton. The straight line is drawn simply to show the theoretical slope $t^{2/5}$.

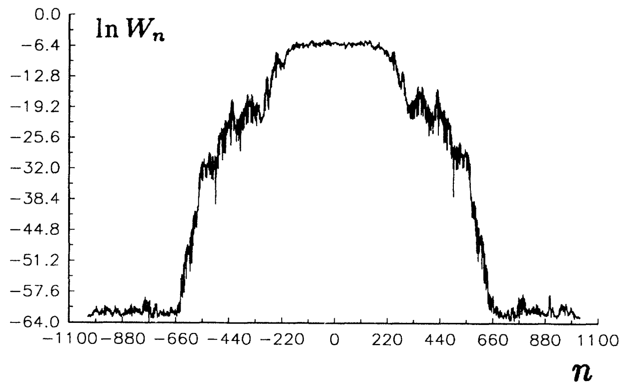

FIG. 2. Probability distribution over unperturbed levels $W_n = |A_n|^2$ in the KNR model (2) for parameters of Fig. 1 at $t = 10^7$.

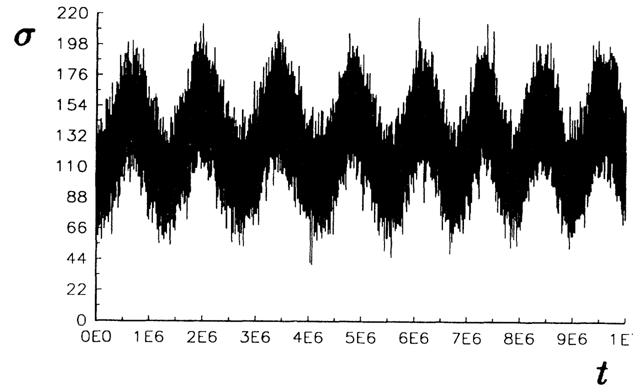

FIG. 3. The square width of the distribution over unperturbed levels $\sigma = \langle n^2 \rangle$ in the KNR model (2) as a function of time t measured in the number of kicks: $k = 5$, $T = 1$, $\beta = 0.03$.

as in the problem of the destruction of localization by noise $D \sim l^2/t_c$ [14] but now the coherence time $t_c = \Gamma_c^{-1}$ is determined by the nonlinear interaction. Let us stress that although Δn grows unlimitedly the rate of growth is much slower than the classical diffusion rate $(\Delta n)^2 \approx 2lt$. In this sense the suppression of chaos by quantum interference is not completely destroyed. [However, the soliton solution of (1) is destroyed in this delocalized phase.]

The results of numerical investigations of the KNR model (2) are presented in Figs. 1 and 2. From Fig. 1 we see that the spreading continues during 10 million kicks and the power law of dependence on time is in good agreement with the theoretical expression (4). The excitation in the kicked NSE model is also in agreement with (4). However, here the interaction time is much shorter and further more careful checks are required. The two cases presented in Fig. 1 give approximately the same value of the constant $\gamma \approx 3$ in (4). The distribution of probability over unperturbed levels in model (2) presented in Fig. 2 shows a tendency to form a flat plateau in the center around the initially excited level. The size of the plateau grows with time according to the law (4). Let us note that the distribution is not exactly symmetric with respect to the change $n \rightarrow -n$ due to exponential growth of roundoff errors in the chaotic delocalized phase.

An absolutely different type of behavior takes place in the localized phase $\beta < \beta_c$ (Figs. 3 and 4). Here the width of the distribution remains finite and the probability distribution is exponentially localized over unperturbed basis. Therefore, in this case nonlinearity is too weak and it does not destroy quantum localization. According to the numerical data the value of β_c is of the order of 0.1. However, it is difficult to find its exact value due to slow growth of Δn in the delocalized phase.

It is interesting to make a few remarks about the local stability of motion in the KNR model. For $\beta \ll \beta_c$ the motion is integrable practically everywhere except in exponentially narrow chaotic layers. Therefore the Lyapunov exponent is zero and motion is locally stable. For $\beta > \beta_c$ the motion is chaotic and the Lyapunov exponent λ is of the order of the width of the resonance $\delta\omega \sim \beta/\Delta n$. Since Δn increases quite slowly, the exponential local instability for nearest trajectories takes place for relatively short time intervals (which may, however, contain many kicks). This explains the observation of practical irreversibility in time in [11]. However, asymptotically in time Δn grows according to (4), the distance between nearby trajectories grows like $\exp(\alpha t^{0.8})$, and the Lyapunov exponent is formally equal to zero ($\alpha > 0$ is some constant).

Another interesting question is about a type of excitation in the classically integrable regime $K = kT < 1$. Numerical simulation shows that the transition to this regime leads to a sharp decrease of the excitation. For example, for the case of Fig. 1 the decrease of T up to 0.08 (with other parameters being the same, $K = 0.4$) gives a value of $(\Delta n)^2$ at $t = 10^7$ that is 20 times smaller than that in Fig. 1. The dependence of $(\Delta n)^2$ on time can be approximated by the power law $t^{0.2}$. In fact, the discussed regime corresponds to a regime where the quan-

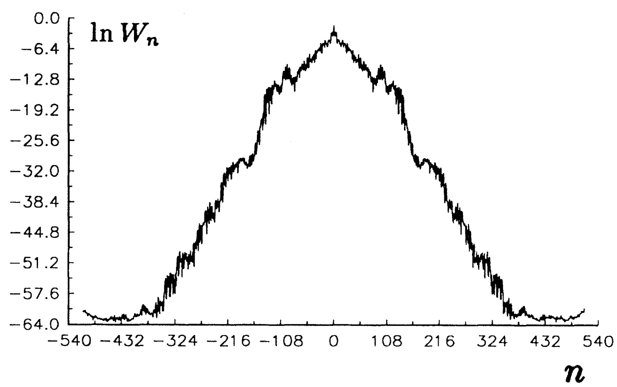

FIG. 4. The same as Fig. 2 for parameters of Fig. 3.

tum tunneling between classical islands of stability is effected by nonlinear interaction. In some sense this case resembles the situation considered in [10]. However, now that the chain of islands (resonances) is infinite this can lead to an infinite spreading. Further investigations are required for this regime.

In summary, it was demonstrated how and under what conditions quantum localization of chaos is destroyed by weak nonlinearity. Such an effect can arise in the propagation of waves through nonlinear disordered media. Since the type of nonlinearity in these systems is usually the same as in the nonlinear Schrödinger equation [and as in the introduced KNR model (2)], it implies that the obtained results are quite universal. For example, the existence of high-order nonlinear corrections to NSE (e.g., $|\psi|^6$) will not change the law of excitation (4). Equation (4) gives the asymptotic law of spreading in the regime where the soliton is completely destroyed.

As a result of the similarity between the localization of quantum chaos and Anderson localization the same type of behavior will take place in the Anderson model with nonlinear interaction:

$$i\frac{\partial \psi_n}{\partial t} = E_n \psi_n - \beta |\psi_n|^2 \psi_n + \psi_{n+1} + \psi_{n-1}, \qquad (5)$$

where E_n are randomly distributed in the interval $(-W, W)$ with $W < 1$. For the model (5) the above arguments based on the overlapping of resonances work in the same way giving the same estimate for β_c and the same law of spreading (4) in the delocalized phase. The same type of arguments can be also applied for higher dimensions in the localized regime with localization length $l > 1$. Here it is necessary to mention the situation when the random potential changes smoothly compared to the size of the soliton. In this case the lifetime of the soliton can be quite large (but always finite), which can give effective propagation along the chain. However, after the destruction of the soliton the spreading (excitation) is given by the law (4). If the potential changes sharply on a distance comparable with the soliton size then the soliton lifetime is small (as for the case of Fig. 1) and the regime (4) starts immediately.

The obtained results, contrarily to the experiments [9], show that the localization is destroyed by nonlinearity for a coupling strength exceeding a critical value. This discrepancy is probably connected with the significantly nonlocal character of the nonlinear interaction in the experimental system [9]. It will be interesting to make experiments on the destruction of Anderson localization by weak local nonlinear interaction.

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