Chaotic and ballistic dynamics for two-dimensional electrons in periodic magnetic fields

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We investigate the classical properties of electrons moving in a superposition of a uniform and periodically oscillating magnetic field. The most interesting dynamics occurs in the case where the uniform and periodic fields are of comparable order of magnitude and the periodic component originates from a planar arrangement of spins. For small energies almost the complete phase space is regular. With increasing energy the fraction of irregular orbits increases and eventually the phase space becomes completely chaotic. For higher energies we observe the appearance of a ballistic mode which allows the electrons to travel with high velocity through the magnetized spin lattice. This regular ballistic mode might be of relevance for transport processes in solid-state physics.

Recently the *t-J* model which describes the behavior of charged excitations on a spin lattice with nearest-neighbor interaction has attracted much attention[1–3] because of its possible relevance in superconductivity. The arrangement of spins is expected to create a spatially varying magnetic field and the charged excitations, i.e., electrons or holes, are moving within this field. Therefore the natural question arises: what does the two-dimensional (2D) dynamics of an electron in, for example, a period magnetic field look like? This is precisely the subject of investigation of the present paper. Apart from its relevance in solid-state physics this question is also of importance for plasma and accelerator physics where the stability of the motion of charged particles in spatially oscillating external magnetic fields is an inherent problem. Finally, the above question is also of interest on its own.

The nonrelativistic motion of a charged particle in a homogeneous magnetic field is known since the work of Landau.[4] Classically the particle performs a circular motion whose radius is inverse proportional to the field strength. Therefore the classical trajectories of, for example, an electron in a uniform magnetic field are confined, i.e., experience only a bounded region of coordinate space. The following interesting questions arise now: How does an additional periodic magnetic field change the dynamics and, in particular, the confinement properties of the motion? Will confinement be preserved or does the additional presence of a spatially oscillating magnetic field allow for an unbound motion in coordinate space? Does both chaotic and regular dynamics occur and what portions of the phase space do they occupy depending on the values of the parameters, i.e., energy and field strengths? In a very recent investigation[5] on the motion of fast particles in strongly fluctuating magnetic fields it was shown that an approach via classical dynamics yields, to some extent, a reasonable description of the dynamics of the system. In Ref. 5 the authors considered fermions which move with large velocity, i.e., close to the Fermi surface, in a spatially randomly fluctuating magnetic field with constant correlation function. In this paper we investigate the classical dynamics of 2D electrons in periodic magnetic fields which might give relevant information for the understanding of the physics of 2D electrons.

Let us now discuss the explicit form of our two-dimensional model. According to the above argumentation the magnetic field B is assumed to take on the following appearance:

$$B = B_0 + B_1 \cos x + B_2 \cos y \ , \tag{1}$$

which contains a uniform component B_0 and a spatially varying part ($B_1 \cos x + B_2 \cos y$). The magnetic field in Eq. (1), which is assumed to point along the z direction, possesses an infinite number of minima, maxima, and saddle points. The Hamiltonian for a charged particle with mass m and charge e in the magnetic field (1) reads

$$H = \frac{1}{2m}\left[\mathbf{p} + \frac{e}{c}\mathbf{A}\right]^2 . \tag{2a}$$

For the vector potential $\mathbf{A}$ in the symmetric gauge we have

$$\mathbf{A} = [(\tfrac{1}{2})\cdot B_0 y + B_2 \sin y, (-\tfrac{1}{2})\cdot B_0 x - B_1 \sin x] \ . \tag{2b}$$

The resulting Hamiltonian equations of motion have the form

$$\dot{\mathbf{p}} = -\frac{e}{mc}\begin{bmatrix} 0 & \partial_x A_y \\ \partial_y A_x & 0 \end{bmatrix}\left[\mathbf{p} + \frac{e}{c}\mathbf{A}\right] , \tag{3a}$$

$$\dot{\mathbf{r}} = \frac{1}{m}\left[\mathbf{p} + \frac{e}{c}\mathbf{A}\right] , \tag{3b}$$

where $\mathbf{r}$ and $\mathbf{p}$ are the position vector of the particle and its canonical conjugated momentum. The model given by Eqs. (2) contains three parameters: the amplitudes of the uniform (B_0) and periodically oscillating (B_1,B_2) magnetic-field strengths. In the following we use the units $m=e=c=1$.

For small amplitudes of the oscillating field strength (B_1,B_2), i.e., for $B_0 \gg B_1,B_2$, the motion of the particle can be described and understood in the framework of classical perturbation theory. For a particle in a uniform magnetic field the so-called guiding center, which is the center of its classical Larmor circle, is a conserved quantity and therefore fixed in space. In an inhomogeneous field the gradient of the field strength is responsible for a drift of this guiding center. The resulting drift velocity is given by the following expression:[6,7]

$$\mathbf{v}_D = -\frac{(\mathbf{F}\times\mathbf{B})}{B^2} , \tag{4}$$

where $\mathbf{F}=-\mu\cdot\nabla B$ is the force averaged over a gyro orbit. $\mu=(1/2B)\cdot\dot{\mathbf{r}}_\perp^2$ is the magnetic moment and $\mathbf{r}$ is the position vector of the guiding center ($\mathbf{r}_\perp$ means the component perpendicular to the magnetic-field vector). In the drift approximation μ is an adiabatic invariant and is assumed to be well conserved.

Let us consider the case $B_0 \gg B_1,B_2$ of our model Eqs. (2). Then the above mentioned drift approximation holds for an arbitrary energy E. The Larmor circles given by the uniform field B_0 perform a drift with the velocity $\mathbf{v}_D$ which is perpendicular to the field and to the field gradient [see Eq. (4)]. In particular the entire phase space is regular. In addition, if we have $B_1 \neq B_2$ we obtain a certain subset of classical trajectories which travel along the potential lines through the lattice and therefore perform an unbounded regular motion. This ballistic motion of the Larmor circle always takes place in the direction of the axis of the smaller field component, i.e., along the x axis if $|B_1|<|B_2|$ and along the y axis if $|B_1|>|B_2|$. The amount of traveling trajectories in phase space depends on the ratio (B_2/B_1). However, a physically more relevant situation is the case $B_1=B_2$. Indeed since the periodic magnetic field has its origin in the planar arrangement of spins the field should be symmetric with respect to the x and y coordinates of the plane, i.e., we have $B_1=B_2=B_s$ for the two independent directions. For $B_1=B_2$ only the trajectories along the separatrix can go to infinity and almost all trajectories are confined. From the expression for the drift velocity (4) it follows that in the case of $B_0 \gg B_1,B_2$ the particle is moving along the equipotential lines of the potential $V(x,y)=(B_1/B_0)\cos x+(B_2/B_0)\cos y$. This can also be considered as a motion in the phase space of the Hamiltonian $H_{\text{eff}}=V(x,p)$, where $y=p$ plays the role of the momentum. This Hamiltonian is the Hamiltonian of the Harper problem[8] and, therefore, we come to the conclusion that our model reduces to the Harper model in the limit $B_0 \gg B_1,B_2$.

Let us now consider the situation $B_1=B_2=B_s$ for $B_0 \approx B_s$ in more details. Here no obvious answer to the question of the dynamical behavior of the system exists. The drift approximation is generally not valid and it is expected that the underlying classical dynamics will exhibit a great variety of phenomena. This case is precisely the subject of investigation of this paper. In the following we will choose the nearby values $B_0=1$ and $B_s=1.4$ for the two field strengths. (One atomic unit of B corresponds to 2.35×10^5 T.) We remark that the ratio (B_s/B_0) is the relevant quantity for the classical dynamics, whereas the absolute order of magnitude of the field strengths is of minor importance, i.e., does not change the general conclusions drawn in this paper.

We investigate the classical dynamics of the model (2) depending on the values of the total energy E. With the help of Poincare sections we will show the different types of motion in phase space and will learn about the physical behavior and properties of our model system (2). Two types of Poincaré sections will be constructed: sections in the (x,y)-coordinate plane for vanishing velocity $v_y=0$ in the y direction and sections in the (x,v_x) and (y,v_y) plane for $y=\pi$ and $x=\pi$, respectively. Due to the periodicity of the magnetic field the (x,y) coordinates are always taken modulo 2π. Let us begin with small energies, i.e., $E<1$. Figure 1 shows a Poincaré section in the (x,y) plane for an energy $E=0.03$. To a large extent phase space is dominated by regular motion and there exist large regular islands of more or less circular shape around the positions of the minima and maxima of the field strength. Only close to the separatrix there exists a layer of chaotic motion which becomes thinner with decreasing energy. This layer comes much closer to the saddle points than to the minima and maxima of the magnetic field. However, the chaotic layer does not reach the boundary of the square ($x\in[0,2\pi],y\in[0,2\pi]$). This is only correct for sufficiently small energies which correspond to a small Larmor radius of the drift motion. In particular, almost all trajectories, the dominating regular ones as well as the chaotic ones, only experience a bounded range of coordinate space, i.e., are confined. Only along the separatrix it is possible to travel through the lattice. The regular small-amplitude motion in the vicinity of the maxima and

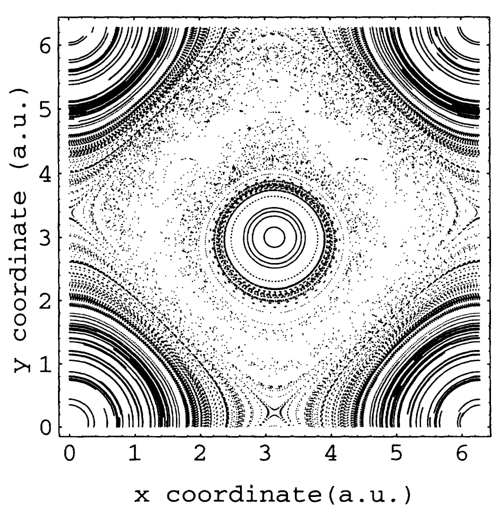

FIG. 1. The Poincaré section in the (x,y) plane for $v_y=0$. (x and y are taken mod 2π.) The values of the total energy E, uniform field component B_0 and oscillating field component B_s are $E=0.03$, $B_0=1.0$, and $B_s=1.4$ in atomic units.

minima of the magnetic field B can still be described in the drift approximation, i.e., as a drift of the Larmor circles along the effective-potential lines described above.

With increasing total energy E the fraction of chaotic trajectories in the phase space increases. Above a certain, still rather small, value of the energy the chaotic layer in the (x,y)-Poincaré section reaches the boundary of the square $([0,2\pi],[0,2\pi])$, which means that the chaotic trajectories are allowed to move through the lattice. With increasing time they experience an increasing part of the lattice (see below for an illustration). If we proceed to still larger values of the energy the chaotic component takes over more and more of the phase space and finally covers the whole phase space. This happens if the Larmor radius, which is due to the uniform field B_0, becomes comparable to the periodicity length of the oscillating field component. Figure 2 shows the Poincaré section in the (y,v_y) plane for a corresponding energy of $E=8.0$. No regular structures survived.

For even higher energies we observe around $E\approx 10$ in, for example, the (x,y)-Poincaré section the reappearance of islands of regular motion which are embedded in a sea of chaotic trajectories. With increasing energy these islands grow and eventually occupy a substantial part of the phase space. The important observation now is that there exists a new type of regular islands and trajectories which is not present for energies $E<10$. These trajectories perform a ballistic motion, i.e., travel through the whole lattice with a high velocity. This ballistic motion can take place along the direction of the x as well as y axis. The corresponding regular ballistic island in the surface of section persists and even grows with increasing energy. To illustrate these results we have plotted the Poincaré sections in the (x,y) and (x,v_x) planes in Figs. 3 and 4, for an energy $E=28.0$, respectively. Let us first have a closer look at Fig. 4. Apart from the dominating sea of chaotic trajectories large regular islands exist. The island located at the upper border of the allowed velocity scale v_x, whose maximal value is $v_{x_{\max}}\approx 7.48$, is a ballistic island and the corresponding regular trajectories travel with high velocity v_x through the lattice (see below for an example). The central island which is located at the x axis around $x\approx 2.6$ corresponds to a ballistic motion in the y direction [this can be seen by looking at the Poincaré section in the (y,v_y) plane]. The remaining regular structures of the section in Fig. 4 correspond to regular trajectories which are confined, i.e., perform oscillations in a bounded range of coordinate space. Figure 3 shows the Poincaré section in the (x,y) plane for $v_y=0$. The ballistic island of high-v_x velocity presented in Fig. 4 appears as a broad regular structure covering the whole range of x coordinates. In Fig. 5 we have plotted a ballistic trajectory with high-v_x velocity in the (x,y) plane. The motion in the x coordinate is almost pure translational, i.e., $x\approx v_x\cdot t$, whereas the motion in the y coordinate consists of quasiperiodic oscillations with a small

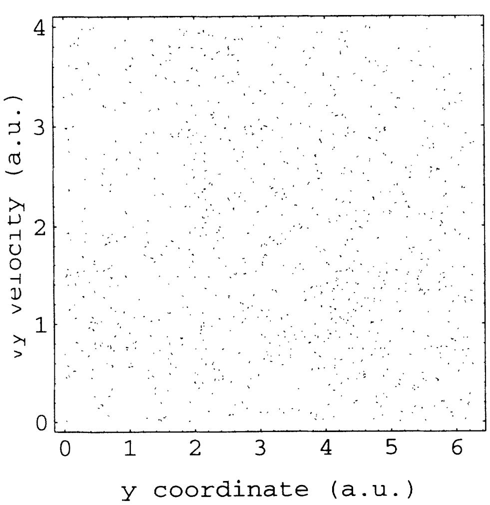

FIG. 2. The Poincaré section in the (y,v_y) plane for $x=\pi$. The parameter values are $E=8.0$, $B_0=1.0$, and $B_s=1.4$.

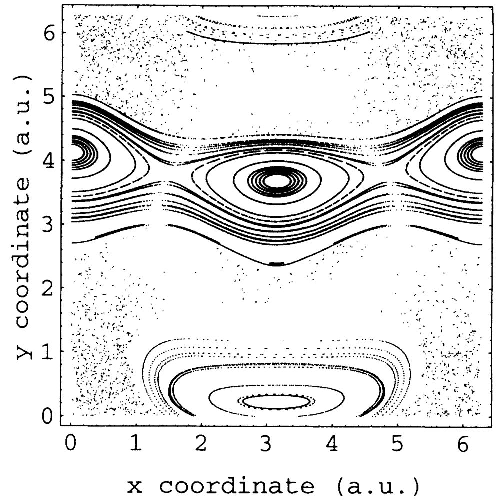

FIG. 3. The Poincaré section in the (x,y) plane for $v_y=0$ and $E=28.0$, $B_0=1.0$, and $B_s=1.4$. (x and y are taken mod 2π.)

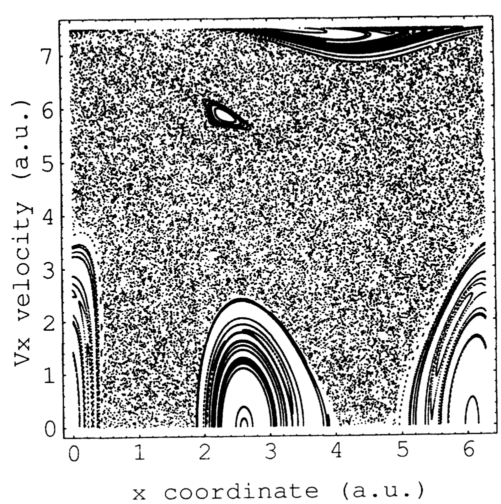

FIG. 4. The Poincaré section in the (x,v_x) plane for $y=\pi$. Same parameter values as in Fig. 3. The regular island at the upper border of the velocity v_x arises from ballistic motion in the x direction.

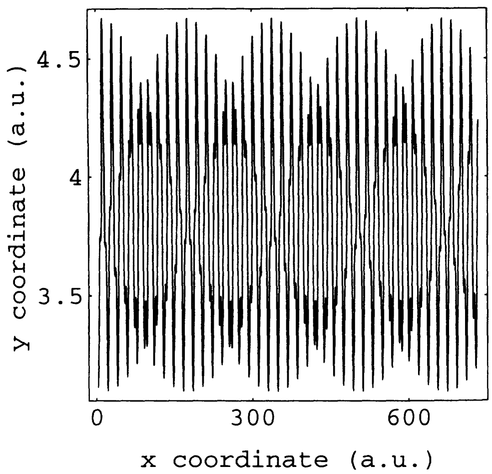

FIG. 5. A ballistic trajectory with high velocity v_x and small-amplitude oscillations in the transversal y direction. The underlying energy and field strengths are the same as in Figs. 3 and 4. The trajectory contributes to the regular ballistic island in Fig. 4 and correspondingly to the broad regular structure in the Poincaré section of Fig. 3. The propagation time for the trajectory is $T=100$ (in atomic units).

amplitude. In order to demonstrate the difference between the unbounded chaotic motion and the unbounded high-velocity translational motion we show additionally in Fig. 6 a chaotic trajectory in the (x,y) plane for the same energy $E=28.0$. Due to the random-walk-like behavior of the chaotic trajectories the mean distance is for the same propagation time much smaller for the chaotic (diffusive) case than for the ballistic one.

The occurrence and stability of a ballistic mode for high energies and a field configuration $B_s > B_0$ can be explained in a qualitative way via the underlying equations of motion of the particle in the magnetic field B. Let us assume we have a ballistic motion in the x direction. Effectively, i.e., to lowest order, this motion can be described by a pure translation $x=v_x \cdot t$ with high velocity v_x. Then the Newtonian equation of motion in the y direction reads as follows:

$$d^2y/dt^2 = -v_x\{B_0 + B_s \cdot [\cos(v_x \cdot t) + \cos y]\} . \quad (5)$$

Since v_x is large we have a fastly oscillating purely time-dependent term $\cos(v_x \cdot t)$, which is averaged out if we are only interested in the behavior of the mean of the transversal degree of freedom y. As a result we obtain the following effective equation of motion in the y direction:

$$d^2y/dt^2 = -v_x(B_0 + B_s \cdot \cos y) . \quad (6)$$

Therefore, the effective potential $V=v_x(B_0 y + B_s \cdot \sin y)$ describes the averaged dynamics in the transversal y direction for ballistic motion in the x direction. In Fig. 7 we have illustrated the potential V with the values for v_x, B_0, and B_s according to the ballistic island in Fig. 4. It exhibits alternating minima and maxima, arising from the cosine, at the positions given by $y=\arccos(-B_0/B_s)$. A necessary condition for the existence of these alternating extrema and the resulting potential wells is, of course, $B_0 < B_s$, i.e., the uniform field component must be smaller than the amplitude of the periodic field component. The small-amplitude oscillatory motion in the y direction takes place inside the potential wells of V, which are illustrated in Fig. 7. In particular for our trajectory in Fig. 5 the transversal y motion is restricted to the encircled potential well which has its minima at $y \approx 3.92$.

For constant energy and increasing values of the uniform field component B_0 the size of the ballistic island decreases and finally for some critical value $B_0 = B_{cr}$ the ballistic mode disappears completely. In our case of

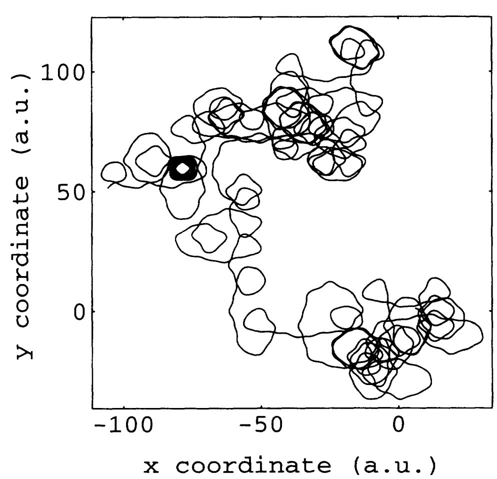

FIG. 6. A chaotic trajectory with energy and field strengths as given in Figs. 3 and 4. The propagation time is $T=500$. With increasing time the trajectory experiences an increasing volume of the (x,y) coordinate plane. The trajectory contributes to the chaotic sea in the sections of Figs. 3 and 4.

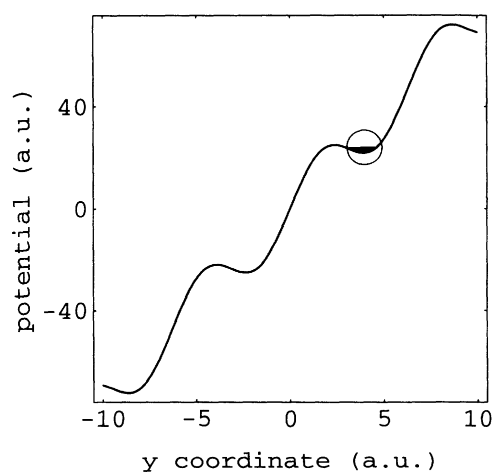

FIG. 7. An illustration of the effective potential $V=v_x(B_0 y + B_s \sin y)$ which is responsible for the transversal part of the ballistic motion. The field strengths are $B_0=1.0$ and $B_s=1.4$ and we have chosen $v_x=7.49$ according to the island of ballistic motion in Fig. 4. The transversal y motion of the trajectory presented in Fig. 5 takes place in the encircled potential well.

$B_s=1.4$ we obtain $B_{cr} \approx B_s$. Another condition for the appearance of the ballistic mode is that the energy must be larger than some critical value which is approximately determined by the condition that the Larmor radius must be larger than the length scale α of the variation of the magnetic field (α is 2π in our case). This condition is equivalent to the requirement that the frequency of the force in Eq. (5) of the motion in x is much higher than the frequency of the motion in the minimum. This condition gives the estimate for the critical energy above which the ballistic mode appears: $E > E_{cr} \approx (\alpha B)^2$. It is important to notice that the above observed ballistic motion continues to exist not only for a broad range of high energies but also if an additional external electric field is switched on. This means that the ballistic mode might be used as a guide for particles in a regular channel which can have large energies. In particular, the particles will stay in this regular mode if they are accelerated by an external electric field. This behavior is somehow similar to the autofocusing effect well known in accelerator physics.

One can ask the question whether the ballistic mode exists in the case when the classical electron is moving in a periodic static potential $U(x,y)$ (which we assume to be symmetric in the x-y plane) and a homogeneous magnetic field. It is known that in the limit of a strong field (that means the Larmor frequency is much higher than the frequency of oscillations in the potential U) the problem can be reduced to the Harper model[8] with the critical parameter value (regime of one Landau level). In this case the classical trajectories are confined. In the general case the conditions for the existence of a ballistic mode are given by the following estimate of the energy: $(\alpha B)^2 < E < U^2/(\alpha B)^2$. The first inequality is the same as in the case of a purely magnetic field, i.e., the Larmor radius should be larger than the periodic scale. The second inequality is equivalent to the condition $v_x B_0 > v_x B_s \cong dU/dx \cong U/\alpha$ which follows from the equation analogous to Eq. (6). From these estimates it follows that in the case of a static potential the ballistic mode will disappear in the limit of high energy. In the case of a periodic magnetic field the upper boundary for the energy disappears. We mention that small ballistic islands at intermediate energies in a static potential and a homogeneous magnetic field were found recently in the corresponding numerical simulations (see Ref. 9).

In conclusion we have investigated the classical dynamics of a charged particle moving in a spatially periodic magnetic field superimposed to a uniform field. The most interesting case arises if the two field amplitudes are of comparable order of magnitude. For small energies the Larmor radius, which belongs to the uniform field component, is small in comparison with the periodicity length of the oscillating field component. Therefore the phase space is dominated by regular motion and chaos appears only as a thin layer in the vicinity of the separatrix. In particular regular motion is confined in coordinate space whereas the chaotic trajectories above a certain energy value are allowed to travel in a diffusive way through the lattice. With increasing energy the Larmor radius grows and eventually becomes comparable to the periodicity length. In the latter energy regime the phase space is completely chaotic. If we go to even higher energies regular structures reappear but additionally a new kind of regular motion appears: the ballistic mode. A ballistic trajectory propagates with high velocity in the direction of one of the two Cartesian coordinate axes. Along this axis it can approximately be described by a translational motion with constant high velocity. For the transversal degree of freedom an effective potential picture was given which is qualitatively able to describe the small-amplitude transversal motion. We emphasize that instead of the fact that the average value of the magnetic field is nonzero the particle motion is not confined and we obtain a type of unbounded regular motion. The latter is an effect of the periodic fluctuations of the field. It is interesting to discuss the case when the periodic component of the magnetic field is replaced by a random-varying magnetic field with approximately the same amplitude and correlation length. In that case for $B_0 < B_s$ and $E > E_{cr}$ still some ballistic modes will appear. However, the randomness of the magnetic field will act like some high-frequency noise which will try to throw out the particle from the local minimum corresponding to ballistic propagation. Of course, the ballistic mode will finally be destroyed but since the frequency of the noise is very high, the lifetime τ_B in the local minimum (ballistic mode) will be exponentially large [$\exp(r_L/\alpha)$ with $(r_L/\alpha) \gg 1$]. Here r_L is the Larmor radius and α is the typical scale of variation of the random-field component. The exponential estimate follows from the fact of a large frequency difference and from the analyticity of the motion. The large value of τ_B implies a very large value of the diffusion rate (conductance) which is proportional to τ_B. Further investigations are required for this interesting regime.

The appearance of a ballistic mode in the dynamics of our model Hamiltonian might suggest its relevance in different areas of physics. For accelerator physics this regular ballistic channel might serve, as already mentioned, as a guide for controllable and focusable fast-particle jets. In solid-state physics the above investigation might indicate that, instead of the existence of a spatially oscillating magnetic field with a nonzero averaged value which is due to, for example, a certain arrangement of spins on a two-dimensional lattice, a mode of fast regular motion exists. This mode might be of relevance for transport processes in the bulk.

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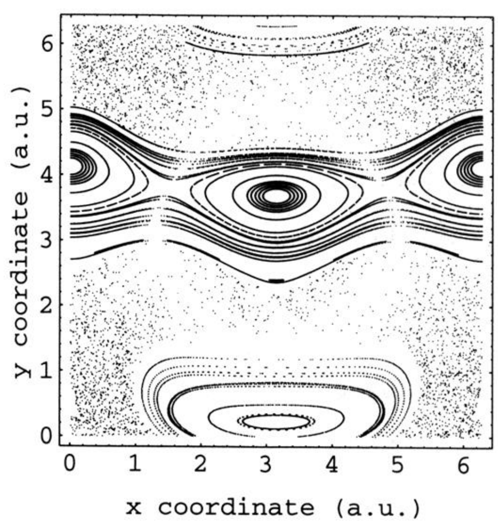

FIG. 3. The Poincaré section in the (x,y) plane for $v_y = 0$ and $E = 28.0$, $B_0 = 1.0$, and $B_s = 1.4$. (x and y are taken mod 2π.)

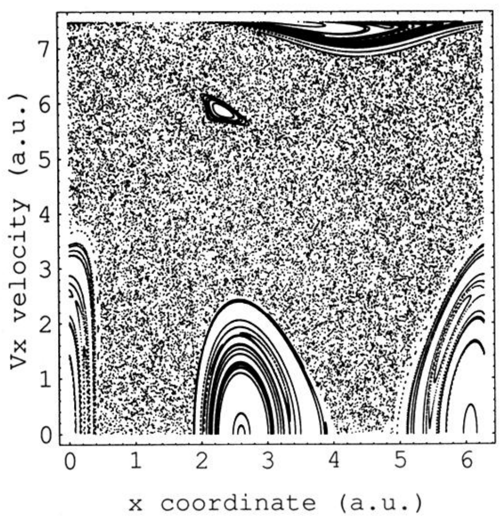

FIG. 4. The Poincaré section in the (x, v_x) plane for $y = \pi$. Same parameter values as in Fig. 3. The regular island at the upper border of the velocity v_x arises from ballistic motion in the x direction.