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Physica D 86 (1995) 45–52

PHYSICA D

Quantum localization, chaos and nonlinear interactions

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Abstract

The results of analytical and numerical investigations of destruction of quantum localization of chaos by a nonlinear interaction are presented. Two different cases are considered. One is the one-particle case which corresponds to nonlinear wave propagation in disordered media. Another case corresponds to the destruction of localization by multi-particle self-consistent interaction at small but finite particle density. It is shown that delocalization and unlimited spreading over the lattice takes place above a critical value of nonlinear interaction. The spreading can be described by anomalous diffusion with the exponent smaller than in the diffusive case.

1. Introduction

One of the most interesting effects discovered in the domain of quantum chaos during last decade is the quantum localization of classical chaotic diffusion [1–4]. In many respects this phenomenon is analogous to Anderson localization in one/quasi-one-dimensional disordered lattice [3,4]. However, in the case of quantum chaos there is no randomness and diffusion on some initial time interval appears as the result of chaotic dynamics in the corresponding classical system. In this sense we have here a dynamical localization in a completely deterministic system. The basic model in which such phenomenon was studied is the model of kicked rotator [1–4] that is obtained by quantization of the Chirikov standard map [5]. This simple model was widely used not only for investigations of the properties of quantum chaos itself but also for explanation of localization in such physical systems as microwave ionization of hydrogen atom in a microwave field [6] and a linear wave propagation in a waveguide [7].

For waves an interesting question arises if the media through which the propagation takes place is a nonlinear media. The same type of effect appears in the case of nonlinear interaction of an electron with the lattice. A model which allows to understand the properties of such motion in the limit of large times has been introduced in [8]. Such type of behaviour is discussed in the Section 2. In this case the packet width Δn growth is unlimited if the constant of nonlinear coupling exceeds some critical value. The growth itself is described by some anomalous subdiffusion law. Another physical situation is discussed in the Section 3. There we try to model the situation when localization is affected by multi-electron interaction which is taken into account in a self-consistent way.

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SSDI 0167-2789(95)00086-0

2. One-particle model with nonlinear interaction

To analyze the manifestation of nonlinear effects on the quantum localization of chaos for a one-particle case a simple model has been introduced in [8]. The model is given by the map:

$$\bar{A}_n = \sum_m (-i)^{n-m} J_{n-m}(k) A_m \times \exp(-i\tfrac{1}{2}Tm^2 + i\Delta\phi_m),$$
$$\Delta\phi_m = \beta |A_m|^2. \quad (1)$$

For $\beta = 0$ it is the map for the kicked rotator [1–4] where the Bessel function J_{n-m} appears as the result of kick which gives $\bar{\psi}(x) = \exp(-ik\cos x)\psi$. The bar denotes the new values of the amplitudes after one period of perturbation. The Fourier harmonics A_n are connected with the wave function by the relation $\psi(x) = (1/\sqrt{2\pi})\sum_n e^{inx} A_n$. The shift of the phase of the amplitude A_n during the free rotation between kicks is determined by the unperturbed energies of the rotator $E_n = n^2/2$ and the time interval between the kicks T. Finally the free propagation is simply given by multiplication of A_n by $\exp(-iTn^2/2)$. The total probability is conserved and is equal to $\sum_n |A_n|^2 = 1$.

Here we have chosen the standard system of units [2,4] where $\hbar = 1$ so that quasiclassical limit corresponds to $T \sim \hbar \to 0$ and $k \sim 1/\hbar \to \infty$ so that the product kT gives the chaos parameter $K = kT$ in the standard map. Chaotic classical diffusion takes place for $K > 1$ and for large values of K the diffusion rate in n is approximately equal to $D = (\Delta n)^2/t \approx k^2/2$ [4] where the time t is measured in the number of kicks. In the quantum system the interference leads to suppression of classical diffusion and localization of chaos [2–4]. Due to this localization the quasienergy eigenstates $u_m(n)$ decay exponentially with the level number n as $u_m(n) \sim \exp(-|n-m|/l)$ with the localization length $l = D/2$ [4]. The total number of effectively excited levels Δn is determined by the localization length $\Delta n \sim l$.

For non-zero β the phase shift of the amplitude of each level during the free rotation depends also on the probability on a given level. Due to that the equations of motion are nonlinear and the question arises what will be the influence of this nonlinearity on the localization of chaos. The nonlinear dependence of the phase shift is of the same type as in the nonlinear Schroedinger equation and corresponds to the case of four wave interaction in nonlinear media. It is also the same type of nonlinearity as in the Hubbard model.

The analysis of the kicked nonlinear rotator model (1) (KNR model) has been carried out in [8]. There it is shown that localization survives for the nonlinear interaction smaller than the critical value ($\beta < \beta_c \sim 1$). Above the critical value the localization is destroyed and unlimited excitation (spreading over unperturbed levels) takes place. This spreading is described by anomalous subdiffusion:

$$(\Delta n)^2 \approx \gamma \beta^{4/5} l^{4/5} t^{2/5} \quad (2)$$

where γ is some constant.

On the first glance it seems surprising that unlimited growth of Δn is possible. Indeed, due to probability conservation we can estimate the probability on the excited levels as $|A_n|^2 \sim 1/\Delta n$. Therefore, the nonlinear phase shift decreases when the number of excited levels (modes) increases: $\Delta\phi_n \approx \beta|A_n|^2 \approx \beta/\Delta n$. However, at the same time the distance between the linear frequencies (resonances) in the spectrum is $\Delta\omega \sim 1/\Delta n$ since all quasienegies of the linear problem ($\beta = 0$) are homogeneously distributed in the interval $(0, 2\pi)$. On the other side, the nonlinear width of the resonance is $\delta\omega \approx \Delta\phi_n \approx \beta/\Delta n$. Therefore, the Chirikov parameter of overlapping resonances [5] is given by $S = \delta\omega/\Delta\omega \approx \beta$ and is independent on Δn. For small overlapping parameter the resonances are isolated and localization is not destroyed. On the contrary for the strong resonance overlapping ($\beta > \beta_c$) chaotic transitions between localized modes (resonances) take place leading to delocalization and unlimited spreading. The law of spreading can be obtained from the estimate for the transition rate from one localized quasienergy state in the linear case ($\beta = 0$) to another due to nonlinear interaction: $\Gamma_c \sim \beta^2/(\Delta n)^3$ [8]. Since the size of the transition is of the order of localization length l we get: $(\Delta n)^2/\Delta t = D_\beta \sim l^2\Gamma_c \sim l^2\beta^2/(\Delta n)^3$ that finally gives the law (2). It is interesting to note that in fact the diffusion rate D_β is given by the same expression as in the problem of the destruction of localization by noise $D \sim l^2/t_c$ [9] but now the coherence time $t_c = \Gamma_c^{-1}$ is determined by the nonlinear interaction and it grows with Δn.

The numerical simulations carried out in [8] confirmed the estimates presented above. They also allow to understand the asymptotic behaviour in the model of kicked nonlinear Schroedinger equation [10] where the numerical results for the energy growth are in the good agreement with the law (2).

The estimates presented above were made for the case when the classical motion is chaotic ($K = kT \gg 1$). Another situation arises in the case of integrable motion ($K = kT \ll 1$). Here the quantum tunneling between the resonances is affected by nonlinear interaction. The first investigations of this phenomenon have been done in [11]. They showed that tunneling is strongly affected by nonlinearity. However, in the KNR model (1) the chain of integrable islands is infinite and at the moment it is not quite clear what will be the asymptotic regime of spreading in this case (the case of [11] corresponds only to two resonant levels).

The obtained results for the destruction of localization by nonlinear interaction are of a general nature and can be also applied for the case of particle motion on a discrete random lattice with nonlinear interaction. Such situation can be described by the equation for the Anderson model with nonlinear interaction:

$$i\partial\psi_n/\partial t = E_n\psi_n - \beta|\psi_n|^2\psi_n + \psi_{n+1} + \psi_{n-1} \qquad (3)$$

where E_n are randomly distributed in the interval $(-W, W)$ with $W < 1$. The nonlinear term takes into account the nonlinear self interaction of the electron via the lattice. In the case when the localization length $l > 1$ the same type of arguments as for the KNR model (1) could be applied giving $\beta_c \sim 1$ and the same anomalous subdiffusion (2) in delocalized phase should take place [8]. It will be very interesting to check these theoretical predictions in numerical simulations. It is also interesting to note that the chosen power for nonlinear interaction ($|\psi_n|^4$ in the Hamiltonian) is the critical one. Indeed, for the power higher than 4 the overlapping parameter S will decrease with the growth of Δn that will make impossible unlimited growth of Δn.

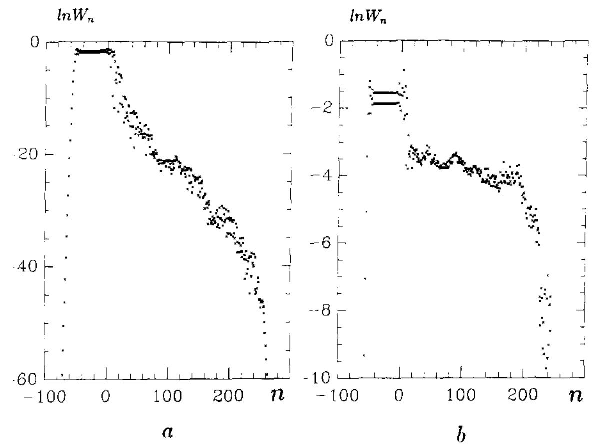

Fig. 1. (a) Probability distribution W_n over unperturbed levels n for wave propagation in the KNR model (1) (see the text) after 10^7 kicks with $k = 5$, $T = 1$, $\beta = 0.1$, NF = 50, NL = 250; (b) the same as in (a) but after $3\ 10^7$ kicks and $\beta = 1$.

Another physical situation where the discussed phenomenon can take place is the wave propagation through random nonlinear media. In this case there is also the question about the distruction of localization of linear waves by nonlinearity which arises due to dependence of dielectric constant on the wave amplitude. This problem is also connected with the question about the wave penetration through a finite layer with a random nonlinear media (see e.g. the review [12]). One of the first attempts to resolve this problem was made in [13,14]. However, there the authors considered only the properties of stationary solutions in the equation of type (3) that led them to the conclusion that the probability of penetration through the layer can decay in a power law instead of the exponential decay in the linear case. In fact the significance of the stationary solutions is very questionable, as it was also noted in [12]. Due to nonlinearity these solutions become unstable and the properties of real time-dependent dynamics become absolutely different. According to the results of this section and Ref. [8] the scenario of wave propagation through the nonlinear layer is the following. For small nonlinearity the probability penetration decreases exponentially with the length of the layer. However, when the nonlinearity exceeds some critical value delocalization happens and the slow probability propagation of type

(2) starts. After some time the front reaches the other side of the layer and approximately after this time the probability distribution inside the layer becomes uniform. This scenario has been confirmed recently [15] in the numerical simulations with the model (3).

Here I would like to discuss another numerical investigation of wave propagation through nonlinear layer based on the KNR model (1). The model is described by the map (1) in which after each kick the value $A_0 = 1$ is fixed. Also for $n < 0$ only free propagation takes place that was reached by putting in this region $T = \beta = 0$ in the interval $-\mathrm{NF} < n < 1$. The nonlinear layer was located in the interval $0 < n < \mathrm{NL}$ and its size was much larger than the localization length in the linear case. All probability outside the interval $[-\mathrm{NF}, \mathrm{NL}]$ was completely absorbed after one kick. The boundary and initial conditions chosen in this way model the situation in which some perturbation excites the waves on the boundary of nonlinear layer $n > 0$ while free wave propagation takes place for $n < 0$ (this is reached by putting $T = \beta = 0$). The numerical results for the probability distribution $W_n = |A_n|^2$ after large number of kicks ($t \sim 10^7$) are presented on the Fig. 1 for subcritical $\beta < \beta_c$ and supercritical $\beta > \beta_c$ cases. The data clearly demonstrate the exponentially small penetration in the first case (Fig. 1a) while in the second one (Fig. 1b) a plateau with approximately constant probability is formed inside the nonlinear layer. These results clearly confirm the above described scenario of wave propagation in nonlinear random media.

3. Multi-particle model with nonlinear interaction

In the previous section we analyzed the dynamics of one particle in a random potential with nonlinear interaction. Formally such problem corresponds to the case of zero density since the size of the system can be infinite while the particle is only one. This problem corresponds to the nonlinear wave propagation or to nonlinear interaction of the electron with the lattice. Absolutely another situation takes place in the solid-state multi-particle problems. There we have many particles with finite density which are localized in a random potential. One of the way to take into account the interaction between particles is the mean field approximation or self-consistent field. In this approximation the energy E_n on a site n in (3) depends on the local particle density at this site $\rho_n = \sum_\mu |\psi^\mu{}_n|^2$ where μ is the particle index. In the case of small density we can assume that the variation of E_n due to local density fluctuations will be linear in ρ_n so that now Eq. (3) will have the form

$$\begin{aligned} i\partial\psi^\mu{}_n/\partial t &= E_n\psi^\mu{}_n - \beta\rho_n\psi^\mu{}_n + \psi^\mu{}_{n+1} + \psi^\mu{}_{n-1}, \\ \rho_n &= \sum_\mu |\psi^\mu{}_n|^2 \end{aligned} \tag{4}$$

where β is some parameter measuring nonlinear interaction. Such multi-particle model at finite density $\rho = \langle\rho_n\rangle$ takes into account the changes of local potential due to local density fluctuations but neglects the correlations between nearest particles. In this sense the model also neglects the statistics of the particles assuming that the density is small, average energy of particles is high enough and due to that the particles can be treated as nonidentical.

From comparison of (3) and (4) it is easy to note that the model (3) is in fact the one-particle limit of model (4). For the model (3) the rigorous mathematical results [16] shows that there is some finite value of β below which localization remains. The simple arguments based on the resonance overlapping allow to estimate the critical value of nonlinear interaction. For the model (4) with finite density no rigorous results are known. Probably it is possible to obtain result analogous to [16] in the regime when the particles are well separated and $\rho l \ll 1$. However, the case $\rho l > 1$ is much more delicate and it will be interesting to have some mathematical proofs in this domain. One of the reasons for that is that the model (4) corresponds to the case of finite density and is much more close to the multi-particle solid-state problems than the model (3).

The situation with finite density can be numerically simulated in the model of kicked rotator by the following kicked multi-rotator (KMR) map:

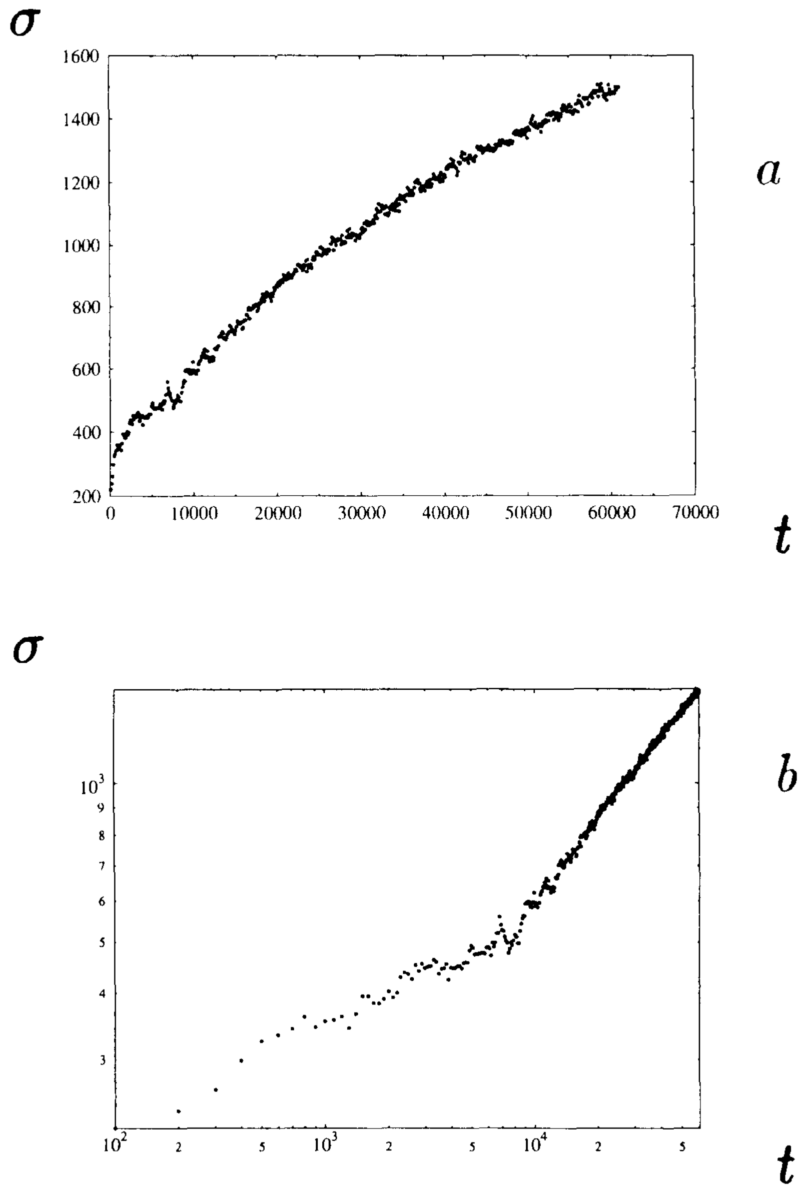

Fig. 2. (a) The square width of the distribution over unperturbed levels $\sigma = \langle (\Delta n)^2 \rangle$ in the kicked multi-rotator model (5) as a function of time t measured in the number of kicks: $k = 5$, $T = 1$, $\beta = 0.03$. The size of the system is $N = 801$, the number of particles (rotators) is $M = 80$; (b) the same as (a) but in log–log scale.

$$\bar{A^{\mu}}_n = \sum_m (-i)^{n-m} J_{n-m}(k) A^{\mu}{}_m \times \exp(-i\tfrac{1}{2}Tm^2 + i\Delta\phi_m),$$
$$\Delta\phi_m = \beta \sum_{\mu} |A^{\mu}{}_m|^2 \qquad (5)$$

Here the index μ corresponds to different rotators (particles). For $\beta = 0$ all this rotators are decoupled and localization takes place for each of them. At nonzero β there is the nonlinear phase shift $\Delta\phi_n$ at a given level for each amplitude $A^{\mu}{}_n$ which is determined by the local particle density on this level $\rho_n = \sum_{\mu} |A^{\mu}{}_n|^2$. This phase shift gives a self-consistent coupling between different rotators which can lead to delocalization.

The nonlinear interaction in the kicked multi-rotator model (5) is of the same type as in (4) and this makes both models quite similar. However, the model (5) is very efficient in the numerical simulations and allows to analyze long time dynamics with many particles. Let us mention another numerical advantage of self-consistent approximation: the time of computation grows only linearly with the number of particles.

To model the situation with the constant particle density the numerical experiments with the map (5) were made on a ring in the momentum space n of size N with M rotators on it. The average density was equal to $\rho = M/N$. Initially rotators were homogeneously distributed on a ring so that each of them was located on one of the unperturbed levels. To characterize the spreading over unperturbed levels the average square width of the distribution $\sigma = \langle (\Delta n)^2 \rangle$ was calculated as

$$\sigma = (1/M) \sum_{\mu} \sum_{n} |A_n^{\mu}|^2 (n - \langle n^{\mu} \rangle)^2,$$

where

$$\langle n^{\mu} \rangle = \sum_{\mu} n |A_n^{\mu}|^2$$

were average positions of the rotators. The average displacement of each rotator was near zero, normalization condition had the form $\sum_n |A_n^{\mu}|^2 = 1$.

The typical examples of the square width σ growth are presented on Figs. 2, 3. The case of Fig. 2 corresponds to the localized case of one-particle problem $M = 1$ (Fig. 3 in [8]). Here we see that multi-particle interaction at finite density $\rho = 0.1$ leads to a definite destruction of localization. However, another important feature of Fig. 2a is that the growth of σ is not diffusive. Approximately it can be characterized by anomalous subdiffusion (see Fig. 2b) with the exponent $\nu = 0.5$ ($\sigma \sim t^{\nu}$). In some sense it is possible to say that interaction destroy localization but a suppression of diffusion remains. This suppression is also observed for a much stronger nonlinear coupling on Fig. 3 in the region where localization was destroyed in the one-particle case (Fig. 1 in [8]). The exponent of anomalous diffusion $\nu = 0.85$ in this case is larger but still is definitely less than 1.

These numerical results put an interesting question why in the system with many particles ($M \sim 100$) interacting in the nonlinear self-consistent way diffusion is zero. Indeed, this result looks to be unusual for multi-particle system where any finite temperature gives finite diffusion coefficient (conductance). Analogously, any finite noise in kicked rotator leads to a finite diffusion rate [9]. One of the possible reasons for this diffusion suppression is that the increase of the distribution width $\Delta n = \sqrt{\sigma}$ leads to the decrease of fluctuations of the nonlinear phase shift $\Delta\phi$. The average value $\Delta\phi$ is not important while the fluctuations decrease as $1/\sqrt{\Delta\mu}$ where $\Delta\mu \approx \rho\Delta n$ is the effective number of particles which determine the local density at a given value of n. However, such simple estimate gives only the value of phase shift at a given moment of time while one needs to know effect of nonlinear decoherence over many iterations.

4. Concluding remarks

The results presented above allow to understand the effects of nonlinear interactions on localization. Two main situations have been considered. One-particle case (Section 2) corresponds to a wave propagation in a random nonlinear media. Nonlinearity can appears as the result of dependence of dielectric constant on the wave amplitude [12]. Another physical situation in which such effect can appear is a particle propaga-

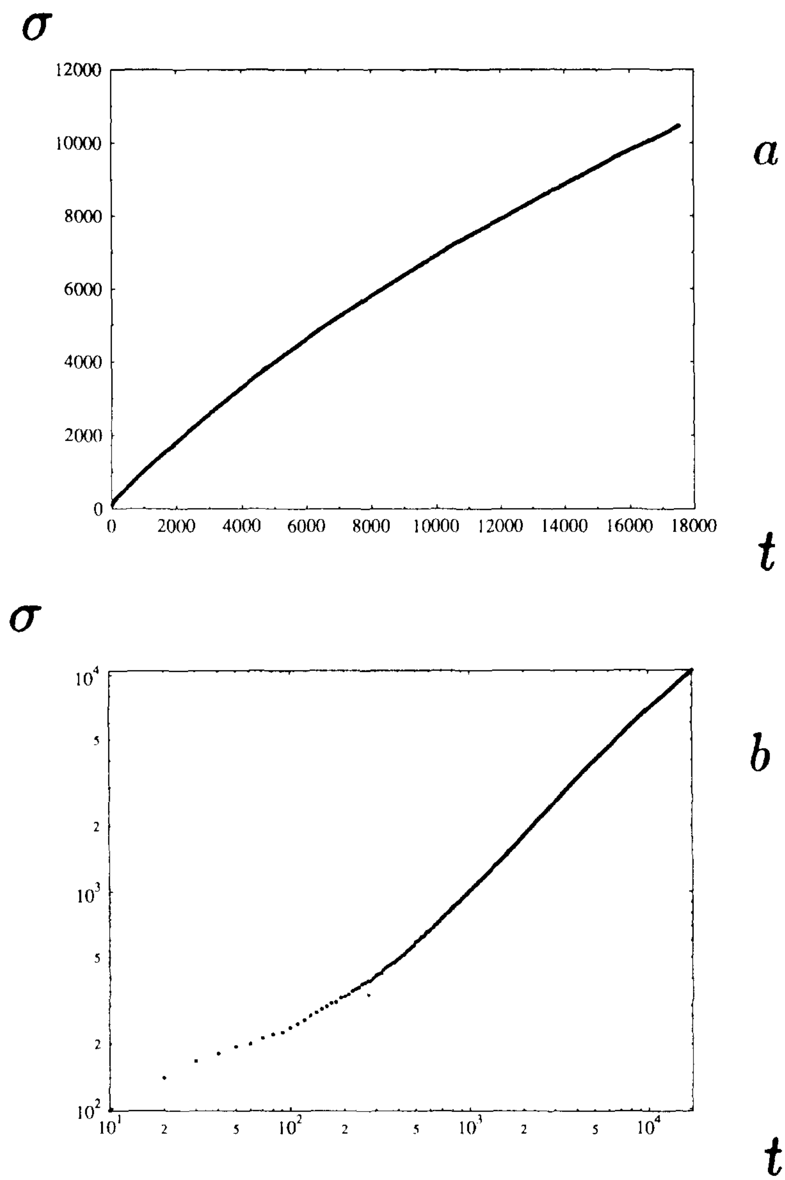

Fig. 3. (a,b) The same as (a,b) of Fig. 2 but with $\beta = 1$, $N = 1601$, $M = 160$.

tion in a disordered lattice where nonlinearity arises as the result of nonlinear deformation of the lattice by the particle. For investigation of this problem we used the model of kicked nonlinear rotator (1) which is very efficient in numerical simulations. The analogy of this model with solid state problems [3,4] and the problem of linear wave propagation in waveguides [7] allows to understand physics of these problems.

The obtained numerical results and analytical estimates (see also [8]) show that there is a critical level of nonlinear interaction. Below it localization is preserved and penetration of a wave through a random nonlinear layer decreases exponentially with its length. Above this level localization is destroyed and slow propagation of a wave front through the layer takes place finally leading to a homogeneous amplitude distribution inside the layer. This picture is quite different from a power law decay inside the layer which was discussed in [13,14]. The reason due to which the power law decay is absent is because it was obtained on the basis of stationary solutions which become unstable in a nonlinear case. In such situation only analysis of time-dependent solutions can give a correct picture of wave propagation through a nonlinear layer. The presented picture of nonlinear wave localization is in agreement with the rigorous mathematical results [16] according to which localization is not destroyed if nonlinear perturbation is sufficiently small. However, the results [16] are obtained in the spirit of the KAM theorem and can be used only for unrealistically small perturbations. The estimates based on the Chirikov criteria of overlapping resonances allow to obtain much more realistic estimates for the critical perturbation strength above which delocalization takes place. However, even in the delocalized phase some manifestation of localization remains since the rate of wave spreading is very slow and it is characterized by anomalous subdiffusion.

Another type of nonlinear interaction and its influence on localization was considered in Section 3. This situation corresponds to a case of multi-particle nonlinear interaction in the case when particles are homogeneously distributed on a random lattice (density of particles is finite) and they interact with each other in a self-consistent way. For example, such situation takes place when parameters of the lattice depend on the local density value at a given site. At my knowledge, no rigorous mathematical results analogous to [16] are known for such case. Numerical investigations of the kicked multi-rotator model (5) show that in the many particle case delocalization takes place in the domain where one-particle model had localization. However, the most surprising result is that even in the case with many particles and strong nonlinearity the spreading is still described by a slow anomalous subdiffusion. So that the suppression of diffusion remains. This result for multi-particle self-consistent interaction looks to be in a contradiction with the fact that any finite temperature gives finite rate of diffusive spreading (conductance) over the lattice. Further analytical and numerical investigations are required for a better understanding of this regime.

Acknowledgemetns

I would like to thank S. Fishman for stimulating discussions and Oren Carmi, Abraham and Laura and Irving Steinhorn Memorial Visiting Professorship for support during my stay in Haifa at Technion.

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