Shnirelman Peak in Level Spacing Statistics

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The first results on statistical properties of quantum quasidegeneracy are presented. A physical interpretation of the Shnirelman theorem which predicted bulk quasidegeneracy is given. Conditions for the strong impact of degeneracy on quantum level statistics are formulated, which allow us to extend the application of the Shnirelman theorem to a broad class of quantum systems.

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Energy level statistics is one of the most important and well studied characteristics of quantum systems. Particularly, it is commonly assumed by now [1] that in the limit of classically completely integrable systems the distribution of nearest-neighbor level spacings is Poissonian (independent levels) [2]. In the opposite limit of classically chaotic systems this distribution is characterized by level repulsion and, for ergodic eigenfunctions, is generally given by the Wigner-Dyson law [3]. In the intermediate nearly integrable [Kolmogorov-Arnol'd-Moser (KAM)] region various expressions were suggested to describe a smooth transition between the above statistics [4,5]. Such behavior was well confirmed by many numerical experiments with various dynamical models (see, e.g., [6] and references therein).

However, this picture seems to be in a sharp contradiction with an old theorem due to Shnirelman [7]. This theorem states that for a classically nearly integrable system at least each second level spacing in the corresponding quantum system becomes exponentially small in the quasiclassical domain. This would imply a big narrow peak in the distribution of nearest-neighbor level spacings (level clustering). This result is especially surprising as no special symmetry was assumed in a particular model considered by Shnirelman. However, the time-reversal symmetry holds in such a model. Formally, the theorem states that the spectrum λ_k is asymptotically multiple, i.e., for each $M > 0$ there exists $C_M > 0$ such that $\min(\lambda_k - \lambda_{k-1}, \lambda_{k+1} - \lambda_k) < C_M \lambda_k^{-M}$. In the first formulation the theorem had been proved for a geodesic flow on a two-dimensional torus (some nearly integrable billiards), while in the second formulation its applicability had been extended to a broader class of two-dimensional nearly integrable systems with at least four invariant tori [7]. To the best of our knowledge no physical interpretation of this theorem has been given as yet.

In this Letter we present the first numerical results on this new phenomenon which allows us to give a plausible interpretation of the theorem and to extend its implications onto a broad class of quantum systems. Our interpretation is based on the conception of quasiclassical degeneracy destroyed by tunneling. Similar phenomena in the presence of spatial symmetry were studied in many papers (see, e.g., [8] and references therein) but the effect of time reversibility on level statistics in absence of spatial symmetry was not considered to our knowledge. In some sense the degeneracy between the states connected by time-reversal symmetry is destroyed by *tunneling between the future and the past.* Such a situation corresponds to a double well in momentum space.

As a simple model we use the kicked rotator on a torus [9] described by the following unitary matrix:

$$U_{nm} = \frac{1}{N} \exp\{-i\tfrac{T}{4}[(n+\alpha)^2 + (m+\alpha)^2]\} \times \sum_{j=-N_1}^{N_1} \exp[-iV(\theta_j) - i(n-m)\theta_j]. \quad (1)$$

Here $V(\theta_j) = k(\cos\theta_j - \gamma \sin 2\theta_j)$, k is the perturbation strength of the kick, T is the period of the perturbation, $N = 2N_1 + 1$ is the total number of states, $\theta_j = 2\pi j/N$ is spatial variable, and n is momentum ($\hbar = 1$). The time-reversal invariance corresponds to $U_{n,m} = U_{-m,-n}$.

The quasiclassical region we are interested in corresponds to big quantum parameters k and N and small quantum parameter T. The classical parameter $K = kT$ determines the type of classical motion, $K \ll 1$ corresponding to nearly integrable motion while $K \gg 1$ describes chaotic motion. The second classical parameter, integer $r = TN/2\pi$, determines the number of primary classical resonances on a torus. The parameter α has the meaning of magnetic field violating the time-reversal symmetry. Another parameter γ controls the spatial symmetry which is completely destroyed for $\gamma \sim 1$ [10]. Usually we consider the cases with $\alpha = 0$ and $\gamma = 1/2$, when only time-reversal symmetry remains. To analyze the properties of the level spacing statistics $p(s)$ we diagonalized the matrix U_{nm} for different values of the parameter k from a fixed interval, so that the total spacing statistics was always equal to 10 000. The normalized level spacing s was defined as a difference between nearest quasienergies divided by the average spacing for all levels $2\pi/N$.

Our results for level spacing statistics in the classical KAM region without spatial symmetry are presented in

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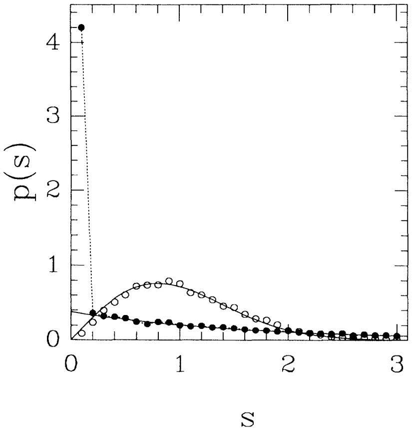

FIG. 1. Level spacing distribution $p(s)$ in model (1) with $\alpha = 0$ and $\gamma = 1/2$: points connected by the dashed line are for $k = 6 - 10$, $T = 4\pi/N \approx 0.025$, $r = 2$, and $N = 501$; the solid line gives Poisson distribution with a 62% fraction of all spacings; the circles are for $k = 25-30$, $T = 40\pi/N$, $N = 501$, and $D/N \approx 1.5$; and the full line shows the Wigner distribution. Total spacing statistics is 10 000.

Figs. 1 and 2 ($\gamma \neq 0$). A huge peak in the first bin of the histogram (Fig. 1) clearly demonstrates the existence of global quasidegeneracy in a qualitative accordance with the Shnirelman theorem [7]. We emphasize that such a peak appears only when one considers *all* level spacings without fixing any symmetry, as was proposed in [11]. The separation of levels by symmetry is the usual practice in the studies of level statistics which was apparently the main cause of missing this peak in previous numerical studies. It is important to distinguish two qualitatively different situations. If there is exact level degeneracy due to some continuous symmetry then such a peak has the trivial shape of a delta function. However, as is well known, an exact discrete symmetry does not imply generally the exact degeneracy but only a quasidegeneracy. In this case the peak has a finite width and contains important information about the structure of the quantum system.

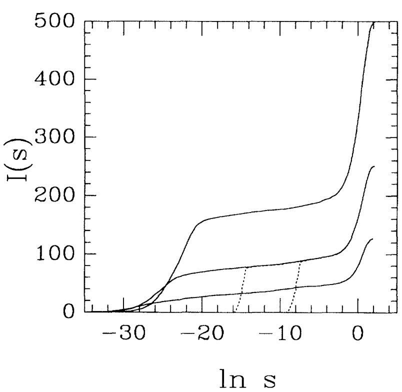

FIG. 2. Normalized integral level spacing distribution $I(s) = NP(s)$: the full lines are for $\alpha = 0$; $N = 501$, $k = 6-10$; $N = 251$, $k = 3-5$; and $N = 127$, $k = 1.5-2.5$. The dashed lines are for $\alpha = 1.41 \times 10^{-9}$ (left) and 1.41×10^{-6} (right), $N = 251$. In all cases $r = 2$.

The distribution out of the peak can be fitted by a renormalized Poisson distribution $p(s) = \sigma^2 \exp(-\sigma s)$. The quantity $1 - \sigma$ has the meaning of the fraction of degenerate levels which form the Shnirelman peak while σ gives the fraction of the states in the Poissonian tail. For the case in Fig. 1 $\sigma \approx 0.62$, so that the total fraction of levels in the peak is approximately 0.38. This leads to the increase of the average level spacing in $1/\sigma$ times, hence, σ in the exponent.

A visible difference of the total probability in the peak from 50% predicted by the Shnirelman theorem can be understood on the following grounds. The origin of this difference, as far as we understand it, is due to the fact that the theorem was proved for a particular geodesic flow where all trajectories are actually rotating even though this was not explicitly formulated. In our model (1) in addition there are also oscillating trajectories within the main resonance. In the quasiclassical case the splitting between the different directions of rotation is exponentially small because of the tunneling between these two classically separated trajectories as was shown in [8] for the time and spatial symmetry case. However, for oscillating trajectories both directions of motion are coupled classically and therefore the corresponding level splitting is big. Hence, the oscillating states in the main resonances at $n = 0, N/2$ do not contribute to the peak. The fraction of such states in the quasiclassical case is determined by the relative phase space area of the main resonances which is equal to $\rho = 6.64\sqrt{K/2}/\pi^2$, for the case of Fig. 1 with $r = 2$. The averaging of $\sqrt{K}$ over the interval of K variation in Fig. 1 gives $\langle\sqrt{K}\rangle \approx 0.45$, so that the fraction of the oscillating states is $\rho \approx 0.21$. Then, the expected fraction in the peak is $1 - \sigma = (1 - \rho)/2 \approx 0.40$, that is in good agreement with the numerical value 0.38.

The resolution of the peak is presented in Fig. 2 where the spacing integral probability $P(s)$ is normalized to the full number of levels in each matrix $[I(s) = NP(s)]$. Three different regions are clearly seen. The rightmost steep increase of $I(s)$ corresponds to the Poissonian tail in Fig. 1. The leftmost steep drop is apparently due to numerical errors. The most interesting for us is the middle region which represents the structure of the Shnirelman peak. Approximately, the dependence of I on $\ln s$ is linear here which corresponds to the exponential splitting of the levels

$$s \approx A \exp(-2n/l_{\rm sp}), \tag{2}$$

where $2n$ is the distance in momentum between the two states $-n, +n$ related by the time reversal. This is a usual rough estimate for the energy splitting ΔE due to tunneling with $\Delta E \sim \exp(-S/\hbar)$ and classical action along the path $S \sim \hbar n$. The maximal distance on the torus is $2n = N/2$ that determines the minimal value of s. The splitting is characterized by the parameter $l_{\rm sp}$, which in this case has the meaning of tunneling length, and A is some constant. With such a definition of $l_{\rm sp}$ the slope is $dI/d\ \ln s \approx l_{\rm sp}$, since the integral probability $P \approx 2n/N$. As is seen from Fig. 2, the slope for the fixed classical structure ($K = \text{const}, r = 2$) is approximately independent on quantum parameters N, k, T. The tunneling length is measured in the number of quantum states and is equal, $l_{\rm sp} \approx 1.8$.

The parameter $l_{\rm sp}$ can be roughly estimated from the splitting ΔE as a single-kick effect due to the coupling matrix element $U_{n,-n} \sim J_n(k/2)$. This gives $l_{\rm sp} \approx 2/\ln(16\pi n/eKN)$. Notice that $l_{\rm sp}$ slowly depends on classical parameters only, including n/N. For the parameters in Fig. 2 and $n/N = 1/8$, we obtain $l_{\rm sp} \approx 0.8$, which gives a correct order of magnitude. The dependence $l_{\rm sp}$ on $n \sim \ln s$ (2) implies very slow variation $l_{\rm sp}(s)$ as $\ln\ln s$.

The differential distribution of level spacings is given by

$$p(s) \approx l_{\rm sp}/Ns$$

whence the average spacing in the peak $\langle s\rangle = l_{\rm sp}s_{\max}/N$, where $s_{\max} \sim 1/N$ corresponds to the crossover of the distribution (3) with the Poissonian one $[p(s) \approx 1]$. Notice that the average spacing in the peak decreases only as a power law of the quantum parameter N in spite of exponential tunneling (2).

The high sensitivity of the Shnirelman peak to the violation of time-reversal symmetry is demonstrated in Fig. 2. A small α produces a sharp cutoff of the distribution on a small spacing s_c, while for larger s the distribution $I(s)$ remains practically unchanged. The estimate for s_c can be obtained from the comparison of the unperturbed level splitting, $\Delta E \sim T\alpha N/2$ [see Eq. (1)], with the critical level splitting $2\pi s_c/N$, that gives $s_c \sim \alpha TN^2/4\pi$, in a good agreement with the data in Fig. 2.

In the KAM region the motion is integrable for most initial conditions. This means that the Shnirelman peak is essentially determined by the quasidegeneracy of integrable motion. Hence, the effect must generally persist in a completely integrable system as well. Indeed, such quasidegeneracy occurs, for example, in the simple (not kicked) pendulum as is clear from the well known solutions to the Mathieu equation. In this system there are both spatial and time-reversal symmetries. However, the peak is produced only by time-reversal symmetry since the states of the opposite spatial symmetry are not separated in the phase space. Also in our model (1) the peak disappears if only spatial symmetry remains ($\gamma = 0, \alpha \neq 0$). Another similar situation corresponds to elliptic billiards. Indeed, here tunneling between different directions of rotations should give exponentially small splitting between levels. This degeneracy is connected with time-reversal symmetry and not with spatial symmetry. Therefore, the degeneracy will not disappear after nonsymmetric spatial deformation of the billiards. Of course, such deformation should not be very strong to keep the system quasi-integrable and to have different directions of rotations separated.

Much more interesting is the opposite limit of classically chaotic motion. In this case the quasidegeneracy depends on the structure of eigenfunctions. If they are ergodic as in the classical limit [12] then the states with the opposite angular momentum $(-n, +n)$ are directly related by the diffusion and hence the splitting is comparable with the average level spacing. This case is demonstrated in Fig. 1 (open circles). The peak is absent in spite of the time-reversal symmetry. However, if the quantum eigenstates are strongly localized (the localization length $l \ll N$), the exponential degeneracy reappears [13] (see also [11] and references therein). An example of level statistics in this case is given in Fig. 3. Again, the dependence of I on $\ln s$ is approximately linear in the region of the peak but the splitting parameter $l_{\rm sp}$ is now much bigger and related to the localization length ($l_{\rm sp} \sim l$). The latter is determined by the classical diffusion rate $D \sim l$ [14]. In our model for $\gamma = 1/2$, the diffusion rate $D \approx k^2$, so that according to the data in Fig. 3 the ratio $l_{\rm sp}/D \approx 1.8$.

Our results allow us to formulate more general conditions for the appearance of the Shnirelman peak in the level spacing distribution. First, the quantum system must have a discrete symmetry. Second (a new condition), the states with opposite symmetry must be separated in the phase space either classically (as in the KAM region in our example in Fig. 2) or quantum mechanically (as for the strongly localized chaotic eigenfunctions in Fig. 3). The second condition was not explicitly formulated in the Shnirelman theorem [7] but was discussed in detail in [8]. On the one hand, the latter condition restricts the applica-

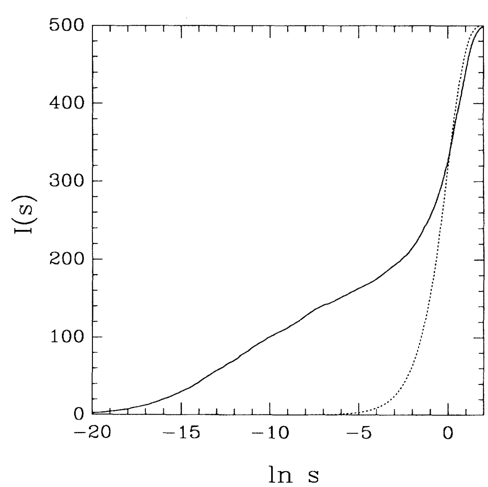

FIG. 3. Integral level spacing distribution for $N = 501$, $k = 2.5-3$, $T = 320\pi/N$, and $D/N \approx 0.015$ (full line); Poisson distribution is shown by the dashed line.

bility of the Shnirelman theorem. On the other hand, our results show that the effect itself can be extended on both completely integrable and chaotic systems (the latter case was also conjectured in [11]).

In usual random matrix models the Shnirelman peak is absent in spite of the time-reversal symmetry of the corresponding Hamiltonian. This can be understood in the following way. For full matrices the second condition is violated because of the ergodicity of the eigenfunctions. For band matrices with localization the time symmetry is usually fixed. To recover the effect in the latter case one needs to introduce explicitly the symmetry by the condition $H_{n,m} = H_{-m,-n}$, in addition to the usual condition $H_{n,m} = H_{m,n}$. An interesting related example is the Anderson localization in a random but spatially symmetric potential: $V(x) = V(-x)$. On the other hand, the time-reversal symmetry in this example does not help due to violation of the second condition: the states with opposite momenta are strongly coupled by the backscattering in a random potential.

An interesting direction of further studies of the Shnirelman effect is related to many-dimensional systems where we would expect a much more rich structure of quasidegeneracy.

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