EUROPHYSICS LETTERS 10 January 1995

Europhys. Lett., **29** (2), pp. 117-122 (1995)

72

Spectral Variety in the Kicked Harper Model.

F. BORGONOVI (*)(**) and D. SHEPELYANSKY (***)

Laboratoire de Physique Quantique, Université Paul Sabatier
118, route de Narbonne, 31062 Toulouse Cedex, France

(received 15 July 1994; accepted in final form 5 December 1994)

PACS. 05.45+b – Theory and models of chaotic systems.
PACS. 73.20Dx – Electron states in low-dimensional structures (inc. quantum wells, superlattices, layer structures and intercalation compounds).

Abstract. – We study the properties of eigenfunctions and quasi-energies in the kicked Harper model. The analysis of eigenfunctions clearly shows the transition from localized to delocalized states in the parameter region $K < L$. In the delocalized phase numerical evidence is given for the simultaneous presence of pure point, singular continuous and absolutely continuous spectrum.

In this letter we study the spectral and eigenfunction properties of the kicked Harper model (KHM) [1]. This model has recently attracted a great deal of attention (see references in [1]). One of the reasons for such interest is due to the fact that this model lies on the intersection between the field of quantum chaos and the physics of incommensurate systems. Indeed, in the study of quantum chaotic systems the effect of quantum localization of classically chaotic diffusive excitation has been established [2]. On the other hand, in the quasi-crystal domain the typical situation is characterized by a multifractal spectrum of eigenenergies [3]. A physical example of such a system is represented by the Harper equation which describes electrons in a 2d lattice in the presence of a magnetic field [4,5]. However, differently from the Harper model (HM), which is integrable in the classical limit, the classical dynamics of KHM is chaotic. This puts an interesting question about the relation between the quantum localization of chaos and the properties of motion in incommensurate potentials. This question was firstly addressed in [6] where it was found that localization can be destroyed leading to unlimited excitation over unperturbed levels. Further studies showed that the spectrum is generally characterized by multifractal properties [7,8] and anomalous diffusion along the lattice.

The interesting property of this anomalous diffusion has been established in [9]. There it was found that while the width of the distribution over the unperturbed levels grows in time

(*) Permanent address: Dipartimento di Matematica, Università Cattolica, via Trieste 17, 25121 Brescia, Italy.
(**) Also at INFN, Sezione di Pavia, via Bassi 7, Pavia, Italy.
(***) Also at Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia.

without any bound, the probability to stay near the initial state does not decay to zero. Since without perturbation the quasi-energies are homogeneously distributed in the interval $[0, 2\pi]$, the situation seems to be quite different from the usual solid-state picture where a mobility edge separates the absolutely continuous spectrum (a.c.s.) from the pure point one (p.p.s.). For a better understanding of this interesting phenomenon we carried out a detailed analysis on both the quasi-energy eigenspectrum and the eigenfunctions.

The KHM is described by the unitary evolution operator

$$\hat{U}_{L,K} = \exp\left[-i\frac{L}{\hbar}\cos(\hbar\hat{n}+\beta)\right]\exp\left[-i\frac{K}{\hbar}\cos(\theta)\right], \tag{1}$$

where $\hat{n} = -i(\partial/\partial\theta)$ and β is the quasi-momentum. The parameter L characterizes the free rotation while K represents the strength of the kick potential. Quasi-energy eigenvalues λ and eigenfunctions ψ_λ are given by

$$\hat{U}_{L,K}\psi_\lambda = \exp[-i\lambda]\psi_\lambda\,. \tag{2}$$

The parameter $\hbar/2\pi$ characterizes the incommensurate structure of the model itself. In our study we mainly concentrate on an irrational $\hbar/2\pi$ with golden tail in the continuous fraction expansion. For concreteness we choose $\hbar = 2\pi/(6+\rho)$ and $\rho = (\sqrt{5}+1)/2$.

The phase diagram which describes the dynamical properties of KHM can be represented in four different regions in the (K, L)-plane, see for instance fig. 11 in [1]. The most interesting region in such a plane corresponds to $K \leqslant L$ where a transition from localization to unbounded excitation takes place.

To investigate the properties of KHM we approximate the irrational value of $\hbar/2\pi$ with its best rational approximants obtained from the continuous fraction expansion. This is a standard procedure in the study of incommensurate systems, see for instance [1]. Properties of irrational systems are then supposed to be recovered in the limit $q \to \infty$, p/q being the rational approximant to $\hbar/2\pi$. The evolution operator is then described by a finite matrix of

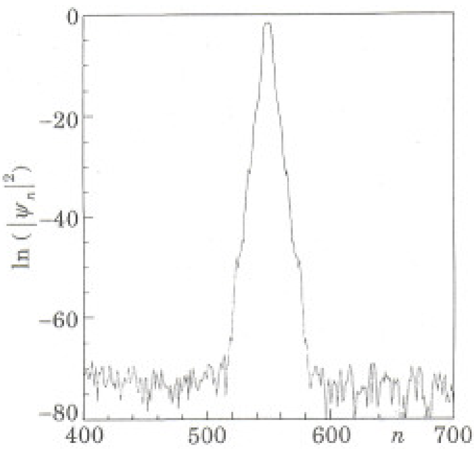

Fig. 1.

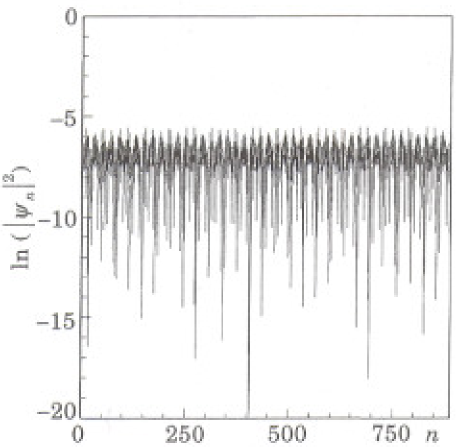

Fig. 2.

Fig. 1. – Localized eigenstate with maximum inverse participation ratio for $K = 1$, $L = 7$ and $\hbar/2\pi = 233/1775$.

Fig. 2. – Delocalized eigenstate with maximum inverse participation ratio for $K = 4$, $L = 7$ and $\hbar/2\pi = 233/1775$.

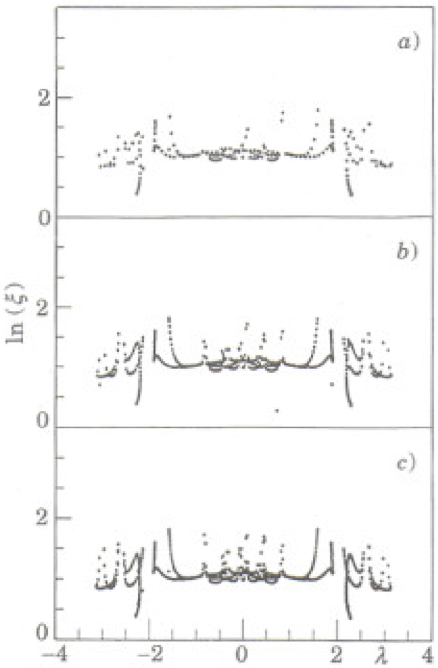

Fig. 3.

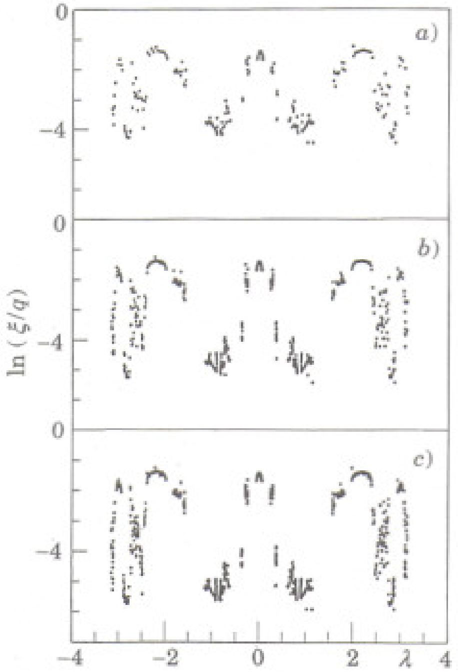

Fig. 4.

Fig. 3. – Inverse participation ratio ξ as a function of the quasi-energy λ for $K = 1$, $L = 7$ and three different approximants: *a*) $\hbar/2\pi = 55/419$, *b*) $\hbar/2\pi = 144/1097$, *c*) $\hbar/2\pi = 233/1775$.

Fig. 4. – Scaled inverse participation ratio ξ/q as a function of the quasi-energy λ for $K = 4$, $L = 7$ and three different approximants: *a*) $\hbar/2\pi = 55/419$, *b*) $\hbar/2\pi = 144/1097$, *c*) $\hbar/2\pi = 233/1775$.

size q [10]. This kind of approximation is equivalent to consider the KHM on a torus instead of a cylinder.

By diagonalizing this matrix, we analysed the properties of the eigenspectrum and the eigenfunctions. To reduce the computational difficulties we studied only symmetric eigenstates ($\psi_\lambda(n) = \psi_\lambda(-n)$). Two typical eigenstates in the localized and delocalized regions of the phase diagram are represented in fig. 1 and 2. It is interesting to note that both eigenstates correspond to the region $K < L$ of the (K, L)-plane where the HM [11] has only localized states. To analyse the properties of the eigenstates we characterized them by their inverse participation ratio $\xi = \sum_n |\psi_n|^2 / \sum_n |\psi_n|^4$. We also checked that other types of eigenstates length measure, like the entropy [12] or the mean width defined as $\Delta n = (\langle (n - \langle n \rangle)^2 \rangle)^{1/2}$, gave the same results.

The behaviour of the inverse participation ratio for different approximants and for different values of K corresponding to the localized and delocalized regions of the (K, L)-plane is shown in fig. 3 and 4. Figure 3 gives clear evidence that all eigenstates are localized. The increase of the rational approximant q in four times does not change the overall structure of the $\xi(\lambda)$ distribution. This gives another confirmation of the p.p. character of the spectrum in this region. It is interesting to note the presence of gaps in the spectrum and the apparent crossover of different $\xi(\lambda)$ branches. Probably these branches are related to the folding of some effective Hermitian Hamiltonian H_{eff} on the quasi-energy interval $[0, 2\pi]$, so that the unitary operator (1) can be represented as $U_{L,K} = \exp[-iaH_{\text{eff}}]$. Then the branches can appear for sufficiently large values of a.

Another situation, correspondent to $K = 4$, is presented in fig. 4. These pictures, for different approximants q, definitely demonstrate evidence for the existence of a.c.s. at those parameters values. Indeed there are *finite* intervals in λ in which ξ grows linearly with the size q of the matrix. This means that the corresponding eigenstate cannot be normalizable in the limit $q \to \infty$. We then show evidence for unnormalized states in a finite region of spectrum, *i.e.* an a.c.s. That is particularly stressed by using the rescaled variable ξ/q (see fig. 4). Another interesting feature is that the lower bound of the distribution corresponds to those ξ values unchanged under the increase of q. This means that there are localized states with an inverse participation ratio independent of the matrix size q. Such states are associated with the p.p. part of the quasi-energy spectrum. It is also important to remark that there are sharp variations (about 50 times) of ξ in relatively short λ intervals of approximate size 0.05.

Even if we do not have a physical interpretation for this sharp variation in the $\xi(\lambda)$ distribution, from the mathematical point of view they should correspond to a singular continuous spectrum (s.c.s.) with multifractal quasi-energies and eigenfunctions, superimposed both with a p.p.s. (lower bound in fig. 4) and with an a.c.s. (finite intervals of spectrum scaling as the size of the matrix: see fig. 4). At the same time from this consideration it is impossible to state whether these three components are dense in the spectrum or located in different non-overlapped intervals of λ.

We also investigated the statistical distribution of inverse participation ratios $P(\xi)$ in the localized phase as well as in the mixed one (we use here the terminology introduced in [1]). Two examples, corresponding to fig. 3*c*) and 4*c*), are shown in fig. 5. In the localized regime (fig. 5*a*)) the distribution is very narrow and the sharp peaks correspond to an accumulation of eigenfunctions with approximately the same value of ξ. On the contrary the mixed phase is characterized by a broad distribution in ξ which shows an almost constant plateau in the tail. The narrow peak for small ξ corresponds to the p.p. part of the spectrum with exponentially localized eigenstates. The formation of the plateau without gaps gives one more indication of the presence of multifractal spectrum with different length scales for eigenfunctions. Computations were performed here by varying randomly the quasi-momentum β in the interval $[0, \pi]$ and for different approximants. This was done to exclude the possibility of

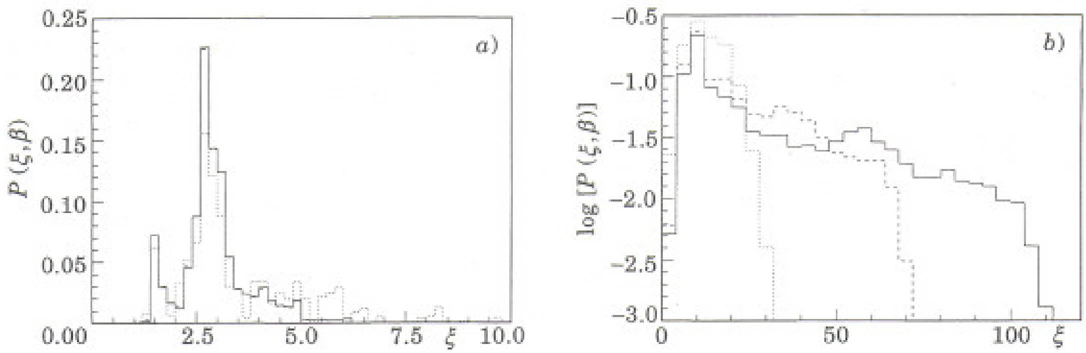

Fig. 5. – Probability distribution of the inverse participation ratios for $L = 7$ in the under critical case $K = 1$ (*a*)); and in the overcritical case $K = 4$ (*b*)) (semilog histogram). Quasi-momentum β has been taken randomly in the interval $[0, \pi]$. Different approximants to $\hbar/2\pi$ are chosen: dotted line $\hbar/2\pi = 34/259$, dashed line $\hbar/2\pi = 55/419$, full line $\hbar/2\pi = 144/1097$. In *a*) dashed and full histograms are superimposed showing the asymptotic distribution. In *b*) the right cut-off is moving linearly with q (a.c.s.).

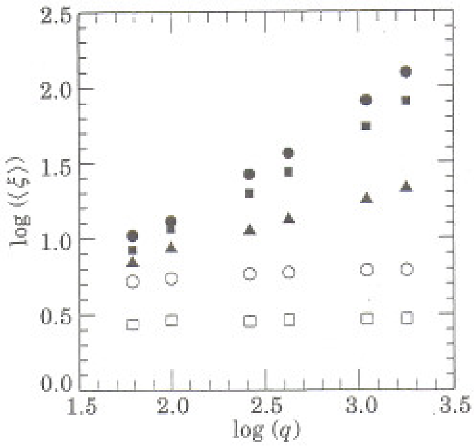

Fig. 6. – Average inverse participation ratio $\langle\xi\rangle$ as a function of the matrix size q, for $L = 7$ and different K values: $K = 1$ (open squares), $K = 2$ (open circles), $K = 2.5$ (triangles), $K = 3$ (full squares), $K = 4$ (full circles).

finite-size edge effects and to prove the persistence of peaks and the broadening of the tail in the limit $q \to \infty$.

Another kind of information about the multifractal structure of the eigenstates can be extracted from the analysis of the momenta of the distribution $P(\xi)$ and their dependence on the approximant q. For the first moment $\langle\xi\rangle$ the dependence on q is presented in fig. 6 for a fixed L value. This picture clearly indicates the existence of transition between localized and multifractal spectrum. Indeed, for $K \leqslant 2$, the average $\langle\xi\rangle$ does not grow with q, while for $K > 2$ an approximate power law growth with q is found: $\langle\xi\rangle \sim q^{\alpha_{\rm I}}$. The values of $\alpha_{\rm I}$ for different K values are given in table I. In the localized phase the values of $\alpha_{\rm I}$ are very small, thus indicating the exponential decay of eigenfunctions (see fig. 1). On the other hand, for $K > 2$ the exponent $\alpha_{\rm I}$ changes with K and this marks the presence of multifractal structure. Another confirmation of this structure is the difference between $\alpha_{\rm I}$ and $\alpha_{\rm II}$, where $\alpha_{\rm II}$ is defined as $\langle\xi^2\rangle \sim q^{2\alpha_{\rm II}}$ (see table I). Even if the errors associated with the determined $\alpha_{\rm I}$ values are small, a direct inspection of fig. 6 definitely shows that $\log(\langle\xi\rangle)$ cannot be fitted by a straight line for $K = 3$ and $K = 4$. This is probably due to the fact that the asymptotic regime has not been reached for these q values. Indeed the results of fig. 4 clearly indicate that there are states for which $\langle\xi\rangle$ scales as q. An interesting point would be the local analysis of the spectrum based upon the definition of a local scaling exponent η from $\xi(\lambda) \sim q^{\eta(\lambda)}$. This work is still in progress and it will be reported in a future paper.

TABLE I. – *First and second momentum growth exponents as a function of the parameter K, for L = 7. Each exponent has been derived by the usual best-fit procedure, for rational approximants of* $\hbar/2\pi$ *up to 233/1775. The exponents are defined by* $\langle\xi\rangle \sim q^{\alpha_{\rm I}}$ *and* $\langle\xi^2\rangle \sim q^{2\alpha_{\rm II}}$.

K	$\alpha_{\rm I}$	$\alpha_{\rm II}$
1	0.02(1)	0.02(1)
2	0.05(1)	0.05(1)
2.5	0.32(1)	0.48(3)
3	0.66(2)	0.77(5)
4	0.75(3)	0.86(5)

In conclusion, we have analysed the properties of eigenfunctions in KHM. This analysis confirms the existence of a transition [9], in the region $K < L$, from a p.p.s. to a mixed spectrum both characterized by gaps. Below this transition all eigenstates are exponentially localized, while above the transition different types of states have been found. Some of them are exponentially localized and correspond to p.p.s. A second part of these states corresponds to the s.c. part of the spectrum with an inverse participation ratio ξ growing as a power law of the rational approximant q. The exponent defining such growth is less than 1. We also found a third kind of states characterized by a linear dependence of ξ on q (see fig. 4). This fact indicates the possible existence of an a.c. part of spectrum itself for $K < L$. At the same time the sharp variation of ξ as a function of λ supports the presence of mixing between s.c.s. and p.p.s. We finally remark that, differently from the HM, where the p.p.s., s.c.s. and a.c.s. are obtained for different parameter values, here all these regimes seem to be realized for just one value of the parameters. One of the reasons for this rich situation is the possibility of folding of some effective Hamiltonian with mobility edges over the quasi-energy interval $[0, 2\pi]$. Indeed, in this case, the unitary operator $U = \exp[-iaH_{\text{eff}}]$ can have a mixing of all three kinds of spectrum for large values of a. However, the stability of such kind of spectrum under a small perturbation remains an open question.

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We thank J. BELLISSARD for useful and stimulating discussions. Computer assistance from Dr. A. COMINELLI is greatly acknowledged. One us (FB) would like to thank the Laboratoire de Physique Quantique, URA 505 CNRS, Université Paul Sabatier for the kind hospitality during his visit. Support from the Centre du Calcul Vectoriel pour la Recherche (CCVR) Palaiseau, France is acknowledged.

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