Chaotic Landau Level Mixing in Classical and Quantum Wells

Dima L. Shepelyansky*,† and A. Douglas Stone
Applied Physics, Yale University, P.O. Box 208284, New Haven, Connecticut 06520
(Received 2 December 1994)

We analyze the 3D motion of a charged particle between parallel planar potential barriers with a magnetic field applied at an angle θ to the barriers and an electric field normal to the barriers. A nonlinear map is derived for the classical system which gives analytic conditions for the occurrence of a chaotic energy exchange between the cyclotron and longitudinal motion as a function of θ and other system parameters. This energy exchange can lead to a population of very high Landau levels. Quantizing this problem suppresses energy exchange up to a critical angle determined by the localization transition.

PACS numbers: 73.20.Dx, 05.45.+b, 71.55.Jv

Several recent experiments in semiconductor heterostructures have studied the tunneling current of electrons through planar potential barriers in a magnetic field which is tilted at an angle θ with respect to the normal to the barriers [1,2]. When a variable voltage is applied across the barriers these systems exhibit oscillations in the I-V characteristic which were found to have a strong sensitivity to the tilt angle θ, magnetic field B, and driving voltage ΔV. Theoretical analysis has shown that tilting the field induces a classical transition from integrability (for $\theta = 0$) to chaos [1,2]; Fromhold *et al.* [1] showed that the Gutzwiller periodic orbit theory of the density of states oscillations [3] could account for the dominant features of resonances in the I-V curve in the strongly chaotic regime. This work has drawn attention to a new dynamical system for the study of classical and quantum chaos which should have other experimental signatures in semiconductor quantum wells and experimental realizations outside of solid-state physics. It therefore seems worthwhile to analyze the dynamics of this system from the point of view of the global phase space structure and obtain the relevant parametric criteria for the onset of chaos in different regimes. Since the real-space motion is three dimensional and depends on a large number of parameters (θ, B, ΔV, E_i, the initial kinetic energy, and d, the distance between the barriers), it is not obvious that this system can be reduced to familiar models which have been previously analyzed. Below we will show that, in fact, the transition to chaos in this system can be described by a two-dimensional map with strong similarities to the Fermi acceleration model [4] in the limit $\Delta V \to 0$, and the Haake "kicked top" [5] and the Chirikov standard map [6] for $\Delta V \gg E_i$. In both limits it is possible to obtain parametric conditions for the onset of chaos which agree with numerical simulations and may be tested experimentally.

The basic physics of the classical motion is the following. Between collisions with the walls the particle executes cyclotron motion around the magnetic field direction and either ballistic ($\Delta V = 0$) or uniformly accelerated motion along the field direction; hence the motion is integrable. If the field is not tilted, collisions do not mix the cyclotron and longitudinal motion and integrability is maintained. Tilting the field causes each collision to mix longitudinal and cyclotron motion and allows energy exchange between these degrees of freedom. The direction and magnitude of the energy exchange depend in a sensitive manner on the *phase* of the cyclotron motion at the time of collision. Hence narrow chaotic layers arise for arbitrarily small tilt angles; however, for large energy exchanges to occur from repeated collisions the tilt angle must exceed the Kolmogorov-Arnold-Moser (KAM) threshold for global chaos. Determining this chaos threshold is the central question for the classical analysis. We find that for typical parameter values the chaos threshold occurs at small tilt angles; it follows that above threshold the energy transfer will occur diffusively (i.e., will require many collisions). In this case quantum dynamical localization effects can suppress the energy transfer even though it is allowed classically [7,8]. For the quantum system the conditions for this suppression become the key question. If global chaos *is* generated then it becomes possible to transfer huge amounts of energy from the electric field to the cyclotron motion. Although this process is inhibited in quantum wells due to optical phonon emission, one can imagine experiments with electron beams in vacuum which would not have this limitation.

We define our coordinates such that the (y,z) plane is the plane defined by the electric field $\boldsymbol{\epsilon} = \epsilon\hat{z}$ and magnetic field $\mathbf{B}$, while the planar barriers are parallel to the (x,y) plane at $z = 0, d$. $\mathbf{B}$ is tilted with respect to $\boldsymbol{\epsilon}$ by a rotation angle θ around the x axis. Choosing the vector potential $\vec{A} = (-By\cos\theta + Bz\sin\theta, 0, 0)$ the Hamiltonian is

$$H = \frac{(p_x - By\cos\theta + Bz\sin\theta)^2}{2m^*} + \frac{p_y^2}{2m^*} + \frac{p_z^2}{2m^*} - \epsilon z\,, \tag{1}$$

where p_i are the canonical momenta and we used atomic units so that for [1,2] $m^* = 0.067$. The cyclotron frequency is $\omega_c = B/m^*$ where B is measured in terms of $B_c \approx 2.35 \times 10^5$ T.

The Hamiltonian of Eq. (1) is independent of x, therefore p_x is conserved and can be eliminated by substi-

 0031-9007/95/74(11)/2098(4)$06.00

tution $y \rightarrow y - p_x/B\cos\theta$. However, $p_x \neq m^* v_x$ so that motion is not free in the x direction and the actual three-dimensional orbits may look quite complicated [1]. Nonetheless the underlying dynamics is that of a conservative system with two degrees of freedom [1,2]. Between collisions the electron motion is integrable and we analyze it in the frame (y', z') rotated by θ around x axis so that the z' axis is parallel to B. The z' motion is then uniform acceleration in the electric field $\epsilon\cos\theta$. The motion in the (x', y') plane is still the cyclotron rotation except that the center of the orbit is displaced by an amount $\delta y' = -\epsilon\sin\theta/m^*\omega_c^2$ which determines the drift velocity in the x direction.

Collisions with the barriers reverse the sign of v_z; it is easily shown that this corresponds to a reflection and rotation of the velocities $v_{z'}, v_{y'}$,

$$\begin{gathered}\bar{v}_{z'} = -\cos 2\theta v_{z'} + \sin 2\theta v_{y'}, \\ \bar{v}_{y'} = \sin 2\theta v_{z'} + \cos 2\theta v_{y'}, \qquad \bar{v}_{x'} = v_{x'},\end{gathered} \tag{2}$$

where the bar denotes the values of the velocities after the collision. Here y', z' velocities are just the momenta divided by m^*, but $v_{x'} = -\omega_c(y' - \delta y')$ is the velocity in x' without the drift component [9]. Between collisions separated by a time Δt the vector $(v_{x'}, v_{y'})$ simply rotates with frequency ω_c accumulating a phase angle $\varphi = \omega_c \Delta t$ while $v_{z'}$ will undergo uniform acceleration. The analysis of the general case $\Delta V \sim E_i$ is complicated, and so to determine the chaos boundary we consider the limiting cases $\Delta V \gg E_i$ and $\Delta V \ll E_i$.

If $\Delta V \gg E_i$ then it is very unlikely that the particle will retain enough longitudinal energy on collision with the barrier at $z = d$ to return to $z = 0$, and almost all trajectories collide repeatedly with the wall at $z = d$. If $\Delta V \ll E_i$ then the z' motion is ballistic between collisions which occur alternatingly between the two barriers. In both of these limiting cases the magnitude of the velocity v_0, determined by the total energy $E_0 = \Delta V + E_i = m^* v_0^2/2$, is unchanged at collisions and the map connecting the velocity vector at one collision to that at the next simply consists of a product of two velocity rotations, first by 2θ in the (y', z') plane (the collision) then by φ in the (x', y') plane (cyclotron rotation). The nonlinear evolution is just a sequence of many such rotation pairs on the sphere of radius v_0 [10].

Consider first the case $\Delta V \gg E_i$ in which the particle collides many times with the barrier at $z = d$ (so the motion is as in an infinite triangular well). Then the phase change due to the cyclotron rotation between collision is $\varphi = 2\omega_c m^* \bar{v} z'/\epsilon\cos\theta + \delta\varphi$. The correction $\delta\varphi$ is due to the fact that the longitudinal motion is tilted with respect to the barriers and so collisions occurring at $z = d$ do not occur exactly at $z' = d/\cos\theta$. This correction is small compared to the first term if $\theta \ll 1$ or if the first term is large; in this case it will affect the detailed trajectories but not the parametric conditions for chaos. Under these conditions we obtain the velocity change between collisions

$$\begin{gathered}\bar{v}_{z'} = -v_{z'}, \qquad \bar{v}_{y'} = \cos\varphi v_{y'} + \sin\varphi v_{x'}, \\ \bar{v}_{x'} = -\sin\varphi v_{y'} + \cos\varphi v_{x'}, \qquad \varphi = \beta\bar{v}_{z'}/v_0\cos\theta,\end{gathered} \tag{3}$$

where $\beta = 2\omega_c m^* v_0/\epsilon = 2^{3/2}\omega_c\sqrt{m^*}\,d/\Delta V^{1/2}$. The map obtained by composing (2) and (3) becomes identical to the kicked top map introduced by Haake [5]. Note that the four external parameters $(B, \Delta V, m^*, d)$ describing the system at fixed θ appear *only* through the dimensionless parameter β. This implies that the curves $\beta =$ const define the surfaces of constant classical dynamics in this 4D parameter space and more specifically that the transition to chaos must occur along a parabolic boundary in the B-ΔV plane. In Fig. 1 we show that the kicked top map does reproduce the exact dynamics generated by Eq. (1).

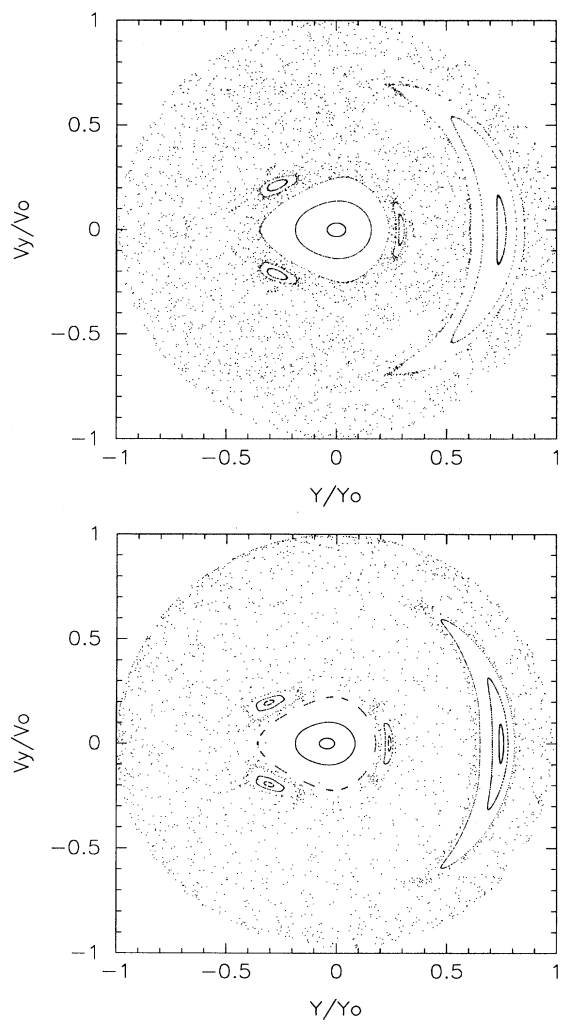

FIG. 1. Phase portraits of Landau dynamics (1) in triangular well (top) and Haake kicked top map (2), (3) (bottom) for $\beta = 9, \theta = 11°$; $z = d$; $y_0 = v_0/\omega_c\cos\theta$; the injection point is at $(0.05, -0.23)$; similarity becomes evident after a 90° rotation.

For large values of β a KAM transition to chaos takes place for $\theta \ll 1$. In this case it is convenient to represent the dynamics by the $(v_{z'}, \phi)$ map which may be approximated in the vicinity of a particular value of $v_{z'} = v'$ by the Chirikov standard map [11],

$$\bar{v}_{z'} = v_{z'} + 2\theta\eta v_0 \sin\phi, \qquad \bar{\phi} = \phi + \beta\bar{v_{z'}}/v_0, \quad (4)$$

where $\eta = (1 - v'^2/v_0^2)^{1/2} = (E_c/E_0)^{1/2}$, E_c is the instantaneous energy in cyclotron motion, and ϕ is the *total* accumulated cyclotron phase. The chaos boundary for this local standard map is given by $K = 2\theta\beta\eta > 1$, which yields the following condition for chaos as an explicit function of all system parameters:

$$\omega_c^2 > \Delta V E_0/32 m^* d^2 \theta^2 E_c \,. \quad (5)$$

Poincaré sections of the map (see Fig. 2) are in good agreement with this estimate. For $\theta \ll 1$ but above the chaos boundary energy exchange between the two degrees of freedom occurs diffusively over many collisions requiring a time $t_D \sim \beta/\omega_c\theta^2$.

Because of this slow relaxation, quantum effects can be quite important for $\theta \ll 1$ and should lead to the suppression of diffusion due to localization [7]. In order to determine the localization length it is necessary to express the map in terms of canonically conjugate variables [8]. In our case the appropriate variables are the Landau level number $n = E_c/\hbar\omega_c$ and cyclotron phase ϕ, and the resulting quantum dynamics differs from the quantum kicked top [5]. The classical diffusion rate per collision in these variables is $D_L = (\Delta n)^2/\Delta t \approx 8\theta^2 n(n_L - n)$ ($n_L = E_0/\hbar\omega_c$ is the total number of Landau levels energetically accessible). The interesting (quasiclassical) regime corresponds to $D_L > 1$. In this limit the localization length [7,8] is given by $\xi_L \approx D_L$ and increases linearly with n for $n \ll n_L$. This growth of ξ_L with n prevents localization unless the initial $\xi_L < n$, which can occur for sufficiently small angles θ. The condition for delocalization then becomes

$$D_L/n = 8\theta^2 n_L > 1 \,. \quad (6)$$

Note here n_L^{-1} is setting the quantum scale and acts like an effective $\hbar$. When (6) is not satisfied localization effects suppress the large energy exchanges predicted by the chaotic classical dynamics and eigenstates are power-law localized [8].

Now consider the case $E_i \gg \Delta V$ in which the particle moves ballistically along the field between alternate collisions at $z = 0, d$. The time between collisions is $\Delta t \approx d/v_{z'}\cos\theta$ so that the cyclotron phase change is $\varphi = \omega_c\Delta t = \omega_c d/v_{z'}\cos\theta$ (where again we neglect the small shift $\delta\varphi$). The free rotation given by (3) now has a phase $\varphi = \gamma v_0/\mid \bar{v_{z'}}\mid$, controlled by a different dimensionless parameter $\gamma = \omega_c d/v_0 \cos\theta$ and $\bar{v_{z'}} = v_{z'}$ [10]. Again a local standard map of the form (4) may be obtained, but now the phase change depends inversely on $v_{z'}$ just as in the Fermi acceleration model [4]. Since large phase changes between collisions promote chaos there is always a chaotic region of phase space as $v_{z'} \to 0$. As above we may define a local chaos parameter near $v_{z'} = v'$, $K = 2\theta\gamma\eta v_0^2/v'^2$. The requirement $K > 1$ gives

$$\omega_c^2 > (E_0 - E_c)^2/2m^* d^2\theta^2 E_c \quad (7)$$

as the condition for chaos. This estimate is in agreement with numerical simulations of the map. Note that (7) differs from (5) only in the replacement $\Delta V E_0 \to (E_0 - E_c)^2$; hence in both cases only the combination $\omega_c d\theta\sqrt{m^*}$ appears implying that the classical dynamics is invariant if this quantity is kept fixed.

We now discuss the relation of this work to experiments. For the quantum well systems $\Delta V \gg E_i$ and the classical dynamics is well described by the kicked top map for angles $\theta < 15°$ and $\beta > 3$ (Fig. 1). The first experiments [1] explored the regime $\beta > 9$ where we find mainly hard chaos. However, the recent experiments [2] study small enough B to be in the transition region. With the experimental parameters $d = 1200$ Å and $\epsilon = 6.5 \times 10^4$ V/cm for $\Delta V = 1$ V [12] we find $\beta = 6.2, n_L = 45$ at $B = 10$ T. The resonances at $\theta = 0$ correspond to the quantum well subband edges and Landau index $n = 0$ (the emitter state). For tilt angles $\theta = 27°, 45°$ the data [2] show a doubling of the number of resonance peaks in a region of parameter space bounded by a parabola $\Delta V \propto \omega_c^2$ consistent with (5). The electrons are injected from the $n = 0$ state of the emitter whereas $e\Delta V \sim 10^2\hbar\omega_c$, so $E_c/E_0 \sim 10^{-2}$. For such small E_c the dependence on θ in Eq. (5) cannot be used. However, analysis of the full dynamics (1) shows that the transition to chaos occurs at $\beta = 3.9$ for $\theta = 27°$ (Fig. 3) and $\beta = 3.5$ for $\theta = 45°$ which give parabolic boundaries in good agreement with the observed peak-doubling boundary. For $\theta < 24°$ the experimental data are more complicated with peak doubling appearing and then disappearing as B increases. We find

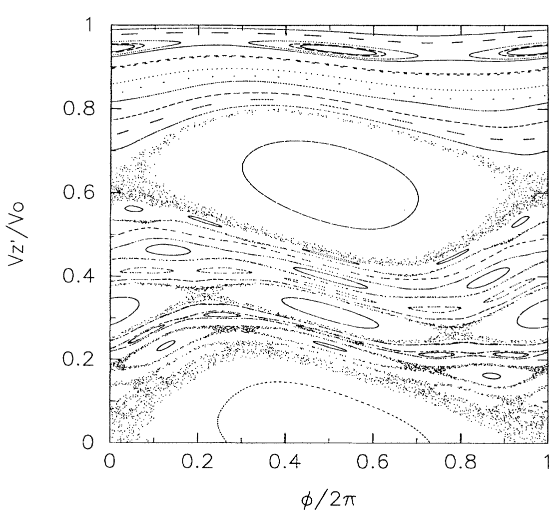

FIG. 2. Phase plane for the Haake kicked top (2) and (3) with $\theta = 0.05$, $\beta = 10$, near the critical regime $K \approx 1$.

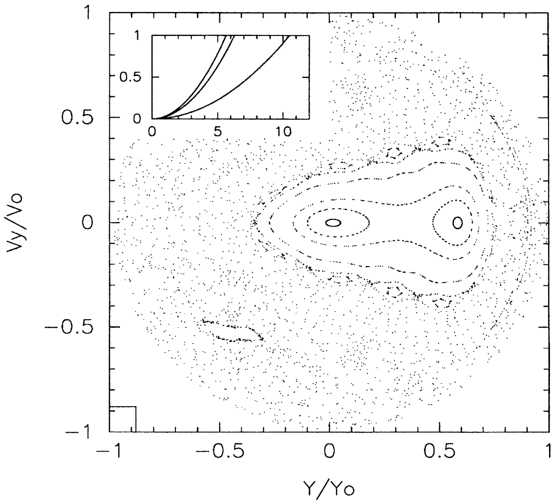

FIG. 3. Same as Fig. 1(a) for $\theta = 27°$, $\beta = 3.8$ near transition to chaos via period doubling, small box at $(-1,-1)$ shows effective $\hbar$ for $\Delta V = 1$ V, $B = 10$ T; orbit injecting at $z = 0$ with $E_i = 0$ has coordinates $(0.33, -0.22)$. Inset shows the lines of transition to chaos in $(B[\mathrm{T}], \Delta V[\mathrm{V}])$ plane for $\theta = 11°, 27°, 45°$ with $\beta = 6.5, 3.9, 3.5$ correspondingly.

the transition to chaos in the region of phase space near the injection point (which is calculated assuming $E_i \approx 0$ at $z = 0$) is much more complicated with several bifurcations of the initial period-one stable island followed by restabilizations of this island. For $\theta = 11°$ the kicked top map describes well the interesting region $3 < \beta$, and it shows that the injection point first enters the chaotic sea at $\beta = 6.5$ and then reenters the stable island near $\beta = 9$ (see Fig. 1). The transition to chaos at $\beta = 6.5$ roughly bounds the region of resonant peak doubling in the spectrum but a more detailed connection with [2] will require further work. Finally, the estimate $n_L \approx 45$ at $B = 10$ T indicates that localization effects may be observed, e.g., in the parameter range $d = 200$ Å and $B = 20$ T.

Finally, we discuss the case $\Delta V \gg E_i$ (triangular well) when the electric field ϵ is tilted in the (y,z) plane by an angle ψ with respect to z. In this case by the change of variables $y \rightarrow y + m^* \epsilon \sin\psi/(B\cos\theta)^2$ and $E_0 \rightarrow E_0 + m^* \epsilon^2 \sin\psi^2/2(B\cos\theta)^2$ the problem is reduced again to the Hamiltonian (1) with an effective electric field $\epsilon \rightarrow \epsilon \cos(\theta - \psi)/\cos\theta$. After all the substitutions it follows that tilting the electric field corresponds to the case $\psi = 0$ and effective $\beta \rightarrow \beta\cos\theta[1 + 4\sin^2\psi/(\beta\cos\theta)^2]^{1/2}/\cos(\theta - \psi)$. Our numerical simulations confirm this result. The renormalization of β with ψ can be used in experiments to increase the effective values of β and reach values inaccessible at available magnetic fields and $\psi = 0$. The tilted electric field leads to a drift along x with the average velocity $\langle v_x \rangle = \epsilon \sin\psi/(B\cos\theta)$ that may be detected experimentally. Varying ψ also changes the injection point in phase space allowing a direct test of its influence on the resonance spectrum without changing the effective β.

We acknowledge helpful discussions with G. Boebinger, H. Mathur, and R. Wheeler. Work at Yale was partially supported by NSF Grant No. DMR-9215065 and by ARO Grant No. DAAH04-93-G-0009.

*On leave from Laboratoire de Physique Quantique, Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse, France.

†Also at Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia.

[1] T.M. Fromhold, L. Eaves, F.W. Sheard, M.L. Leadbeater, T.J. Foster, and P.C. Main, Phys. Rev. Lett. **72**, 2608 (1994).

[2] G. Müller, G.S. Boebinger, H. Mathur, L.N. Pfeiffer, and K.W. West (unpublished).

[3] M.C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, Berlin, 1990).

[4] A.J. Lichtenberg and M.A. Lieberman, *Regular and Chaotic Dynamics* (Springer-Verlag, Berlin, 1992).

[5] F. Haake, *Quantum Signatures of Chaos* (Springer, Berlin, 1991).

[6] B.V. Chirikov, Phys. Rep. **52**, 263 (1979).

[7] D.L. Shepelyansky, Physica (Amsterdam) **28D**, 103 (1987).

[8] F. Benvenuto, G. Casati, I. Guarneri, and D.L. Shepelyansky, Z. Phys. B **84**, 159 (1991).

[9] The collision dynamics apparently compensates this drift and the particle remains confined in the x direction (see also [1]).

[10] For $\Delta V \gg E_i$ the free motion also inverts the longitudinal velocity.

[11] F. Haake and D.L. Shepelyansky, Europhys. Lett. **5**, 671 (1988).

[12] G.S. Boebinger (private communication).