Universal Diffusion near the Golden Chaos Border

S. Ruffo* and D. L. Shepelyansky†
Laboratoire de Physique Quantique, Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse, France
(Received 6 July 1995)

We study the local diffusion rate D in a Chirikov standard map near the critical golden curve. Numerical simulations confirm the predicted exponent $\alpha = 5$ for the power law decay of D as approaching the golden curve via principal resonances with period q_n ($D \sim 1/q_n^\alpha$). The universal self-similar structure of diffusion between principal resonances is demonstrated, and it is shown that other resonances may also play an important role. [S0031-9007(96)00063-4]

PACS numbers: 05.45.+b

During the last few years intensive investigations have allowed one to understand the structure of critical invariant curves at the chaos border in Hamiltonian dynamical systems with divided phase space [1,2]. Usually the analysis is carried out for two-dimensional (2D) area preserving maps, the paradigm being the Chirikov standard map [3]. The critical invariant curve is characterized by the rotation number and its continued fraction expansion. It has been shown that the small scale structure near invariant curves with the same tail in this expansion is universal for all smooth Hamiltonians with two degrees of freedom and for 2D maps [1,2,4]. Among all invariant curves, better studied are those with the golden rotation number $r_g = (\sqrt{5} - 1)/2$, whose expansion is a series of 1's. This r_g is the most irrational number, and therefore it is believed that invariant curves with golden tails are locally the most robust ones. The structure of the critical golden curve has been studied by means of the renormalization group approach, and it has been shown that the phase space structure is self-similar and universal on small scales.

Different scaling exponents have been found in this critical regime, and they were successfully used to determine the diffusion rate through the destroyed invariant curve called a cantorus [3,5]. However, no flux passes through the golden curve at the critical value of the perturbation parameter $K = K_g$. In this case the trajectory has only a local diffusion rate D, which depends on its distance Δr_n to the golden curve. This diffusion characterizes the motion of a particle in the vicinity of the invariant curve r_g at different levels n of convergents $r_n = p_n/q_n$ of the continued fraction expansion of r_g ($r_1 = 1/1, r_2 = 1/2, r_3 = 2/3, \ldots$). One can expect a power law dependence of D on the resonant approximant q_n, namely, $D \sim q_n^{-\alpha}$. Chirikov gave a simple argument for $\alpha = 5$ [6]. According to him, $D \sim (\Delta r_n)^2/t_n$, where $\Delta r_n = |r_g - r_n| \sim q_n^{-2}$ and t_n is the typical inverse frequency Ω_n of small oscillations around the principal resonance q_n. Then, $\Omega_n \sim q_n \Delta\omega_n$, where $\Delta\omega_n$ is the width of resonance q_n [3]. In the critical case the Chirikov overlap criterion [3] implies $\Delta\omega_n \sim \Delta r_n$, which gives [6]

$$D \approx AD_0/q_n^5 \sim (\Delta r_n)^{5/2} \sim (\delta y_n)^\nu, \tag{1}$$

where $D_0 = K^2/8\pi^2$ is the quasilinear diffusion rate and $\delta y_n = |y_n - y_g|$ is the distance of unstable periodic orbit y_n with rotation number r_n to the golden curve y_g along the symmetry line. The exponent $\nu = 2.14699\ldots$ can be determined from the exponent σ for $\delta y_n \sim 1/q_n^\sigma$ found in [2] ($\nu = 5/\sigma$).

The fast decay of the diffusion rate near the chaos border r_g means that a diffusing particle will never reach the border itself. This slow diffusion gives a long sticking trajectory around stable islands on different renormalization levels. As a result, the statistics of Poincaré recurrences P (integrated probability to return into a given region after a time larger than τ) decays with τ as $P(\tau) \sim 1/\tau^p$. Such decay was first observed in [7], where the average value of the exponent $p \approx 1.5$ has been found. Further investigations have shown that the power law decay of P is a generic property of Hamiltonian systems with divided phase space [8–10]. However, according to numerical results, the power itself is not universal, varying in the range $1 < p < 2$. Moreover, different maps with a golden chaos boundary give different p values [9], which seems to be in contradiction with the universal self-similar structure of phase space near the golden border. Indeed, renormalization arguments give $p = 3$ [11], which is in sharp contradiction with numerical results. One of the possible reasons for the above contradiction is the sticking of particles near stable islands *between* principal resonances q_n. However, attempts to take into account these intermediate resonances gave $p = 2$ [11], which is still too large compared to the numerical results for the golden border $1 < p < 1.35$ [9]. Therefore the problem of Poincaré recurrences remains unsolved, and more detailed investigations of the phase space structure near the golden border should be performed (see also [12]). One of the reasons why the properties of $P(\tau)$ are so important is that the correlation function of dynamical variables $C(\tau)$ and the probability $\mu(\tau)$ staying in a given region for a time $t > \tau$ are related to $P(\tau)$,

$$C(\tau) \sim \mu(\tau) \sim \tau P(\tau)/\langle\tau\rangle \sim 1/\tau^{p-1}, \tag{2}$$

where $\langle\tau\rangle$ is the average return time. The above relations follow from the ergodicity of motion on the chaotic component of the phase space [7–9]. The decay of

 0031-9007/96/76(18)/3300(4)$10.00

correlations with power $p_c = p - 1 < 1$ can lead to a divergence of global diffusion rate and to strong fluctuations in divided phase space.

In this Letter we investigate the behavior of local diffusion rate D near the critical golden curve. Our first aim was to verify Chirikov's heuristic prediction (1) and to analyze the structure of D at different renormalization levels. As a model we have chosen the Chirikov standard map

$$\bar{y} = y - K/(2\pi)\sin(2\pi x), \qquad \bar{x} = x + \bar{y} \bmod 1 \quad (3)$$

with a perturbation parameter corresponding to the critical golden curve $K = K_g = 0.97163540631\ldots$. To measure the local diffusion rate $D = (\Delta y)^2/\Delta t$ we apply the efficient method used for the investigations of Arnold and modulational diffusion in Ref. [13] (t is the number of iterations). This method allows us to measure very small diffusion rates (down to computer noise level) with a relatively small number of iterations. Using this method, we compute D at different resonant Fibonacci approximants $r_n = p_n/q_n$ of the critical curve r_g. We use from $N_p = 10$ to $N_p = 100$ trajectories near unstable periodic orbits of period q_n (these points had been determined by MacKay [2]). Each trajectory is integrated for about $T = 1000q_n$ iterations. The total interval T is divided into $N_w = 10$ windows, where the average y displacement was computed with the smoothing function $f = \sin^{2\beta}(\pi t N_w/T)$. Usually we take $\beta = 4, 6$. Such smoothing allows us to suppress regular oscillations by a factor $\propto (N_w/T)^{4\beta+3}$ [13]. To control the accuracy of our numerical computation of D, we determine D also near stable periodic orbits in the center of resonance q_n, where our method gives a value $D \sim 10^{-34}$, which corresponds to the level of computer round-off error in double precision.

Our results in Fig. 1 confirm the theoretical prediction (1) for the variation of D over more than 20 orders of magnitude. The numerical fit gives $\alpha = 4.99 \pm 0.02$ and $A = 0.0066$, indicating no visible deviations from the theory (1). We attribute the fluctuations at small q_n values to the fact that, for a large number of iterations, trajectories can exit from the chaotic layer corresponding to the initial q_n. This effect disappears for larger q_n, where the local diffusion rate is sufficiently small, or for a shorter number of iterations. Let us note that the numerical value of A is surprisingly small. Our explanation for this fact is the following. According to [3] the action change after a half period of rotation in the chaotic separatrix layer is quite small, $\Delta y \sim 4\lambda^2 \exp(-\pi\lambda/2)$ (here $\lambda = 2\pi$ is the frequency ratio). This gives an order of magnitude estimate for $A = D(q_n = 1)/D_0 \sim (\Delta y)^2/2D_0 \sim 0.003$ and explains its small value. A more accurate estimate of A requires taking into account higher orders of perturbation in K. The measured diffusion rate in the chaotic component is well separated from diffusion in the stable regions produced by numerical round-off errors (Fig. 1).

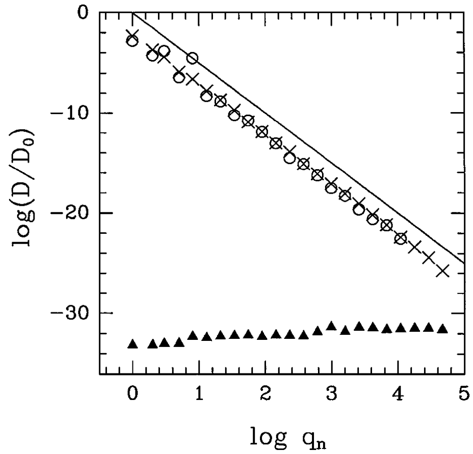

FIG. 1. Local diffusion rate D near principal resonance convergents $r_n = p_n/q_n$ to $r_g = (5^{1/2} - 1)/2$, $D_0 = K^2/(8\pi^2)$; chaotic component near unstable points with (a) $T = 10^3 q_n$, $N_w = 10$, $N_p = 10$ ($\times$); (b) $T = 10^4 q_n$, $N_w = 100$, $N_p = 100$ ($\bigcirc$); (c) regular component near stable points with $T = 10^3 q_n$, $N_w = 10$, $N_p = 10$ (triangles). The straight line shows the theoretical slope $\alpha = 5$.

The above result shows the global structure of the diffusion rate D while approaching the chaos border via resonant approximants q_n. However, an interesting question concerns the behavior of D between $r_n = p_n/q_n$ and r_{n+1}. The comparison of D on these scales should reflect the self-similar structure of phase space on different renormalization levels. To check this self-similarity we measure D on two symmetry lines $x = 0, 0.5$. The symmetry line $x = 0.5$ crosses the main part of the chaotic layers and contains mainly unstable points, while the other line $x = 0$ passes mainly through stable islands. The known structure of periodic orbits on symmetry lines of map (3) [2] allows us to find the basic renormalization intervals on stable $x = 0$ and unstable $x = 0.5$ lines.

For the unstable line the first renormalization level interval is $\Delta y_1 = |y_1 - y_7|$, where y_n is the y value of the unstable periodic orbit with rotation number r_n on the line $x = 0.5$. The next interval $\Delta y_2 = |y_4 - y_{10}|$ lies on the other side of the golden curve. The mth interval is $\Delta y_m = |y_{3m-2} - y_{3m+4}|$. This m series selects the subsequent n values which we will denote by $n_m = 3m - 2$. Intervals with odd values of m lie above the invariant curve r_g and those with even m lie below. The period 6 in the renormalization levels is related to the periodicity 6 of unstable fixed points on the symmetry line as it follows from [1,2]. The self-similarity of the phase space implies that the dependence of diffusion D on the position inside mth interval should be approximately the same as in $(m + 1)$th interval, after $q_{n_m}^5$ rescaling. At larger m values the self-similarity is expected to become better and better. To numerically check this self-similarity we computed the diffusion rate at 320 homogeneously distributed points in the intervals

Δy_m for $m = 1, \ldots, 6$. To compare different intervals we rescale the diffusion rate defining $D_R = q^5_{n_{m+2}} D_m$, where D_m is the diffusion at mth level. The position inside each interval is denoted by $\Delta y_R = |(y - y_{3m-2})|/\Delta y_m$ for y between y_{3m-2} and y_{3m+4} $(0 \le \Delta y_R \le 1)$.

The comparison of renormalized diffusion rate D_R on two levels $m = 4$ and $m = 6$ on the unstable line is shown in Fig. 2. The diffusion rate is self-similar in agreement with the universal phase space structure near the golden curve. The minimal diffusion rate D_R on levels $m = 4, 6$ is determined by computer noise and is different on the two levels due to the different normalization factors. The rare fluctuations in the upper diffusion plateau are presumably due to exits of trajectory from the initial chaotic layer. Points with a high diffusion rate have a chaotic component, while those with minimal diffusion correspond to trajectories in stable islands. The self-similarity was also observed at other m levels, both when intervals were on the same side or on opposite sides of the golden curve. This self-similarity becomes better with the growth of m. The sharp separation between two levels of diffusion allows us to determine the leading resonances in each renormalization interval. The biggest gap in the diffusion is for periodic orbit (labeled b in Fig. 2) with rotation number $\rho_b = [\{111\}_{m-1}, 1, 2, 1]$, where the triplet of 1's in curly brackets is repeated $m - 1$ times. This rotation number is not from the series of principal resonances given by the continued fraction expansion of r_g and labeled by i in Fig. 2 ($\rho_i = [\{111\}_{m-1}, 1, 1, 1, 1, 1]$). While the resonance ρ_b is significantly larger than the principal one ρ_i, there are also other resonances which are of comparable or smaller size than ρ_i: $\rho_c = [\{111\}_{m-1}, 1, 2, 1, 1, 1, 1]$, $\rho_d = [\{111\}_{m-1}, 1, 2, 2, 2, 1]$, $\rho_e = [\{111\}_{m-1}, 1, 2, 2, 1]$, $\rho_g = [\{111\}_{m-1}, 1, 1, 1, 3, 2, 1]$, $\rho_h = [\{111\}_{m-1}, 1, 1, 1, -2, 1, 1, 1]$, $\rho_l = [\{111\}_{m-1}, 1, 1, 1, 1, 1, 2, 1, 1]$. It is interesting to remark that most of these rotation numbers have continued fraction expansions containing mainly 1's and 2's, which is in qualitative agreement with the conjecture made in [9].

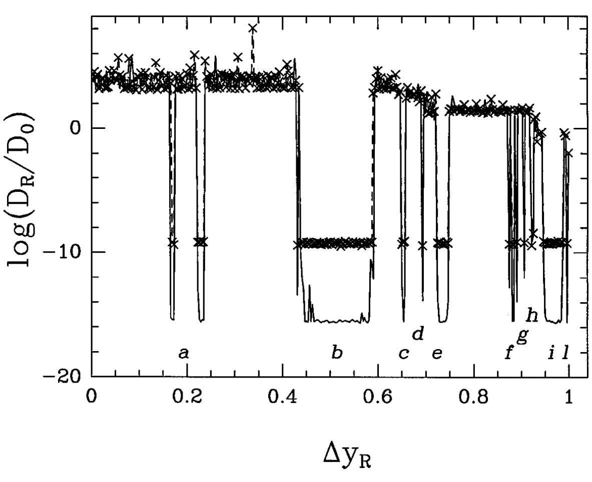

FIG. 2. Renormalized diffusion rate D_R/D_0 vs rescaled position Δy_R in the mth level interval of the renormalization scheme on the unstable symmetry line $x = 0.5$; $m = 4$ (full curve) and $m = 6$ (dashed line with crosses). The letters indicate the drops of diffusion D_R (see text). Other parameters are as in Fig. 1(a).

In Fig. 2 we also see other types of resonances, namely, $\rho_a = p_a/q_a = [\{111\}_{m-1}, 1, 4, 1]$ and $\rho_f = p_f/q_f = [\{111\}_{m-1}, 1, 1, 1, 3, 1]$. However, they have a different structure than previous ones, displaying two and three intersections with unstable line $x = 0.5$ giving rise to double (a) and triple (f) drops in diffusion. Indeed, the periodic orbit inside resonance ρ_a (ρ_f) has period $2q_a$ $(3q_f)$. These orbits are not present in the limit $K \to 0$, and correspond to a new chain of islands in the chaotic layer around resonances ρ_a (ρ_f). An important consequence of the analysis of the structure of these resonances is that, in spite of the fact that the renormalization group describes quite well the convergence to golden curve, we see that, on each renormalization level, resonances other than those of the main series of r_g, and even some which are not present at $K = 0$, occupy a sizable part of phase space. The existence of such nonstandard resonances might explain the lack of universality of the exponent p in (2) and the numerical value of p significantly less than 2. Indeed, very long Poincaré recurrences can be originated by sticking of orbits not only near the main resonances r_n but also around these nonstandard resonances and the chains of islands around them. The general description of the phase-space structure should take into account the presence of these resonances. An interesting question is which is the strongest cantorus on a given renormalization level which will determine the transition time between different renormalization levels. According to Fig. 2 all drops in D are associated with periodic orbits and not to invariant curves, which is in agreement with the conjecture that r_g is the last invariant curve.

We have also studied local diffusion on the stable line $x = 0.0$ (Fig. 3). The size of the renormalization interval n is defined as $\Delta y_n = |y_n - y_{n+2}|$ and the variable $\Delta y_R = |(y - y_n)|/\Delta y_n$. The structure of diffusion is also self-similar. As expected, the main part of the renormalization interval is occupied by stable islands. The largest resonances with rotation number $\rho'_a = [\{1\}_n]$ and $\rho'_i = [\{1\}_n, 1, 1]$ correspond to the main series of r_g. There are also other resonances: $\rho'_c = [\{1\}_n, 3, 1]$, $\rho'_d = [\{1\}_n, 3, 1, 1]$, $\rho'_e = [\{1\}_n, 2, 1]$, $\rho'_f = [\{1\}_n, 2, 1, 1, 1]$, $\rho'_g = [\{1\}_n, 2, 1, 1]$, $\rho'_h = [\{1\}_n, 2, 2, 1]$. Again, the biggest resonances have only 1's and 2's in the continued fraction. Some of the resonances seen on $x = 0.0$ are also observed on $x = 0.5$, and one can easily establish a correspondence. The resonance ρ'_b has the same rotation number as ρ'_a, but it corresponds to a different orbit. This orbit, which does not exist for $K = 0$, corresponds to a chain of nine islands around the golden resonance ρ'_a, and its rotation number around the main sequence chain is $1/9$.

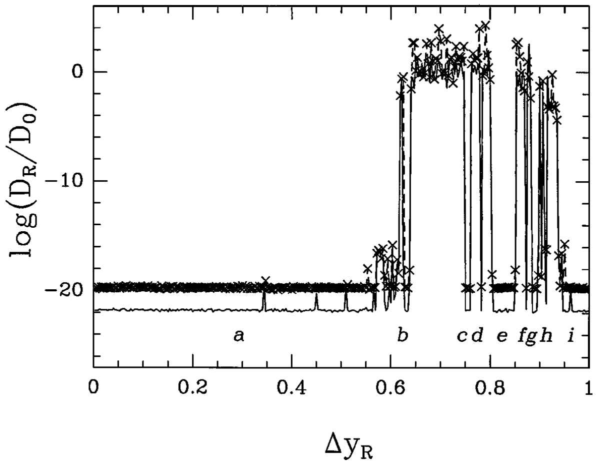

FIG. 3. Renormalized diffusion rate D_R/D_0 vs rescaled position Δy_R in the nth level interval of the renormalization scheme on the stable symmetry line $x = 0.0$; $n = 8$ (full curve) and $n = 10$ (dashed line with crosses). The letters indicate the drops of diffusion D_R (see text); $T = 2 \times 10^4 q_n$, $N_w = 10$, $N_p = 10$.

The above analysis allows us to understand some important properties of the local diffusion rate D near the critical invariant curve r_g. The self-similar structure of D also shows the importance of nonstandard resonances, different from principal approximants $r_n = p_n/q_n$ of r_g. These nonstandard resonances are also self-similar at different renormalization levels. However, their sizes in phase space are comparable or sometimes even larger than those of principal resonances. Therefore trajectories can be trapped for a long time around these nonstandard resonances and diffuse very slowly to internal chaos boundaries surrounding islands of these resonances. Since the sizes of nonstandard and principal resonances are comparable, the contribution of internal chaos boundaries to Poincaré recurrences may be relevant, and thus the decay of $P(\tau)$ may be nonuniversal. The determination of the asymptotic behavior of $P(\tau)$ also requires a better understanding of transition rates between different renormalization levels, which are not directly related to local diffusion and should be studied in more detail. Finally, let us mention that for $\Delta K = K - K_g > 0$ the scales with $q_n < q_{\rm cr} \sim 1/\Delta K$ are unaffected [6,9] and the diffusion on them is still given by (1). For scales with $q_n > q_{\rm cr}$ the diffusion rate is approximately $D \sim AD_0/q_{\rm cr}^5$. The average diffusion rate is $1/\langle D \rangle \sim \int_0^1 dr_n/D_n \sim r_{\rm cr}^{-3/2} \sim \Delta K^3$. The scaling power is in agreement with the result in [5] with better than 1% accuracy.

One of us (S. R.) gratefully acknowledges the hospitality and financial support of the Laboratoire de Physique Quantique, URA 505, CNRS at Université Paul Sabatier, Toulouse, France. This work is also part of European Contract No. ERBCHRXCT940460.

*Also at Dipartimento di Energetica, Università di Firenze, Via S. Marta, 3, 50139 Firenze, Italy and INFN, Sezione di Firenze. Electronic address: ruffo@vaxf.fi.infn.it

†Also at Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia.

[1] J. M. Greene, J. Math. Phys. **20**, 1183 (1979).
[2] R. S. MacKay, Physica (Amsterdam) **7D**, 283 (1983).
[3] B. V. Chirikov, Phys. Rep. **52**, 263 (1979).
[4] D. F. Escande, Phys. Rep. **121**, 165 (1985).
[5] R. S. MacKay, J. D. Meiss, and I. C. Percival, Physica (Amsterdam) **13D**, 55 (1984).
[6] B. V. Chirikov, Lect. Notes Phys. **179**, 29 (1983).
[7] B. V. Chirikov and D. L. Shepelyansky, Novosibirsk Report No. INP81-69, 1981 (translation, Princeton University Report PPL-TRANS-133, 1983); *Proceedings of the IX International Conference on Nonlinear Oscillations, Kiev, 1981* [Naukova Dumka **2**, 420 (1984)]; Phys. Rev. Lett. **61**, 1039 (1988).
[8] C. F. F. Karney, Physica (Amsterdam) **8D**, 360 (1983).
[9] B. V. Chirikov and D. L. Shepelyansky, Physica (Amsterdam) **13D**, 395 (1984); in *Renormalization group*, edited by D. V. Shirkov, D. I. Kazakov, and A. A. Vladimirov (World Scientific, Singapore, 1988), p. 221.
[10] I. Dana, Phys. Rev. Lett. **70**, 2387 (1993).
[11] J. D. Meiss and E. Ott, Phys. Rev. Lett. **55**, 2741 (1985); Physica (Amsterdam) **20D**, 387 (1986).
[12] N. W. Murray, Physica (Amsterdam) **52D**, 220 (1991).
[13] B. V. Chirikov, M. A. Lieberman, D. L. Shepelyansky, and F. Vivaldi, Physica (Amsterdam) **14D**, 289 (1985).