

Breit-Wigner Width for Two Interacting Particles in a One-Dimensional Random Potential

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For two interacting particle (TIP) in a one-dimensional random potential, the dependence of the Breit-Wigner width Γ , the local density of states, and the TIP localization length on system parameters are determined analytically. The theoretical predictions for Γ are confirmed by numerical simulations. [S0031-9007(96)02242-9]

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Recently, the problem of two interacting particles (TIP) in a random potential has attracted the interest of different groups [1–6]. It has been shown that two repulsive/attracting particles can propagate together on a distance l_c much larger than the one-particle localization length l_1 in absence of interaction. The first analytical studies [1,2] for TIP with on site interaction on a one-dimensional (1D) one channel lattice gave the following estimate $l_c/l_1 \sim \Gamma\rho \sim (U/V)^2 l_1$, where U is the strength of the interaction, V is the intersite hopping matrix element, $\rho \sim l_1^2/V$ is the density of the two-particle states coupled by the interaction, and $\Gamma \sim U^2/Vl_1$ is the interaction induced transition rate between these states. The numerical investigations [3,4] definitely confirmed existence of the strong enhancement of l_c due to interaction. However, a direct verification of the above estimate is quite difficult, even for the modern computer facilities, due to the strong increase of required basis with l_1 . Also, the recent numerical results of von Oppen *et al.* [4] and Weinmann and Pichard [7] indicate in the 1D case almost linear growth of the enhancement factor for l_c with U instead of expected U^2 . Because of all these things, it would be important to have a more rigorous derivation of the factor l_c/l_1 for this on a first glance quite simple problem, at least in the 1D case. To reach this aim we started from the computation of the rate Γ , which also characterizes the spread width of the Breit-Wigner distribution for eigenfunctions in the basis of eigenstates of noninteracting particles [8–10]. If the parameter dependence of Γ is known then the ratio l_c/l_1 can be determined from the relation $l_c/l_1 \sim \Gamma\rho$, which has been checked in models of superimposed band ran-

dom matrices [1,8–10]. In the present work for calculation of Γ we use the technique developed in [11] which allows us to account all orders in the interaction.

We consider the 1D Hubbard model with Hamiltonian

$$H = -V \sum_{n\sigma} (a_{n+1\sigma}^\dagger a_{n\sigma} + a_{n\sigma}^\dagger a_{n+1\sigma}) + U \sum_n a_{n\uparrow}^\dagger a_{n\downarrow}^\dagger a_{n\downarrow} a_{n\uparrow}. \tag{1}$$

Here a_n^\dagger is a creation operator of the particle at the site n , V is the hopping matrix element, and U is the on site interaction. We assume that particles are distinguishable and denote the type of particle by spin $\sigma = \pm 1/2$. The single particle eigenstate is plane wave $|p\rangle = \frac{1}{\sqrt{L}} e^{ipn}$ with dispersion $\epsilon_p = -2V \cos p$, $-\pi \leq p \leq \pi$. We set lattice spacing equal to unity. The size of the lattice is denoted by L .

The Breit-Wigner width can be found in the following way. The forward scattering amplitude f for particles with different spins is given by a series of diagrams presented at Fig. 1. The solid line represents a particle, and the wavy line is the matrix element of the interaction $\langle p_3 p_4 | \hat{U} | p_1 p_2 \rangle = \frac{U}{L} \delta_{p_1+p_2, p_3+p_4}$. Because of the optical theorem width of the state, $|p_1 p_2\rangle = |p_1\rangle |p_2\rangle$ is related to the forward scattering amplitude:

$$\Gamma/2 = -\text{Im} f. \tag{2}$$

One can easily check the coefficient in this relation considering Fig. 1(b), which gives the usual Fermi golden rule:

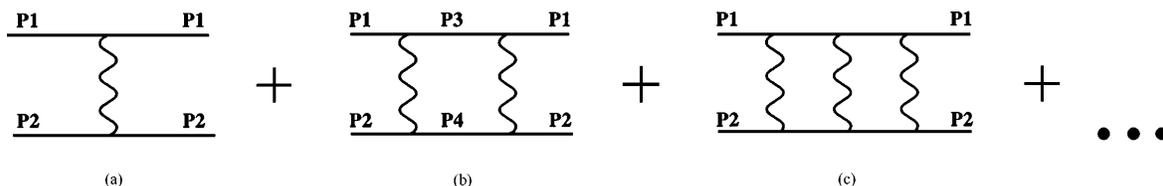


FIG. 1. Diagrams for the forward scattering amplitude f in (2)–(5).

$$\Gamma \approx -2 \operatorname{Im} f_{1b} = -2 \operatorname{Im} \sum_{p_3 p_4} \frac{|\langle p_3 p_4 | \hat{U} | p_1 p_2 \rangle|^2}{E - \epsilon_3 - \epsilon_4 + i0} \quad (3)$$

$$= 2\pi \sum_{p_3 p_4} |\langle p_3 p_4 | \hat{U} | p_1 p_2 \rangle|^2 \delta(E - \epsilon_3 - \epsilon_4).$$

$$f_{1b} = \sum_{p_3 p_4} \frac{|\langle p_3 p_4 | \hat{U} | p_1 p_2 \rangle|^2}{E - \epsilon_3 - \epsilon_4 + i0} = \frac{U^2}{L^2} \sum_{p_3} \frac{1}{E + 2V \cos p_3 + 2V \cos(p - p_3)}$$

$$= \frac{U^2}{L^2} \int_{-\pi}^{\pi} \frac{L dp_3 / 2\pi}{[E + 2V \cos p_3 + 2V \cos(p - p_3)]} = \frac{U^2/L}{\sqrt{E^2 - 16V^2 \cos^2 p/2}}, \quad (4)$$

where $p = p_1 + p_2 = p_3 + p_4$ is the total quasimomentum. Higher orders in Fig. 1 correspond to simple iterations of the box Fig. 1(b). Therefore the summation of the ladder is reduced to geometrical progression and the result is

$$f(E, p) = \frac{U/L}{1 - U/\sqrt{E^2 - 16V^2 \cos^2 p/2}}. \quad (5)$$

The scattering amplitude depends only on the total energy $-4V \leq E \leq 4V$ and the total momentum $-\pi \leq p \leq \pi$. The branch of the square root should be chosen in such a way that $\operatorname{Im} f \leq 0$.

With amplitude (5) one can easily calculate the Breit-Wigner width using the optical theorem (2). But we are interested in the average width at a given energy. So we have to average over momentum p . The density of the two particle states is of the form

$$\rho(E, p) = \int_{-\pi}^{\pi} \frac{L dp_1}{2\pi} \int_{-\pi}^{\pi} \frac{L dp_2}{2\pi} \delta(p - p_1 - p_2)$$

$$\times \delta(E + 2V \cos p_1 + 2V \cos p_2)$$

$$= \frac{L^2/(8\pi^2 V)}{\sqrt{\cos^2 p/2 - E^2/16V^2}}. \quad (6)$$

It is nonzero only if square root is real. After integration over momenta we find

$$\rho(E) = \int_{-\pi}^{\pi} \rho(E, p) \frac{dp}{2\pi}$$

$$\approx \frac{L^2}{2\pi^2 V} \left(\ln \frac{16V}{|E|} + 0.18 \frac{|E|}{4V} \right). \quad (7)$$

The integral in (7) cannot be exactly expressed in terms of elementary functions. The presented approximate formula is valid with accuracy better than 1% in the interval $-4V \leq E \leq 4V$. Now we can find the average width.

$$\Gamma(E) = -2 \operatorname{Im} \int f(E, p) \rho(E, p) \frac{dp}{2\pi} / \rho(E)$$

$$= \frac{8Vu^2/L}{(\ln 4/\epsilon + 0.18\epsilon)\sqrt{|u^2 - \epsilon^2|(1 + u^2 - \epsilon^2)}} F(Z). \quad (8)$$

Here $u = |U/4V|$ and $\epsilon = |E/4V|$ is the interaction and energy expressed in units of bandwidth: $0 \leq \epsilon \leq 1$. The

Here E is energy of the initial state $E = \epsilon_1 + \epsilon_2$.

The Born term in the amplitude f is given by Fig. 1(a) and equals $f_{1a} = U/L$. Calculation of the diagram Fig. 1(b) is also straightforward:

function $F(Z)$ is defined by

$$F(Z) = \begin{cases} \arctan Z, & \text{for } u \geq \epsilon, \\ \frac{1}{2} \ln \frac{1+Z}{1-Z}, & \text{for } u \leq \epsilon, \end{cases} \quad (9)$$

$$Z = \sqrt{\frac{|u^2 - \epsilon^2|(1 - \epsilon^2)}{\epsilon^2(1 + u^2 - \epsilon^2)}}.$$

At small energy ($\epsilon^2 \ll u^2, 1$) formula (8) gives:

$$\Gamma \approx \frac{4\pi V}{L} \frac{1}{\ln 4/\epsilon} \frac{u}{\sqrt{1 + u^2}}, \quad (10)$$

so that at small interaction ($\epsilon^2 \ll u^2 \ll 1$) it is linear in the interaction. In the other limit [$u^2 \ll \epsilon^2, (1 - \epsilon^2)$] the width (8) is quadratic in u with logarithmic correction:

$$\Gamma \approx \frac{8V}{L} \frac{1}{(\ln 4/\epsilon + 0.18\epsilon)} \frac{u^2}{\epsilon} \ln \frac{2\epsilon\sqrt{1 - \epsilon^2}}{u}. \quad (11)$$

The value of Γ in (10) is significantly larger than in (11) due to the growth of two-particle density of states (7) near the center of the band.

If we now add to the Hamiltonian (1) a single particle random potential $H_{\text{rand}} = \sum w_n a_{n\sigma}^\dagger a_{n\sigma}$ with a disorder homogeneously distributed in the interval $-W \leq w_n \leq W$, then the one-particle eigenstates in infinite lattice become localized with localization length $l_1 \approx 24(4V^2 - \epsilon_1^2)/W^2$, where ϵ_1 is one-particle energy. However, as soon as $l_1 \gg 1$, the above calculation of the average width remains valid. The reason for this is that $l_1 \gg 1$ is the only condition which we need to formulate the scattering problem and to use the conventional diagram technique. The distribution of Γ depends on the relation between the size of the box L and the localization length l_1 . If $L \leq l_1$ all values of Γ are of the order of the average value given by (8). For $L \gg l_1$ the average value is still given by (8). However, in this case Γ vanishes for the majority of the states. These are the states in which particles are localized far from each other and practically do not interact. On the other hand, the width for the states with interparticle distance of the order l_1 is approximately the same as for the particles in a box of size $L \approx l_1$ so that Γ is given by Eqs. (8) and (9) with L replaced by l_1 . The two-particle localization length l_c for such states is determined by the relation $l_c/l_1 \sim \Gamma(E)\rho(E)$, with Γ calculated at $L \sim l_1$. This relation is valid if $\Gamma(E)\rho(E) > 1$ [1,2]. In the opposite case $\Gamma(E)\rho(E) \ll$

1 the above relation is not valid [8–10] and the interaction can be treated in a perturbative way. In this regime “Rabi oscillations” between two quasidegenerate levels play an important role [7]. Above we have considered the distinguishable particles. The generalization to the identical particles is rather simple: the width Γ vanishes if the coordinate wave function is antisymmetric, and it is doubled in comparison with Eqs. (8) and (9) if the coordinate wave function is symmetric.

To check the above theoretical formula for Γ we studied numerically the model (1) of two identical interacting particles (symmetric coordinate wave function) in the disordered potential on a ring of size L , which is less or comparable with one-particle localization length $l_1 \approx 24(V/W)^2$. Using the Lanczos technique (see, for example, [12]) we determined the local density of states in the basis of noninteracting eigenstates:

$$\rho_W(E - \epsilon_{m_1} - \epsilon_{m_2}) = \sum_{\lambda} |\psi_{\lambda}(m_1, m_2)|^2 \delta(E - E_{\lambda}). \quad (12)$$

Here E_{λ} is the eigenenergy of TIP, while $\epsilon_{m_{1,2}}$ are one-particle eigenenergies. The dependence of ρ_W on E is well described by the Breit-Wigner distribution

$$\rho_W(E) = \frac{\Gamma}{2\pi[E^2 + \Gamma^2/4]}, \quad (13)$$

an example of which is shown in Fig. 2. The comparison of numerically obtained Γ with theoretical prediction (8) and (9) in the regime $\Gamma(E)\rho(E) > 1$ is shown in Figs. 3 and 4 for different energies as the function of interaction. The theory gives good agreement with numerical results for $15 \leq L \leq 300$ and variation of scaled width $\Gamma L/V$ by more than 2 orders of magnitude. For the states with the energy close to the band center ($E \approx 0$) (Fig. 3), the dependence of $\Gamma\rho$ on U is almost linear for $U < V$ [see (7) and (10)]. Therefore the TIP localization length l_c according to the relation $l_c/l_1 = C\Gamma\rho \approx 2Cl_1(U/V)/\pi$ also varies linearly with U . Here we took the values of Γ and ρ at $L = l_1$ and introduced the numerical coefficient C to take into account the uncertainty of this choice. According to the numerical results [4] at the center of the band $l_c/l_1 \approx 0.2l_1(U/V)$, which is in good agreement with the above theoretical expression and gives $C \approx 1/4$.

For energies away from the band center and small interaction $|U| \ll |E|$ the enhancement factor according to (7) and (11) is $l_c/l_1 \approx l_1 U^2 \ln(2E/U)/(4\pi^2 VE)$, where we have used the above value of C . The dependence on U is almost quadratic in agreement with the first estimate [1,2]. However, due to the logarithmic correction, to observe clearly the U^2 behavior one should go to really small U values and, since the condition $\Gamma\rho > 1$ should be also satisfied, this can be reached only for quite large values of l_1 or L . In this respect our numerical approach based on the measurement of Γ is more efficient than the one used in [4]. It allows us to see the behavior $U^2 \ln U$ away from the band center in agreement with the theory (8) and (9)

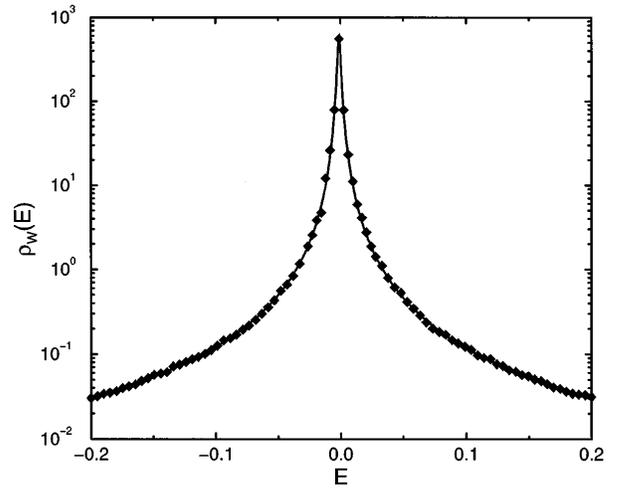


FIG. 2. Local spectral density $\rho_W(E)$ computed for the TIP eigenstates in the energy interval $[-0.1, 0.1]$ for the case $L = 150$, $U = 1$, $V = 1$, and $W = 0.4$. The full line gives the best Breit-Wigner fit (13) with $\Gamma = 0.0073$. The theoretical prediction is $\Gamma = 0.0072$.

(see inset in Fig. 4). At moderate $U/V > 0.3$ values in the presence of numerical fluctuations the dependence of Γ on U is hardly distinguishable from a linear one (see the normal scale in Fig. 4). In our opinion this is the reason why the linear behavior in U had been attributed in [4] also to the states away from the band center. As for the result of Ref. [7], the system size was too small ($L = 25$) and the main part of the data (Fig. 4 with $U/V < 0.4$) corresponds to the different regime $\Gamma\rho < 1$. In this perturbative case

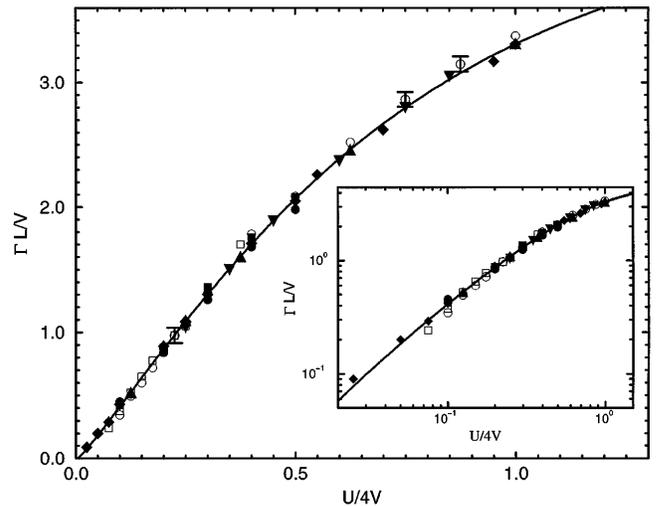


FIG. 3. Scaled Breit-Wigner width $\Gamma L/V$ as a function of the rescaled interaction $U/4V$ computed in the energy interval $E/V \in [-0.1, 0.1]$. The system size is $L = 15$ ($W/V = 1$, empty circles), $L = 25$ ($W/V = 1$, empty squares), $L = 40$ ($W/V = 0.6$, empty diamonds), $L = 60$ ($W/V = 0.5$, full circles), $L = 80$ ($W/V = 0.5$, full squares), $L = 100$ ($W/V = 0.5$, full diamonds), $L = 150$ ($W/V = 0.4$, full triangles up), $L = 200$ ($W/V = 0.35$, full triangles down). The solid line gives the theoretical prediction (8) and (9) multiplied by 2 to take symmetrization into account. Inset shows log-log scale.

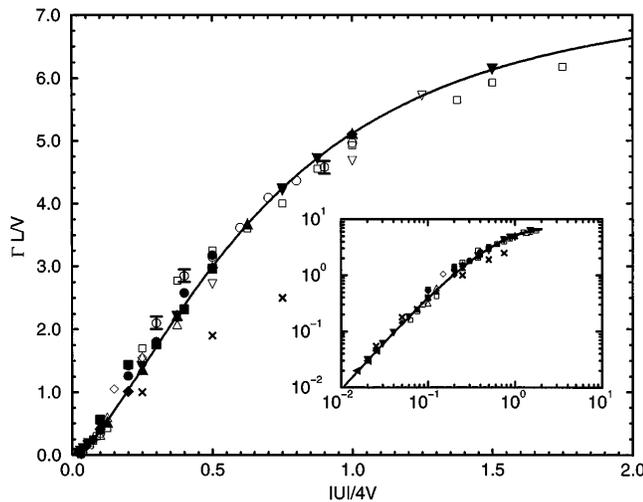


FIG. 4. Same as in Fig. 3 for energy interval $E/V \in [1, 1.2]$. The system size is $L = 15$ ($W/V = 1$, empty circles), $L = 25$ ($W/V = 1$, empty squares and $W/V = 0.5$, empty diamonds), $L = 40$ ($W/V = 0.8$, empty triangles up and $W/V = 0.5$, empty triangles down), $L = 60$ ($W/V = 0.5$, full circles), $L = 80$ ($W/V = 0.5$, full squares), $L = 100$ ($W/V = 0.5$, full diamonds), $L = 100$ ($W/V = 0.5$, negative U , crosses), $L = 150$ ($W/V = 0.4$, full triangles up), $L = 200$ ($W/V = 0.35$, full triangles down), and $L = 300$ ($W/V = 0.25$, full triangles left).

the typical energy scale which determines the change in level statistics is determined by Rabi oscillation frequency in a pair of quasidegenerate states, which is proportional to U [7]. Also, one should keep in mind that the results there are integrated over the whole energy band including the center of the band where the dependence on U is linear even for $\Gamma\rho > 1$.

Turning back to our numerical data (Fig. 4), we would like to mention that there is a significant difference from the theory for negative $U < -V$. Generally, we should expect such difference for $|U| \gg V$ when the spectrum is composed from two separated energy bands and the density of states is not described by (7), while for $|U| < V$ the width Γ is independent of sign U in agreement with the theory. We note that such asymmetry for attraction and repulsion away from the band center and relatively strong interaction $U \approx V$ has been seen in [4] for the ratio l_c/l_1 . Also a change in the behavior of Γ has been observed in [7] for $U > V$.

In summary, taking diagrammatically into account the effects of interaction, we have derived the analytical formula for the Breit-Wigner width Γ which determines the enhancement factor $l_c/l_1 \sim \Gamma\rho > 1$ for TIP in 1D random potential. The analytical calculations were done in the basis of free plane waves without disorder being in good agreement with the numerical results for exact problem. Our interpretation for such nonobvious agreement is the following. For $l_1 \gg 1$ the TIP collisions are local and are well described in the plane waves basis. On the other side, the frequency of collisions according to ergodicity is determined only by the volume of the system. Because

of that, the averaging over all momenta $p_{1,2}$ on the energy surface reproduces correctly collision frequency and finally Γ even if one uses the plane wave basis without disorder. In some sense such an approach assumes that the eigenstates are ergodic and then the frequency of collisions and Γ are fixed by the sum rule. One of the arguments in support of this interpretation in the 1D case is that inside the size $L \sim l_1$ the real eigenfunctions are quite close to the plane waves. The situation in 2D and 3D samples with size $L < l_1$ looks to be less obvious. However, there the standard estimates with ergodic eigenstates [1,2] give $\Gamma \sim U^2/(VL^d)$ and, since the result is again inversely proportional to the volume, one can expect that the correct averaging over momentum in the plane wave basis will also reproduce the correct ergodic result.

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