Emergence of Quantum Ergodicity in Rough Billiards

Klaus M. Frahm and Dima L. Shepelyansky*

Laboratoire de Physique Quantique, UMR C5626 du CNRS, Université Paul Sabatier, F-31062 Toulouse Cedex 4, France

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By analytical mapping of the eigenvalue problem in rough billiards onto a band random matrix model, a new regime of Wigner ergodicity is found. There, the eigenstates are extended over the whole energy surface but have a strongly peaked nonergodic structure. At the same time the level spacing statistics is still given by the Wigner-Dyson distribution. [S0031-9007(97)03972-0]

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In 1974, Shnirelman [1] proved a theorem according to which quantum eigenstates in chaotic billiards become ergodic for sufficiently high level numbers. Later it was demonstrated [2,3] that in this regime the level spacing statistics $p(s)$ is well described by random matrix theory [4]. However, one can ask the question as to how this quantum ergodicity emerges with an increasing level number N. This question becomes especially important in light of recent results [5,6] for diffusive billiards, where the time of classical ergodicity τ_D due to diffusion on the energy surface in the angular-momentum l space is much larger than the collision time with the boundary τ_b. In such a situation quantum localization on the energy surface may break classical ergodicity, eliminating the level repulsion in $p(s)$. The investigation of rough billiards [6] showed that this change of $p(s)$ happens when the localization length ℓ in l space becomes smaller than the size of the energy surface characterized by the maximal $l = l_{\max}$ at a given energy ($\ell < l_{\max}$). For $\ell > l_{\max}$ the eigenfunctions are extended over the whole surface, but as we will see they are not necessarily ergodic (see Fig. 1 with notations explained below).

The measure on the classical ergodic energy surface in the phase space $(\mathbf{p}, \mathbf{q})$ is given by $d\mu = \delta(E - E(\mathbf{p}, \mathbf{q}))d\mathbf{p}d\mathbf{q}$. The usual scenario of ergodicity breaking [7] was based on an image of transition from the quantum eigenstates ergodic on this surface (*Shnirelman ergodicity*) to the exponential localized states. Here we show that this transition between localized [Fig. 1(a)] and Shnirelman ergodic states [Fig. 1(c)] can pass through an intermediate phase of *Wigner ergodicity* [Fig. 1(b)]. In this Wigner phase the eigenstates are nonergodic and composed of rare strong peaks distributed on the *whole* energy surface. Our description and understanding of this case is based on the mapping of the billiard problem with weakly rough (random) boundary onto a superimposed band random matrix (SBRM). This model is characterized by strongly fluctuating diagonal elements corresponding to a preferential basis of the unperturbed problem. Recently, such a type of matrices was studied in the context of the problem of particle interaction in disordered systems [8–11]. There it was found that the eigenstates can be extended over the whole matrix size while having a very peaked structure. The origin of this behavior is due to the Breit-Wigner form [12] of the local density of states, according to which only unperturbed states in a small energy interval Γ_E contribute to the final eigenstate.

Recent optical experiments with micrometer size droplets initiated new theoretical investigations of weakly deformed circular billiards [13]. In this case, the ray dynamics becomes chaotic, leading to a strong directionality of light emission [14]. Here we will consider another type of weakly deformed circles [6], namely, we chose a random elastic boundary deformation which can be represented by $R(\theta) = R_0 + \Delta R(\theta)$ with $\Delta R(\theta)/R_0 = \mathrm{Re}\ \sum_{m=2}^{M} \gamma_m e^{im\theta}$, where γ_m are random

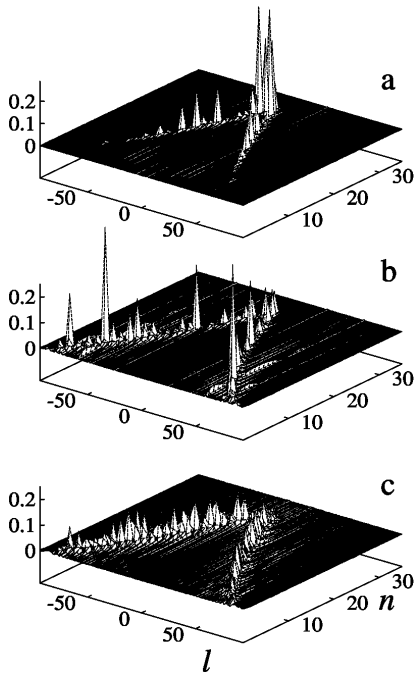

FIG. 1. Transition from localization to Shnirelman ergodicity on energy surface for level number $N \approx 2250$, $l_{\max} \approx 95$, and $M = 20$; shown are the absolute amplitudes $|C_{nl}^{(\alpha)}|$ of one eigenstate: (a) localization for $D(l_r = 0) = 20$, (b) Wigner ergodicity for $D = 80$, and (c) Shnirelman ergodicity for $D = 1000$ (see text).

complex coefficients and M is large but finite. This type of deformation seems to be very generic and may appear in numerous different physical situations [6]. In small droplets such boundary perturbations may be created by temperature induced surface waves. We will restrict ourself to the case of weak surface roughness given by $\kappa(\theta) = (dR/d\theta)/R_0 \ll 1$ and all γ_m being of the same order of magnitude. Then we have for the angle average $\tilde{\kappa}^2 = \langle \kappa^2(\theta) \rangle_\theta \sim M^2(\Delta R/R_0)^2$.

In such a billiard the dynamics is diffusive in orbital momentum due to collisions with the rough boundary provided $\tilde{\kappa}$ is above the chaos border $\kappa_c \sim M^{-5/2}$ [6]. For $l \simeq l_r$ the diffusion constant is determined by the average change of orbital momentum per collision being $D = \langle (\Delta l)^2 \rangle = 4(l_{\max}^2 - l_r^2)\tilde{\kappa}^2$. This D is the local diffusion rate for l close to the resonant l_r [6]. The quantum interference leads to localization of this diffusion with the length $\ell = D$ for $M < \ell < l_{\max}$, while for $\ell > l_{\max}$ the eigenstates are extended over the energy surface [6]. The transition between these two regimes is illustrated in Fig. 1. Here we present the absolute values of eigenfunction amplitudes $C_{nl}^{(\alpha)}$ in the eigenbasis $|nl\rangle$ of a circular billiard as a function of unperturbed radial and orbital quantum numbers n, l, with α marking the eigenenergy E_α. For small roughness $\tilde{\kappa}$ (or D) the states are exponentially localized [Fig. 1(a)] while for large $\tilde{\kappa}$ they are homogeneously distributed [Fig. 1(c)] on the energy surface. The case of Fig. 1(b) corresponds to an unusual regime of Wigner ergodicity, where the eigenstate is extended over the surface but is composed of rare strong peaks. The positions of these peaks on the energy surface of the circular billiard $E = \mathcal{H}(n, l)$ are shown in Fig. 2(a). The equation of the surface, projected on the action plane (n, l), can be found from the Bohr-Sommerfeld quantization $\mu_l(E) = \sqrt{l_{\max}^2 - l^2} - l \arctan(l^{-1}\sqrt{l_{\max}^2 - l^2}) + \pi/4 = \pi(n + 1)$, where $l_{\max}^2 = 4N = 2mR_0^2E/\hbar^2 = k^2R_0^2$ with k being the wave number. A part of the surface is shown in more detail in Fig. 2(b). It is clearly seen that the peaks are large for those integer n, l which are close to the line $\mathcal{H}(n, l) = E_\alpha$. Our understanding of the fact that not all integer values of the (n, l) lattice near this line are populated is based on the concept of the Breit-Wigner structure of eigenstates described below.

According to Refs. [6,15], the internal scattering at the rough boundary can be described by the S matrix,

$$S_{l\tilde{l}}(E) = e^{i\mu_l(E)} \langle l | e^{iV(\theta)} | \tilde{l} \rangle e^{i\mu_{\tilde{l}}(E)}, \qquad (1)$$

with $\mu_l(E)$ being the scattering phases of the circle and $V(\theta) = 2\sqrt{l_{\max}^2 - l_r^2}\,\Delta R(\theta)/R_0$. This quantum rough map [6] is defined with respect to amplitudes a_l in the wave function expansion $\psi(r, \theta) = B \sum_l a_l J_{|l|}(kr) e^{il\theta}$ with Bessel functions J_l and B being a normalization constant. The S matrix gives a local unitary description for l close to a resonant value l_r. The eigenvalue equation reads $\sum_{\tilde{l}} S_{l,\tilde{l}}(E_\alpha) a_{\tilde{l}}^{(\alpha)} = a_l^{(\alpha)}$ so that the eigenvalues E_α are determined by $\det[1 - S(E_\alpha)] = 0$. For $V = 0$, we recover the Bohr-Sommerfeld quantization for eigenvalues E_{nl} of the ideal circle.

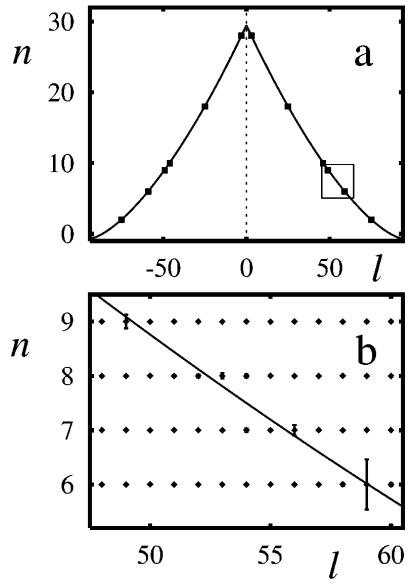

FIG. 2. (a) Main peaks of eigenstate in Fig. 1(b) (squares for $|C_{nl}^{(\alpha)}| \geq 0.1$) shown on the energy surface $\mathcal{H}(n, l) = E_\alpha$ (see text). (b) Rescaled part of (a): Diamonds show the integer (n, l) lattice; the error bar size is $2|C_{nl}^{(\alpha)}|$.

The semiclassical regime of ray dynamics corresponds to the limit $V \gg 1$, where the θ integral can be evaluated in a saddle point approximation giving the classical limit of the quantum rough map [6]. Here we are interested in a different regime, where $V < 1$ corresponds to $D < M^2$. There, by the mapping on an effective solid state Hamiltonian H_{eff} introduced by Fishman, Grempel, and Prange [16], the equation for eigenstates takes the form

$$\tan[\mu_l(E_\alpha)] a_\ell^\alpha) + \frac{1}{2} \sum_{\tilde{l}} \langle l | V | \tilde{l} \rangle a_{\tilde{l}}^{(\alpha)} = 0. \qquad (2)$$

In this way the eigenvalue equation is reduced to a solid state problem with $2M$ coupled sites. The H_{eff} matrix is of the SBRM type with strongly fluctuating diagonal elements produced by scattering phases μ_l. The investigations of such matrices [9–11] showed that the local density of states has the Breit-Wigner width given by the Fermi golden rule $\Gamma_\mu = 2\pi\rho_\mu \langle [V(\theta)/2]^2 \rangle \approx 3D/2M^2$, where $\rho_\mu = 1/\pi$ is the density of diagonal elements and we used the relation between the phase average of $V^2(\theta)$ and D. This expression is valid [17] when Γ_μ exceeds the mean level spacing $(\sim 1/M)$ in the bandwidth M. In the opposite limit $\Gamma_\mu M < 1$, the eigenstates are given by standard perturbation theory. Together with the condition $V < 1$, we find that the Breit-Wigner regime exists for $M < D < M^2$ near the zero energy of H_{eff}. In this regime the localization length is $\ell = D$ [6,8–11,15]. However, the Breit-Wigner structure remains in both localized ($\ell < l_{\max}$) and delocalized ($\ell > l_{\max}$) cases if $M < D < M^2$.

Therefore for $l_{\max} < D < M^2$ the states are extended, but only l with $|\tan(\mu_l)| < \Gamma_\mu < 1$ are mixed, leading to a peaked structure of eigenstates [18]. The fraction of peaks in $\max(\ell, l_{\max})$ is Γ_μ.

The above properties of scattering amplitudes $a_l^{(\alpha)}$ allow one to understand the behavior of eigenfunction coefficients $C_{nl}^{(\alpha)} = \langle \psi_\alpha | nl \rangle$. For this, one has to compute the expansion $J_l(k_\alpha r)e^{il\theta}$ in terms of $|nl\rangle$. Since $\Delta R \ll R_0$, the angular and radial integrals factorize and can be evaluated using the radial eigenvalue equation and the semiclassical expression for $J_l(kr)$. As a result, we obtain

$$C_{nl}^{(\alpha)} \approx \tilde{B} a_l^{(\alpha)} (l_{\max}^2 - l^2)^{1/4} \frac{\sin \Delta\mu}{\Delta\mu}, \qquad (3)$$

with $\Delta\mu = \mu_l(E_\alpha) - \mu_l(E_{nl}) \approx (E_\alpha - E_{nl})/E_b$ and $E_b = dE/d\mu_l(E) = \hbar^2 l_{\max}^2/(mR_0^2\sqrt{l_{\max}^2 - l^2}) = 2\hbar/\tau_b$ being the energy scale related to the ballistic collision time τ_b; $\tilde{B}$ is a normalization constant. The amplitudes $C_{nl}^{(\alpha)}$ determine the local density of states by

$$\rho_W(E - E_{nl}) = \left\langle \sum_\alpha \delta(E - E_\alpha) |C_{nl}^{(\alpha)}|^2 \right\rangle. \qquad (4)$$

The averaging is performed with respect to different roughness realizations and/or over a sufficiently large energy interval. Because of the Breit-Wigner distribution for $\tan(\mu_l)$ in (2), we obtain

$$\rho_{BW}(E - E_{nl}) = \frac{1}{\pi} \frac{\Gamma_E/2}{(E - E_{nl})^2 + \Gamma_E^2/4}, \qquad (5)$$

with

$$\Gamma_E = E_b \Gamma_\mu = E_b \frac{N}{N_W}\left(1 - \frac{l^2}{l_{\max}^2}\right), \qquad N_W = \frac{M^2}{24\tilde{\kappa}^2}. \qquad (6)$$

Equations (5) and (6) are valid for $\Delta E < E_B$ ($\Delta\mu < 1$) and $\Gamma_\mu < 1$ or $N < N_W$. Here N_W is the border of the Breit-Wigner regime in level number N. Remember that the eigenstates are localized for $N < N_e = 1/64\tilde{\kappa}^4$ corresponding to $\ell < l_{\max}$ [6]. As a result, the Breit-Wigner structure can exist in both the localized and delocalized cases. An example of Breit-Wigner distribution is shown in Fig. 3. Our numerical data confirm the theoretical expression (6) for variation of Γ_μ by more than 1 order of magnitude (inset).

For $N > N_W$ the kick amplitude V in (1) is larger than one, and the mapping onto Eq. (2) is not valid. In this case, the scattering phases (eigenphases of S) are homogeneously distributed in the interval $(0, 2\pi)$. If, in addition, $N > N_e$ then, as in the case of the kicked rotator (see Chirikov in [3]), the amplitudes a_l are homogeneous in l space with $|a_l|^2 \approx 1/2l_{\max}$. Using Eq. (3), we obtain the local density of states by

$$\rho_W(E - E_{nl}) = \frac{E_b}{\pi} \frac{\sin^2[(E - E_{nl})/E_b]}{(E - E_{nl})^2}. \qquad (7)$$

This density is normalized to one, and as a result the probability $|C_{nl}^{(\alpha)}|^2$ is ergodically distributed along the energy surface shown in Fig. 2(a). This is the regime of Shnirelman ergodicity which emerges for $N > \max(N_W, N_e)$. For fixed roughness $\tilde{\kappa} > \kappa_{EW} = \sqrt{6}/4M$ we have $N_W > N_e$, and the transition to Shnirelman ergodicity with the increasing level number N crosses the region of Wigner ergodicity for which an eigenfunction is ergodic only inside the Breit-Wigner width $\Gamma_E < E_b$. In this regime our numerical data show that $p(s)$ still has the Wigner-Dyson form even though the eigenstates are nonergodic on the energy surface. In the case $\kappa_c < \tilde{\kappa} < \kappa_{EW}$ the Shnirelman ergodicity emerges directly from the localized phase (the Breit-Wigner regime exists only in the localized phase). The averaging of Eq. (7) over different l values gives the distribution (5) with $\Gamma_E \sim E_b$. This explains why, also for the case $\Gamma_\mu < 1$ in Fig. 3, the distribution (5) remains valid even for $\Delta E > E_b$ (note that $\pi E_b/\Gamma_E \approx 10$).

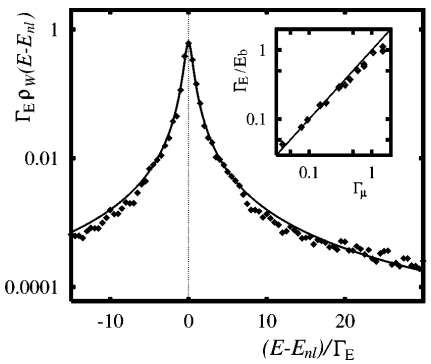

FIG. 3. Breit-Wigner distribution for eigenstates of a rough billiard (diamonds) with the parameters of Fig. 1(b) (five eigenstates for each of ten roughness realizations are used). The solid curve gives the distribution (5) with the theoretical Γ_E value (6), $\Gamma_E/E_b = \Gamma_\mu = 0.3$. The inset shows the variation of Γ_E/E_b (diamonds) as a function of Γ_μ for a parameter range $10 \le M \le 40$ and $M \le D \le M^2$ (E_b and Γ_μ are taken at $l = 0$). The theory (6) is shown by the straight line.

The above analysis shows that in the regime of Wigner ergodicity there are four relevant energy scales: level spacing Δ, Thouless energy for diffusion in l space $E_c = \hbar D/l_{\max}^2 \tau_b$, the Breit-Wigner width $\Gamma_E = 3\hbar D/M^2\tau_b$, and bouncing energy $E_b = 2\hbar/\tau_b$ which are ordered as $\Delta < E_c < \Gamma_E < E_b$. These scales should appear in the level statistics, namely, for the number variance $\Sigma_2(E)$ [3,4]. For $E < E_c$ we find the Gaussian orthogonal ensemble (GOE) and Wigner-Dyson statistics to be valid (Fig. 4), while in the interval $E_c < E < \Gamma_E/2$ the behavior is modified, due to the diffusive dynamics [19], being $\Sigma_2(E) \sim (E/E_c)^{1/2}$. The first investigations of the regime with $\Gamma_E/2 < E < E_b/2$ for SBRM were only recently done [20]. They showed that level rigidity is strongly suppressed with a nearly linear energy behavior in $\Sigma_2(E)$ due to the disappearance of correlations between levels with energy differences larger than Γ_E. However, such local characteristics as $p(s)$ are still described by

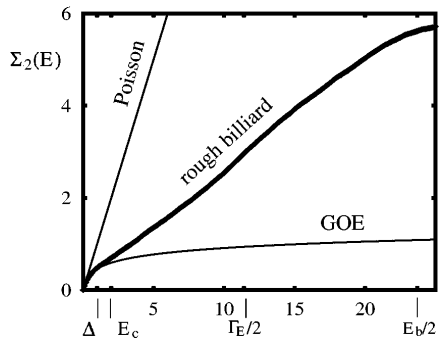

FIG. 4. Dependence of number variance $\Sigma_2(E)$ on energy for a rough billiard compared to Poisson and GOE for $M = 50$, $D = 800$, $l_{\max} \approx 95$. Six roughness realizations in the interval $2150 < N < 2350$ are used for the average. The energy scales are also shown in units of level spacing Δ.

GOE if $E_c \gg \Delta$. Our numerical data qualitatively confirm this picture (see Fig. 4), but quantitative numerical and analytical verifications are still required. In Fig. 4 the above energy scales are not separated by strong inequalities, but parametrically it is possible to have them. In this unusual regime it would be interesting to study other physical properties. We mention, for example, the frequency dependence of dielectrical response [21] which should be sensitive to the above energy scales.

In conclusion, we studied the parameter dependence of the quantum energy surface width in rough billiards. In the limiting case of Shnirelman ergodicity with high level numbers, this width is determined by the typical frequency of collisions with the boundary ($\Gamma_E \sim E_b$). This means that *all* integer points on the (n, l) lattice of quantum numbers with a distance $\Delta l = \Delta n \approx 1$ from the energy line $E_\alpha = \mathcal{H}(n, l)$ are occupied by *one* eigenfunction ψ_α ($N > N_W$ and $N > N_e$). We have found a new regime of Wigner ergodicity, where $\Gamma_E \ll E_b$ so that only points with $\Delta l = \Delta n \leq \Gamma_\mu \ll 1$ contribute to ψ_α. Therefore a lot of holes appear in the energy surface, and ψ_α has a strongly peaked structure on the (n, l) lattice. It would be interesting to study such nonergodic eigenstates with Wigner-Dyson statistics in the experiments with microwave billiards [22] and micrometer droplets [13,14].

*Also at Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia.

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