Quantum Poincaré Recurrences

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We show that quantum effects modify the decay rate of Poincaré recurrences $P(t)$ in classical chaotic systems with hierarchical structure of phase space. The exponent p of the algebraic decay $P(t) \propto 1/t^p$ is shown to have the universal value $p = 1$ due to tunneling and localization effects. Experimental evidence of such decay should be observable in mesoscopic systems and cold atoms. [S0031-9007(98)08273-8]

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The general structure of classical phase space in chaotic Hamiltonian systems displays a hierarchical mixture of integrable and chaotic components down to smaller and smaller scales [1]. This complicated structure leads, in particular, to an anomalous power law decay of Poincaré recurrences $P(t)$ and correlations $C(t)$ inside the chaotic components [2,3]. As it follows from the Poincaré theorem a trajectory always returns to a region around its origin, but the statistical distribution of recurrences depends on the dynamics. For a strongly chaotic motion, e.g., in the Arnold cat map [1], the probability $P(t)$ to return or survive in a given region after time t decays exponentially with t (as for coin flipping). However, in a more general case of systems with hierarchical structure of phase space the decay of $P(t)$ is algebraic (see a more detailed discussion in the related Letter [4]). Physically, such slow decay appears due to a decrease, down to zero, of the diffusion rate for a trajectory when it approaches the chaos border determined by some critical invariant curve [2,3,5–7]. Typically, $P(t) \sim C(t)/t \sim t^{-p}$ with $p \approx 1.5$ [2]. As a result the integrated correlation function, which determines the diffusion rate ($D \sim \int C\,dt$), can diverge thus leading to a superdiffusive propagation [8]. Such effects are important for electron dynamics in superlattices where usually the phase space has a mixed structure [9].

The above anomalous properties had been studied in great detail for classical systems [2,3,5–9]. However, the question how they are affected by quantum dynamics was not addressed up to now. This problem becomes more and more important not only due to its fundamental nature but also in the light of recent experiments with mesoscopic systems. Indeed, different types of ballistic quantum dots can now be studied in laboratory experiments [10] and the phase space in such systems generally has a mixed structure. Since the probability to stay in a given region is directly related with $P(t)$ and $C(t)$, its slow decay can significantly affect conductance properties. In particular it has been proposed that such decay should lead to fractal conductance fluctuations [11], the experimental observation of which has been reported recently [12]. According to [11,12] the fractal exponent σ for conductance fluctuations is directly related to the exponent p as $\sigma = 2 - p/2$.

A different type of system in which such effects should be observable experimentally is given by cold atoms in external laser fields where the Kicked Rotator model of quantum chaos has been built experimentally [13,14]. Possibilities of experimental investigation of slow probability decay in such systems has been discussed recently [15].

The experimental studies of slow power correlation decay in the regime of quantum chaos are also important from the fundamental point of view, since here the typical scale of correlation decay is much larger than the Ehrenfest time scale $t_E \sim \ln 1/\hbar$ on which the minimal coherent wave packet spreads over the available phase space. To the best of our knowledge the comparison of classical and quantum correlations in such a regime has not been investigated so far. Only recently such a comparison has been made in the regime of hard chaos with exponential correlation decay [16]. In this paper we address directly the comparison between classical and quantum Poincaré recurrences (QPR), related to the correlations decay, in the regime with mixed phase space. Our results demonstrate a new universal law for QPR related to localization and tunneling effects.

To investigate the QPR we use the model of kicked rotator with absorbing boundary conditions studied in [16,17]. The evolution operator over the period T of the perturbation is given by

$$\bar{\psi} = \hat{U}\psi = \hat{\mathcal{P}} e^{-iT\hat{n}^2/4} e^{-ik\cos\hat{\theta}} e^{-iT\hat{n}^2/4}\psi\,, \qquad (1)$$

where $\hat{\mathcal{P}}$ is a projection operator over quantum states n in the interval $(-N/2, N/2)$. Here, we put $\hbar = 1$ so that the commutator is $[\hat{n}, \hat{\theta}] = -i$ and the classical limit corresponds to $k \to \infty$, $T \to 0$, while the classical chaos parameter $K = kT$ remains constant. In the classical limit

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the dynamics is described by the Chirikov standard map:

$$\bar{n} = n + k \sin\left(\theta + \frac{Tn}{2}\right); \qquad \bar{\theta} = \theta + \frac{T}{2}(n + \bar{n}), \tag{2}$$

in which orbits are absorbed outside the interval $-N/2 < n < N/2$. In order to study the classical and quantum survival probability $P(t)$ we fixed the ratio $N/k = 4$ and take the classical chaos parameter $K = 2.5$, so that the classical phase space has a hierarchical structure of integrable islands and chaotic components.

A typical example of classical and quantum survival probability decay is shown in Fig. 1. The classical probability $P(t)$ decays with a power law with exponent $p \approx 2$ in a range of 6 orders of magnitude. In this case the exponent is slightly different from the typical value 1.5. Indeed as discussed in [2,3,5,6] the exponent can vary from system to system (and even oscillates with $\ln t$) depending on the local structure of phase space in the vicinity of the critical boundary invariant curve whose rotation number can play an important role.

The quantum survival probability $P_q(t)$ is plotted for different values of N (effective $\hbar$ is proportional to $1/N$) in Fig. 1. In agreement with the correspondence principle, the quantum probability follows the classical value during a rather long time scale which grows with N. For longer times the quantum probability $P_q(t)$ is found to approximately follow the quantum decay law

$$P_q(t) \approx C/t, \tag{3}$$

where C is some constant. This latter decay continues during a rather long interval of time (4 orders of magnitude in time for the case $N = 3^6$). Of course (3) represents an intermediate asymptotic behavior since for $t \to \infty$ the decay will be exponential $P_q(t) \sim \exp(-\Gamma_{\min} t)$, where $\Gamma_{\min}$ is determined by the minimal imaginary part of the eigenvalue λ of the evolution operator $\hat{U}$ ($\hat{U}\psi_\lambda = e^{-i\lambda}\psi_\lambda$).

In order to better understand the origin of quantum behavior (3) we studied the probability decay in a simpler case where the classical dynamics is completely chaotic ($K = 7$) or quasiintegrable ($K = 0.5$). The quantum and classical probability decays are shown in Fig. 2 for $k = 5$ and absorption for $n \le 0$ and $n > N = 500$. Initially the probability is concentrated at $n = 0$. For $K = 7$ the classical probability decays, asymptotically, exponentially with time $P(t) = 0.11 \exp(-\gamma t)$. The value of γ can be found from the solution of the Fokker-Planck equation with absorbing boundary conditions which gives $\gamma \approx D\pi^2/2n^2$, where $D = \beta k^2/2$ is the classical diffusion rate [see Eq. (4) in [17]]. The value of β depends on the classical chaos parameter and for $K = 7$ is $\beta \approx 2.8$ [17]. Therefore the expected theoretical value is $\gamma = 6.9 \times 10^{-4}$ which is close to the numerical value $\gamma = 6.4 \times 10^{-4}$. The asymptotic exponential decay starts after the diffusive time $t_D \approx 1/\gamma$. The quantum probability in this case (Fig. 2) follows the classical one during some interval of time after which it decays according to Eq. (3).

The quantum decay (3) can be understood in the following way. The quantum eigenstates are localized with localization length $\ell \approx D/2$ [18]. Therefore the absorption time for a state at a distance n from the absorbing boundary is $t \sim \exp(2n/\ell)$. Since n is proportional to the total measure $\mu \sim n \propto \ell \ln t$, it follows that the survival probability is $P_q(t) = d\mu/dt \sim \ell/t$ in agreement with (3). The same estimate can be obtained by

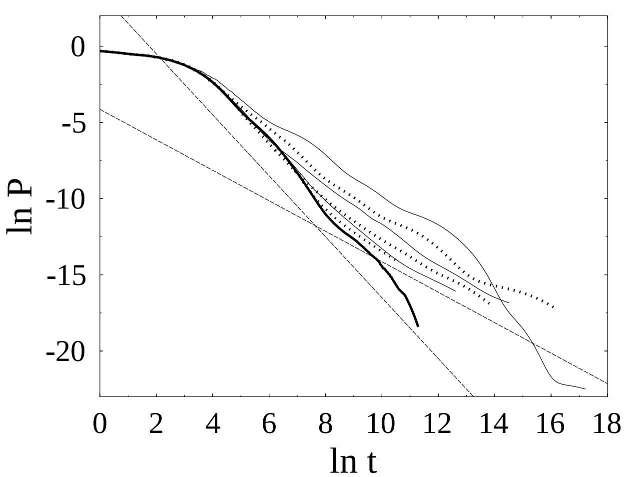

FIG. 1. Classical (thick continuous line) and quantum (thin continuous and dotted lines) probability decays for $K = 2.5$ and $N/k = 4$. The two dashed straight lines show slope 2 and 1. The quantum curves correspond to $N = 3^p$ with p increasing from 5 (upper continuous line) to 10 (lower dotted line). The starting conditions are two symmetric lines at $n = \pm N/3$, both for the classical and quantum evolutions.

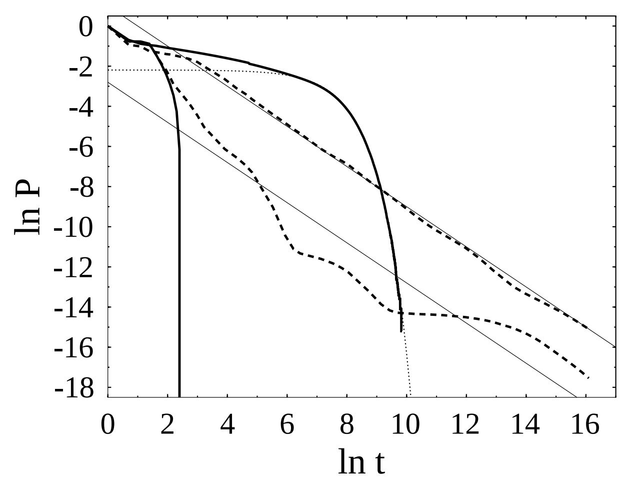

FIG. 2. Classical (continuous lines) and quantum (dashed lines) probability decays for $K = 0.5$ (grey lines) and $K = 7$ (black lines); parameter $k = 5$. The evolution starts at $n = 0$ and the probability is absorbed for $n \le 0$ and $n > 500$. The two thin straight lines have slope one, while the dotted grey curve shows the fit for the classical decay at $K = 7$: $P = 0.11 \exp(-\gamma t)$, $\gamma = 6.4 \times 10^{-4}$.

expanding an initial state over eigenstates of the evolution operator (1) with probabilities $|c_n|^2 \sim e^{-2n/\ell}/\ell$ and ionization rates $\Gamma_n \sim |c_n|^2 \sim e^{-2n/\ell}/D$. Then the survival probability $P_q(t) \sim \int_0^\infty |c_n|^2 e^{-\Gamma_n t}\, dn \sim D/t$. Our data at $K = 7$ for different values of k ($k = 5, 6, \ldots, 10$) give that $C = aD^b$, with $a = 0.27(5)$ and $b = 0.92(13)$.

The above derivation of the decay law (3) refers to the regime of quantum localization of chaos. For the integrable case one still has the rates $\Gamma_n \sim |c_n|^2 \sim e^{-n/\ell_{\mathrm{eff}}}$, where ℓ_{eff} is an effective length determined by tunneling in the classically forbidden region. However here the fluctuations in time are stronger since ℓ_{eff} depends on the local structure of invariant curves and islands in the integrable domain. Nevertheless the global decay at $K = 0.5$ is in agreement with the $1/t$ law (see Fig. 2). In both cases ($K = 7; 0.5$), the quantum decay proceeds in a much slower way than the corresponding classical one. Because of the above reasons even in the case of mixed phase space the decay follows the $1/t$ law in accordance with numerical data of Fig. 1. The $1/t$ behavior can continue up to a time $t_{\max} \propto \exp(N/\ell_{\mathrm{eff}}) \sim 1/\Gamma_{\min}$ which is determined by the minimal decay rate in the system. For $t > t_{\max}$ the quantum survival probability $P_q(t)$ decays exponentially with time $P_q(t) \propto \exp(-t/t_{\max})$.

The ionization rate Γ_λ of an eigenstate ψ_λ localized at a distance n from the absorbing boundary is proportional to $\Gamma_\lambda \propto \exp(-n/\ell_{\mathrm{eff}})$; therefore the number of such states is proportional to the measure $\mu \sim n \sim \ln 1/\Gamma$. As a result the probability to find a value Γ in the interval $d\Gamma$ is $dW/d\Gamma \sim d\mu/d\Gamma \sim 1/\Gamma$. Our numerical data for $dW/d\Gamma$ obtained in the localized regime confirm this $1/\Gamma$ dependence (see [19]).

An interesting question is at what time scale the QPR start to deviate from their classical behavior. To test these deviations we determined them in two different ways. The first one is defined as the time t_q at which the ratio of classical over quantum probability is $P(t_q)/P_q(t_q) = 0.9$. This condition gives the first quantum deviation and was studied in [16] where it was found $t_q \propto \sqrt{N} \propto \sqrt{1/\hbar}$. This time scale can be explained on the basis of analysis of eigenvalue fluctuations of the evolution operator $\hat{U}$ in the complex plane or as the result of weak localization corrections [16]. Our numerical data shown in Fig. 3 confirm this dependence even if the situation is qualitatively different from that in [16], where the integrable component was absent, while here we have power law decay in a mixed phase space (see Fig. 1).

However, at time t_q quantum probability only starts to deviate from the classical one while the behavior (3) can set in after a longer time t_H. We determine this time in two ways: (i) $t_H \approx \ln P(t_q)/\ln P_q(t_q) = 0.9$ (circles and stars in Fig. 3) and (ii) as the crossing point between the two extrapolations for the classical behavior $P(t) \propto 1/t^2$ and the quantum one $P_q(t) \propto 1/t$ (crosses in Fig. 3). The data for stars in Fig. 3 were obtained by averaging over 50 values of N near a given value. We used two methods

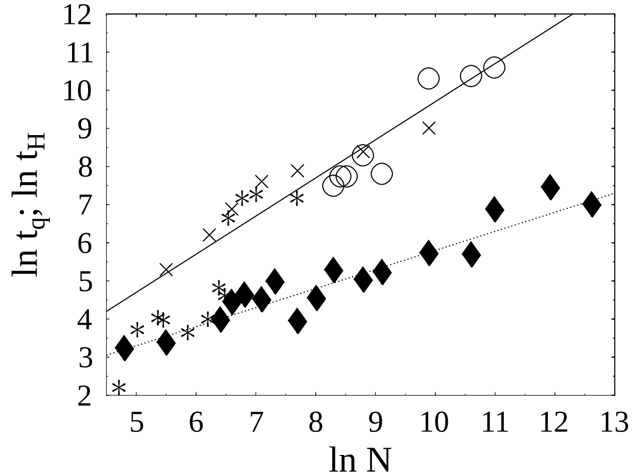

FIG. 3. Dependence of quantum time scales t_q (full diamonds) and t_H (circles, crosses, stars) on the system size N (see the text). The dotted and continuous straight lines show the theoretical slopes 1/2 and 1, for t_q and t_H, respectively.

to determine the time scale t_H because the first definition gives large oscillations related to the oscillations of the classical $P(t)$ in $\ln t$ while the second definition allows one to smooth out the effect of the oscillations. The numerical data are in agreement with the dependence $t_H \sim N \propto 1/\hbar$. This time can be interpreted as the Heisenberg time scale, which is determined by inverse level spacings, and after which the quantum behavior becomes qualitatively different from the classical one.

The above scales t_q and t_H can be also seen in the quantum evolution of the Husimi function obtained from the Wigner function by smoothing over the size of the $\hbar$ cell. This evolution is presented in Fig. 4. For short times $t < t_q = 150$ the classical and the quantum probability distributions in the phase space (n, θ) look rather similar. The quantum Husimi function reflects the underline fractal structure of the classical distribution. For larger times the classical recurrencies are determined by more and more fine scales in the phase space and the classical distribution becomes localized around small islands near critical invariant curves. On such long times $t > t_H = 3000$ the quantum evolution in phase space in proximity of critical invariant curves becomes influenced by quantum interference effects that results in a qualitatively different probability distribution (Fig. 4). The penetration of quantum probability in phase space on smaller and smaller scales proceeds via a very slow tunneling process and as a result the Husimi function remains almost unchanged when the time is increased almost by 2 orders of magnitude (Fig. 4).

At very long times $t > t_{\max}$ the quantum decay becomes exponential; this corresponds to eigenstates which are localized in the center of the island (Fig. 4). We have been able to observe such a localized asymptotic Husimi function for $N = 3^4$ (Fig. 4). However, already

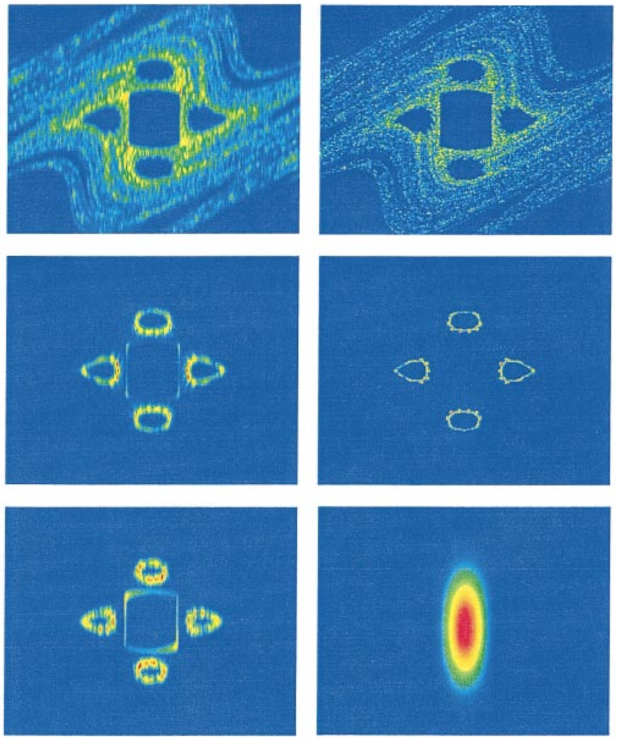

FIG. 4(color). Left column: Husimi function at time 10^2 (top), 5×10^3 (middle), and 3×10^5 (bottom), for the case in Fig. 1 with $N = 3^8$. Right column: classical density plot of orbits surviving up to time 10^2 (top) and 5×10^3 (middle). The right bottom shows the Husimi function for $N = 3^4, t = 10^7$. Distributions are averaged in a small time interval $\delta t = 20$ near the given t values. The size of the phase region (θ, n) is $0 \leq \theta \leq 2\pi$, $-N/2 \leq n \leq N/2$. The color is proportional to the density: blue for zero and bright red for maximal density (the color scale is the same for classical and quantum cases).

for $N = 3^5$ the time $t_{\max}$ is too large to reach it in our numerical simulations where we followed the evolution up to 3×10^7. Apparently this time is already comparable with $t_{\max}$ and this explains the strong oscillations of $P_q(t)$ for $t > 10^6$ (Fig. 1). At this time the quantum probability penetrates inside four small islands while at $t \approx 1.5 \cdot 10^7$ the probability becomes concentrated inside the main central island. We stress that $t_{\max}$ grows exponentially with N ($N \sim 1/\hbar$). As the result, the quantum decay $1/t$ in the semiclassical region proceeds up to enormously long times. In this regime the correlation functions practically do not decay because $C(t) \propto tP(t) \approx$ const.

In conclusion, we established a new universal law for probability and correlations decay in quantum systems both in the exponentially localized regime and in regimes in which classical dynamics has a mixed phase space. It should be possible to observe this universal behavior in the kicked rotator model which has been studied in experiments with cold atoms [13,14]. A similar effect should be also observable in the experiments with micrometer droplet lasing discussed recently in [20]. The quantum decay (3) of $P_q(t)$ has exponent $p = 1$ that should also effect the fractal dimension of conductance fluctuations in soft billiards studied in [11,12]. According to these results the fractal dimension of these fluctuations should be $\sigma = 2 - p/2 = 1.5$. This regime should start at large times $t > t_H$ which requires one to analyze conductance fluctuations with very fine resolution of magnetic field. The above value of σ has not been seen in experiments [11], which apparently indicates that the experimental resolution was not yet sufficient to detect quantum Poincaré recurrences at large times.

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