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ASYMPTOTIC STATISTICS OF POINCARÉ RECURRENCES IN HAMILTONIAN SYSTEMS WITH DIVIDED PHASE SPACE

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Asymptotic Statistics of Poincaré Recurrences in Hamiltonian Systems with Divided Phase Space

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Abstract

By different methods we show that for dynamical chaos in the standard map with critical golden curve the Poincaré recurrences $P(\tau)$ and correlations $C(\tau)$ asymptotically decay in time as $P \propto C/\tau \propto 1/\tau^3$. It is also explained why this asymptotic behavior starts only at very large times. We argue that the same exponent $p = 3$ should be also valid for a general chaos border.

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During last two decades the local structure of phase space in chaotic hamiltonian systems and area-preserving maps had been studied in great detail [1, 2, 3, 4, 5]. These researches allowed to understand the scaling properties in a vicinity of critical invariant curves where coexistence of chaos and integrability goes on to smaller and smaller scales in the phase space. The most studied case is the critical golden curve in the 2D maps with the rotation number $r_g = [111...] = (\sqrt{5}-1)/2$ for which the scaling exponents were found with high precision, and the phase space structure was shown to be self-similar and universal [2]. The most studied map with mixed integrable and chaotic components is the standard map [6] where the golden curve is critical at the chaos parameter $K = K_g = 0.97163540631...$ [2]. It was conjectured that for $K > K_g$ all invariant Kolmogorov-Arnold-Moser (KAM) curves are destroyed [2]. Later on, the upper bound for the critical perturbation K_l, corresponding to the destruction of the last invariant curve, was proved: $K_l - K_g < 0.01$ [7].

While the local structure of divided phase space is now well understood, the statistical properties of dynamics still remain unclear in spite of simplicity of these systems. Among the most important statistical characteristics are the correlation function decay in time $C(\tau)$ and the statistics of Poincaré recurrences $P(\tau)$. The later is especially convenient for numerical simulations due to its natural property $P(\tau) > 0$ and statistical stability. The first studies of $P(\tau)$ in a separatrix map showed that at a large time the recurrences decay as a power law $P(\tau) \propto 1/\tau^p$ with the exponent $p \approx 1.5$ [8]. Investigations of other maps also indicated approximately the same value of p [9, 10] even though it was remarked that p can vary from map to map, and that the decay of $P(\tau)$ can even oscillate with $\ln \tau$ [8, 9, 10, 11]. Such a slow decay of Poincaré recurrences was attributed to the sticking of trajectory near a critical KAM curve [8, 9, 10, 11, 12, 13]. Indeed, when approaching the critical curve with the border rotation number r_b , the local diffusion rate

D_n goes to zero as $D_n \sim |r_b - r_n|^{\alpha/2} \sim 1/q_n^\alpha$ with $\alpha = 5$ [12] where $r_n = p_n/q_n$ are the convergents for r_b determined by the continued fraction expansion. The theoretical value $\alpha = 5$ was derived from a resonant theory of critical invariant curve [12, 13] and was confirmed by numerical measurements of the local diffusion rate in the vicinity of the critical golden curve in the standard map [14]. Such a decrease of the diffusion rate near the chaos border would give the exponent $p = 3$ if to assume that everything is determined by the local properties of principal resonances p_n/q_n given by the convergents of r_b [12, 13, 15, 11]. However, the value $p = 3$ is rather different from the numerically found $p \approx 1.5$. Moreover, the special simulations for the standard and separatrix maps with the border rotation number $r_b = r_g$ have given a different behavior of $P(\tau)$ and different p [8, 11] in spite of the fact that the local structure of the golden critical curve is universal. Various attempts were undertaken to resolve this difficulty. In [16] the authors argued that a contribution from non–principal resonances can reduce the exponent down to $p = 2$. Other arguments based on disconnection of principal resonance scales were proposed in [11], while Murray discussed a possibility that larger times are required to see $p = 3$ decay [17]. During these years different Hamiltonian systems were studied where the values of $p \approx 1 - 2.5$ have been found [10, 18, 19, 20, 21].

The analysis of Poincaré recurrences is interesting not only by itself but also because they are directly related to the correlation function of dynamical variables [9, 10, 11, 12, 13, 14, 15, 16]:

$$C(\tau) \approx \mu(\tau) \approx \tau P(\tau)/ <\tau> \sim 1/\tau^{p-1} \tag{1}$$

Here $\mu(\tau)$ is the normalized probability for a trajectory to remain in a given region for time $t > \tau$, proportional to the measure μ, and $<\tau>$ is the average recurrence time. This relation can be understood as follows. By definition, $P(\tau) = N_\tau/N$ where N is the total number of recurrences and N_τ is the number of recurrences with time $t > \tau$. Therefore, for the total motion time $T = <\tau> N$ we have $P(\tau) = <\tau> N_\tau/T \approx <\tau> \mu(\tau)/\tau$ where, due to ergodicity of motion, the measure $\mu(\tau)$ (probability to stay) is proportional to the ratio of time the trajectory spends in the region ($T_\tau \approx \tau N_\tau$) to the total time T ($\mu(\tau) \approx T_\tau/T$). Inside the sticking region the dynamical variables are correlated so that $C(\tau) \approx \mu(\tau)$ [8, 10, 11, 12]. Since the correlations are directly related to a diffusion rate ($D_c \approx \int C d\tau$) the exponent $p < 2$ can lead to a superdiffusive dynamics [10, 11, 12, 13]. For the standard map such a behavior was indeed observed in [10, 11, 22]. All this shows that the asymptotic decay of Poincaré recurrences is a cornerstone statistical problem

To understand the asymptotic properties of $P(\tau)$ we used, for the first time, a new approach based on the direct computation of the *exit* times from a vicinity of the critical golden curve in the standard map

$$\bar{y} = y - K/(2\pi)\sin(2\pi x) \quad , \quad \bar{x} = x + \bar{y} \tag{2}$$

with parameter $K = K_g$. The properties of this curve had been studied in great detail [2]. In particular, the positions of unstable fixed points of resonances p_n/q_n are known to high precision [2].

To determine the exit time τ_n from the scale q_n we placed 100 trajectories in a very close vicinity of an unstable fixed point and computed the average exit time. For each trajectory the exit time is determined as the time after which the trajectory crosses the exit line. The exit line was fixed as $y = 1$ for the trajectories from the side of the main resonance $q = 1$ or as $y = 0.5 + a\sin(2\pi x)$ for the trajectories from the other side of the critical curve. In the later case the exit line was chosen in such a way to cross the two unstable points of resonance $q = 2$ ($a = 0.0773...$). This allowed us to take into account the deformation of the $q = 2$ resonance. The average exit time τ_n from a given scale q_n is related to the distance of this resonance from the critical curve and is proportional to this distance (measure) $\mu_n = |r_g - r_n| \approx 1/\sqrt{5}q_n^2$ squared divided by the local diffusion rate D_n : $\tau_n \sim \mu_n^2/D_n \sim q_n$. This gives $\mu \approx \mu_n/\mu_0 \sim C \sim 1/\tau^2$ and $p = 3$ where $\mu_0 \approx |r_g - r_0|$ is the normalizing constant (the total measure of the strip between the critical curve (r_g) and the exit line with the rotation number r_0).

The numerical data for dependence of μ_n on τ_n are shown in Fig.1. From both sides of the r_g curve the exit times converge to the asymptotic dependence

$$\mu_n = (\tau_g/\tau_n)^2/\sqrt{5}, \quad \tau_n = \tau_g\ q_n, \quad \tau_g = 2.11 \times 10^5 \tag{3}$$

This dependence corresponds to the scaling near the critical curve [2, 12]. Indeed, the local diffusion rate in y on the scale q_n is $D_n \approx AD_0/q_n^5$, where $D_0 = K^2/8\pi^2$ is the quasilinear diffusion rate [23] and $A \approx 0.0066$ is an empirical constant which is quite small due to a slow diffusion inside the separatrix layer [14]. As a result the sum of transition times between the two scales from r_n to r_{n-2} is approximately equal to the total exit time: $\tau_n \approx \sum_n |r_n - r_{n-2}|^2/D_n \approx 1.4 \times 10^5 q_n$. This estimate gives the value of τ_g close to (3) and allows us to understand the physical origin of its large numerical value. It is interesting to note that the data of Fig.1 show that the convergence of τ_n to its asymptotic value can be satisfactory described as $|\tau_n/q_n\tau_g - 1| \propto 1/q_n$. This indicates a certain similarity between the ratio $\tau_n/q_n\tau_g$ and the residue R_n for periodic trajectories which converges to its

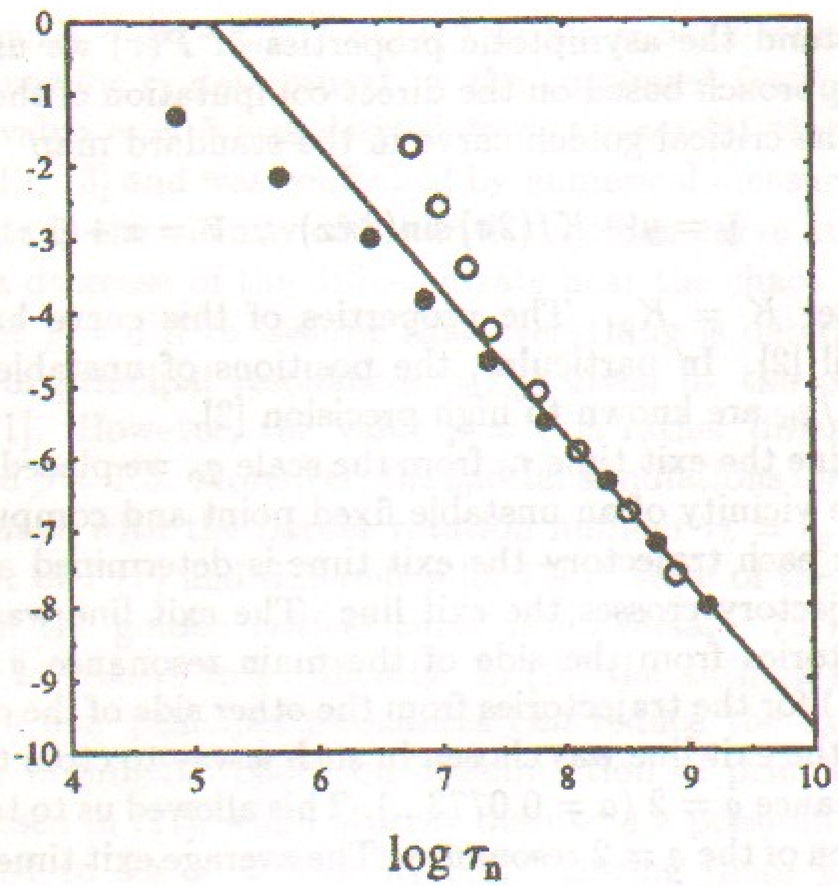

Figure 1: Exit times τ_n from scales r_n with $\mu_n = |r_g - r_n|$ for $q_n = 3, 8, ..., 6765$ (black circles) and $q_n = 5, 13, ..., 4181$ (open circles). The straight line shows asymptotic behavior (3). Error bars are less than the symbol size. Logarithms are decimal in Figs.1-3.

critical value in a similar way [2]. A physical reason of this similarity is the following: R_n is related to the trajectory stability, and the larger it is the more rapidly the trajectory escapes from the scale q_n. Due to that for odd $n(q_n = 1, 3, 8, ...)$ the time τ_n is smaller than the asymptotic expression (3) $(R_n > R_{cr} = 0.250...$ [2]) while for even n $(q_n = 2, 5, 13, ...)$ it is larger than (3) ($R_n < R_{cr}$ [2]). Due to universality of the critical golden curve structure it is natural to expect that the relation (3) and the time τ_g are universal for all area-preserving maps as well as R_{cr} (note that q_n is the period of unstable periodic trajectory on the scale n).

The convergence of the points in Fig.1 from both sides of the critical curve to the same asymptotic dependence (3) supports the above-mentioned conjecture $K_l = K_g$. Moreover, our results allow us to considerably improve the accuracy of this conjecture as compared to that in Ref.[7] mentioned above. Indeed, the scaling of the critical perturbation $K - K_{cr} \sim 1/q_n$ [2, 11, 17] implies that $K_l - K_g < 1/q_n < 2 \times 10^{-4}$ (see Fig.1).

The relation (3) determines the measure of chaotic region at which a trajectory is stuck for a time $\tau \sim \tau_n$. Then, according to (1), the exponent of Poincaré recurrences is $p = 3$, and correlation $C(\tau) \approx \mu$ decays as the inverse square of time. However, this asymptotic decay starts in fact only after a very long time $\tau > 10^6$ due to the large value of τ_g. This strong delay of asymptotic behavior is responsible for the nonuniversal decay observed for $P(\tau)$ in [10, 11] at $K = K_g$. Apparently, to reach the theoretical exponent $p = 3$ one should go to much longer times.

For a test of these theoretical expectations we made extensive numerical simulations of $P(\tau)$ at $K = K_g$ with the recurrences to the exits lines defined above on the both sides of the critical r_g curve. The results are presented in Fig.2 and show a clear change in the decay of $P(\tau)$ for $\tau > 10^5$ (side $q = 1$) and $\tau > 10^7$ (side $q = 2$). To check the relation (1) we computed $P(\tau)$ from the data of Fig.1 taking the recurrence time $\tau = 2\tau_n$ and $P(\tau) = B_q < \tau >_q \mu_n/\tau$ where $q = 1, 2$ mark the side of the critical curve. With the average recurrence time $< \tau >_1 \approx 24.5$ (or $< \tau >_2 \approx 61$) and the fitting constants $B_1 \approx 1$ (or $B_2 \approx 10$) the data in Fig.1 satisfactorily describe the variation of $P(\tau)$ in the interval of 6 (4) orders of magnitude. This gives an additional support for the asymptotic theoretical exponent $p = 3$. Notice, however, that the exponent of $P(\tau)$ decay at side $q = 1$ still does not reach the asymptotic value $p = 3$. Also, the expected values of $B_q = 1/\mu_{0q}$ ($B_1 \approx 2.6$, $B_2 \approx 8.5$, see Eq.(1)) are somewhat different from the above fitting parameters which characterizes the accuracy of Eq.(1).

Another check of the relation (1) was done by computing the diffusion rate in phase $\bar{z} = z + (\bar{y} + y - 2z_q)/2$ with $z_q = 0$ $(q = 1)$ and $z_q = 1/2$ $(q = 2)$. Similar approach was used in [22]. The diffusion rate is $D_c = (\Delta z)^2/\Delta t$, and its dependence on time is determined by the decay of the correlation function of $y(t)$. According to (1) we have $D_c(\tau) = D_{cq} G_q \int \tau P(\tau)/ < \tau > d\tau = D_{cq} \tilde{G}_q \int C(\tau) d\tau$ where $D_{cq} = |r_g - r_q|^2/3 = 0.049$ (0.0046) is the quasilinear diffusion rate [23] for $q = 1$ (2) side, and $G_q, \tilde{G}_q$ are some constants. The correlation $C(\tau)$ was computed from the piecewise linear interpolation of data in Fig.1 assuming that it remains constant up to the first exit time τ_{n1} $(C(0) = C(\tau_{n1}) = \mu_{n1}/\mu_0)$. This follows from the data in Fig.2 which are well approximated by the relation:

$$P_q \approx \frac{A_q}{\tau} \approx \frac{\mu_{q1}}{\mu_{q0}} \cdot \frac{< \tau >_q}{\tau} \tag{4}$$

where the latter expression is found from Eq.(1). Again, there is a difference between numerical $A_1 \approx 2$ $(A_2 \approx 3.1)$ and analytical $A_1 \approx 3.2$ $(A_2 \approx 9.3)$ values of A_q due to approximation (1).

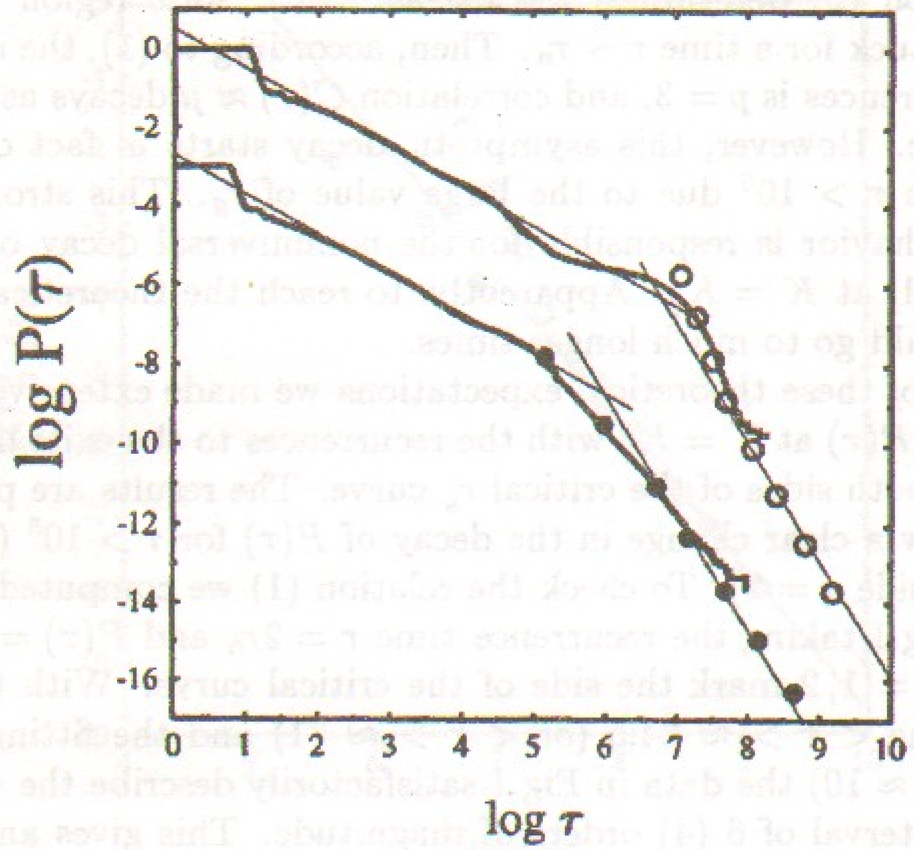

Figure 2: Poincaré recurrences in the standard map at $K = K_g$ from the side of resonance $q = 2$ (upper thick curve) and $q = 1$ (lower thick curve, shifted down by 3 for clarity). Open and full circles show the values of $P(\tau)$ recalculated from the data in Fig.1 (see text). The thin straight lines give the asymptotic decay with theoretical exponent $p = 3$; the initial decay with slope $p = 1$ (Eq.(4)) is shown by the thin straight lines. Data for $P(\tau)$ are obtained from 10 trajectories of length 10^{11}.

For $p = 3$ the rate D_c should be finite. The time dependence of D_c was computed from 100 trajectories initially located near the unstable fixed point of period $q = 1, 2$. Dependence on τ, and the comparison with its computation from $P(\tau)$ and from exit times $\tau_n = \tau/2$ via the above integral relation, are shown in Fig.3. Both methods give a reasonable agreement with $D_c(\tau)$, especially in the case of $P(\tau)$, thus further confirming the approximation (1). The fitting parameters are $G_1 \approx G_2 \approx 2$, $\bar{G}_1 \approx 3.0$, $\bar{G}_2 \approx 1.6$. For $\tau > 10^7$ the asymptotic value $p = 3$ leads to a saturation of D_c growth in time. Even though the asymptotic diffusion rate is constant the distribution function is non-gaussian since the higher moments diverge. For smaller τ the diffusion rate $D_c(\tau)$ grows approximately linearly that corresponds to an intermediate value of $p \approx 1$.

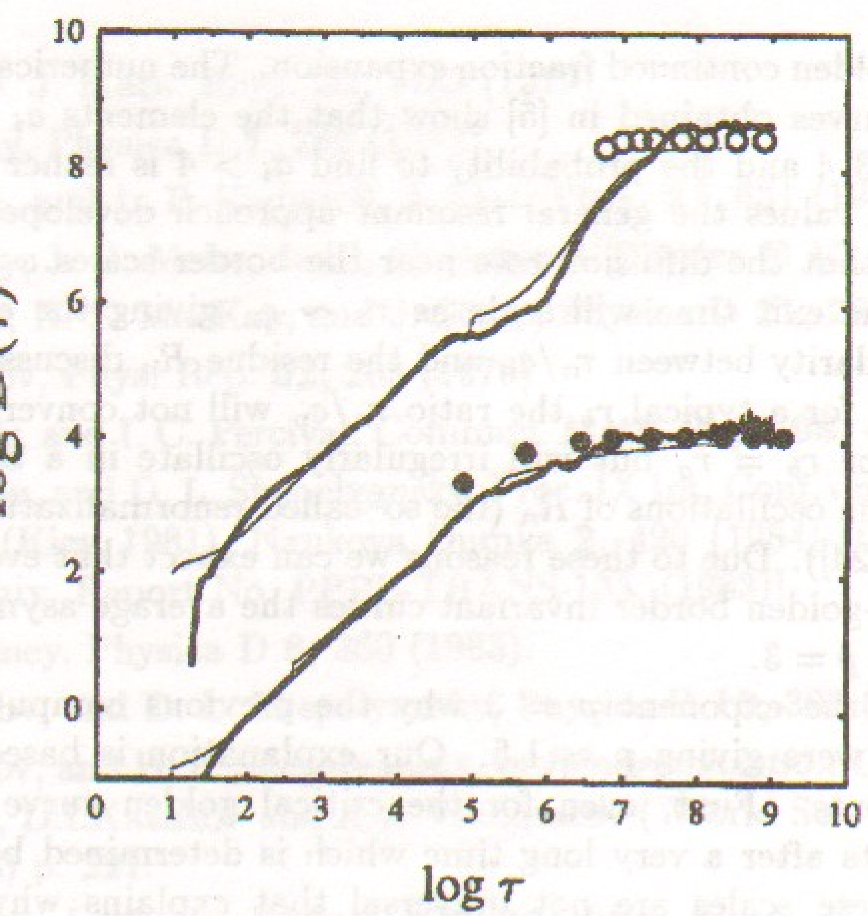

Figure 3: Dependence of diffusion rate D_c on time (thick curves) compared with those computed from the Poincaré recurrences in Fig.2 (thin curves) and from the exit times in Fig.1 (open and full circles) according to relation (1) (see text). Lower curves and circles are for $q = 1$ side, while the upper ones are for $q = 2$ (shifted up by 4 for clarity).

This slow decay is responsible for an enormously large ratio of the asymptotic diffusion rate to its quasilinear value: $D_c(\infty)/D_{cq} \approx 3 \cdot 10^5$ ($q = 1$); 10^7 ($q = 2$).

The ensemble of data in Figs.1-3 shows that the asymptotic decay of Poincaré recurrences and of correlations is determined by the universal structure in a vicinity of the critical golden curve. Thus, the contribution of the borders of other internal islands of stability is not significant at variance with [16]. The hypothesis of dynamical disconnection of scales [11] is also ruled out. On the other hand, our present results are in agreement with the previous numerical observations indicated that long recurrences are related to the trajectories being very close to r_g curve [10, 11].

The next interesting question is how the value of the exponent $p = 3$ would be modified for the case of the main border curve $r_b = [a_1, a_2, ..., a_i, ...]$

with non-golden continued fraction expansion. The numerical data for border invariant curves obtained in [5] show that the elements a_i mainly take the values $1, 2, 3, 4$ and the probability to find $a_i > 4$ is rather small. For such bounded a_i values the general resonant approach developed in [12, 13, 11] still shows that the diffusion rate near the border scales as $D_n \propto 1/q_n^5$ and therefore the exit time will scale as $\tau_n \sim q_n$ giving the exponent $p = 3$. Due to similarity between τ_n/q_n and the residue R_n discussed above we can expect that for a typical r_b the ratio τ_n/q_n will not converge to a constant as it was for $r_b = r_g$ but will irregularly oscillate in a bounded interval, similar to the oscillations of R_n (the so-called renormalization chaos, see e.g. [10, 11, 13, 24]). Due to these reasons we can expect that even in the general case of non-golden border invariant curves the average asymptotic universal exponent is $p = 3$.

Now, if the exponent $p = 3$ why the previous computations of different groups were giving $p \approx 1.5$. Our explanation is based on the following arguments. First, even for the critical golden curve the asymptotic regime starts after a very long time which is determined by first resonance scales. These scales are not universal that explains why p was varying from map to map. If the border curve is non-golden then the ratio τ_n/q_n should oscillate with n and the asymptotic regime will appear even later than for $r_b = r_g$. Also, on the first scales a given map can be often locally approximated by the standard map with the local order parameter $K \approx K_{cr} + (r - r_b)df(r)/dr$, where f is some smooth function of the rotation number [6]. A typical example is the separatrix map [6, 10, 11]. In this approximation the local order parameter is supercritical at the chaotic side of the border: $K - K_{cr} \propto |r_n - r_b| \propto 1/q_n^2$. This scaling is different from the asymptotic one with $K - K_{cr} \propto 1/q_n \propto |r_n - r_b|^{1/2}$ [2, 11, 17], and can give a very long exit time for first scales. Indeed, in the standard map with $K > K_g$ the transition time from $y = 0$ to $y = 1$ is proportional to $1/|K - K_g|^3$ [6, 4] corresponding to the exit time $\tau_n \sim /|K - K_g|^3 \sim q_n^6$. According to (1) this would give $p = 4/3$ that is not far from the average $p \approx 1.5$. In addition, when being close to the critical curve, as in the standard map with $K = K_g$, one should still wait a long time τ_g to reach the asymptotic exponent $p = 3$.

In conclusion, we showed that in the case of the critical golden curve the asymptotic exponent for the decay of Poincaré recurrences is $p = 3$. This implies that the correlation integral converges and the diffusion rate produced by such a dynamical chaos is finite. However, the higher moments of the distribution will diverge. Finally, we expect that the asymptotic exponent should remain the same also in the case of a typical border invariant curve.

References

[1] J. M. Greene, J. Math. Phys. **20**, 1183 (1979).

[2] R. S. MacKay, Physica D **7**, 283 (1983).

[3] S. J. Shenker, and L. P. Kadanoff, J. Stat. Phys. **27**, 631 (1982).

[4] R. S. MacKay, J. D. Meiss, and I. C. Percival, Physica D **13**, 55 (1984).

[5] J. M. Greene, R. S. MacKay, and J. Stark, Physica D **21**, 267 (1986)

[6] B. V. Chirikov, Phys. Rep. **52**, 263 (1979).

[7] R. S. MacKay, and I. C. Percival, Commun. Math. Phys. **98**, 469 (1985).

[8] B. V. Chirikov, and D. L. Shepelyansky, Proc. IX Int. Conf. on Nonlinear Oscillations (Kiev 1981), Naukova Dumka **2**, 420 (1984) [translation, Princeton Univ. Report No. PPPL-TRANS-133, (1983)].

[9] C. F. F. Karney, Physica D **8**, 360 (1983).

[10] B. V. Chirikov, and D. L. Shepelyansky, Physica D **13**, 395 (1983).

[11] B. V. Chirikov, and D. L. Shepelyansky, in *Renormalization Group*, Eds. D.V.Shirkov, D.I.Kazakov and A.A.Vladimirov (World Sci. Publ., Singapore, 1988) p. 221.

[12] B. V. Chirikov, Lect. Notes Phys. **179**, 29 (1983).

[13] B. V. Chirikov, Proc. Intern. Conf. on Plasma Physics, Lausanne, **2**, 761 (1984).

[14] S. Ruffo, and D. L. Shepelyansky, Phys.Rev.Lett. **76**, 3300 (1996).

[15] J. D. Hanson, J. R. Cary, and J. D. Meiss, J. Stat. Phys., **39**, 327 (1985).

[16] J. Meiss, and E. Ott, Phys. Rev. Lett. **55**, 2741 (1985); Physica D **20**, 387 (1986).

[17] N. W. Murray, Physica D **52**, 220 (1991).

[18] T. Geisel, A. Zacherl, and G. Radons, Phys. Rev. Lett. **59**, 2503 (1987).

[19] J. Klafter, M. F. Shlesinger, and G. Zumofen, Physics Today **2**, 33 (1996).

[20] G. M. Zaslavsky, and B. A. Niyazov, Phys. Rep. **283**, 73 (1997).

[21] R. Artuso, Physica D (to appear).

[22] B.V.Chirikov, JETP **83**, 646 (1996) [Zh. Eksp. Teor. Fiz. **110**, 1174 (1996)].

[23] A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics*, Springer-Verlag, New York (1992).

[24] B. V. Chirikov, Chaos, Solitons & Fractals **1**, 79 (1991).

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