Regular and anomalous quantum diffusion in the Fibonacci kicked rotator

G. Casati,[1] G. Mantica,[1] and D. L. Shepelyansky[2]
[1]*International Center for the Study of Dynamical Systems, Università dell'Insubria, Via Valleggio 11, I-22100 Como, Italy and Istituto Nazionale Fisica della Materia, Unità di Milano, and Istituto Nazionale Fisica Nucleare, sez. di Milano, Milan, Italy*
[2]*Laboratoire de Physique Quantique, UMR 5626 du CNRS, Université Paul Sabatier, F-31062 Toulouse Cedex 4, France*
(Received 27 July 2000; published 25 May 2001)

We study the dynamics of a quantum rotator, impulsively kicked according to the almost-periodic Fibonacci sequence. A special numerical technique allows us to carry on this investigation for as many as 10^{12} kicks. It is shown that above a critical kick strength, the excitation of the system is well described by regular diffusion, while below this border it becomes anomalous and subdiffusive. A law for the dependence of the exponent of anomalous subdiffusion on the kick strength is established numerically. The analogy between these results and quantum diffusion in models of quasicrystals and in the kicked Harper system is discussed.

DOI: 10.1103/PhysRevE.63.066217 PACS number(s): 05.45.Mt, 02.30.Tb

I. INTRODUCTION

It can be reasonably argued that the quantum kicked rotator [1] is the oldest and most thoroughly investigated system in the realm of so-called quantum chaos. Yet we pretend in this paper that recourse to this system is still justified, and can provide new dynamical phenomena worthy of investigation. Consider in fact the problem of *transport* in quantum extended systems: for instance, a wave packet is initially localized on a few states of a quantum lattice, and is then left evolve. A deep analysis carried out on model systems [2–7] with nearest-neighbor interactions has shown that this dynamics is either *localized*—this is typically the case of random couplings—or, as it happens in the case of almost periodic couplings, *anomalously diffusive* (i.e., delocalization occurs, but the diffusion coefficient is either zero or infinity, but very rarely finite) and *intermittent* [8–14].

It is well known that the kicked rotator for typical values of the parameters is akin to the first example just presented: its quantum dynamics is (eventually) localized in the lattice given by the free rotator eigenstates. Therefore, it seems to feature a far simpler dynamics than the second class of systems mentioned above, or than its not-too-close relative, the *quantum kicked Harper* model [15], in which anomalous diffusion [16] has been documented. A way to overcome this is offered by an idea employed profitably in spin and few-levels systems [17–19]: rather than acting upon the system with a periodic impulsive force, it is convenient to let the sequence of time intervals between different ''kicks'' be almost periodic. In few-level systems, this idea has led to interesting spectral properties and dynamical behaviors. In this paper, we apply this procedure to the kicked rotator. We hope to combine the complexity of an almost-periodic interval sequence and the richness of the infinite Hilbert space of the unperturbed problem. At the same time, we desire to maintain a sufficient simplicity for the analysis: for this, we elect to investigate the effect of a Fibonacci sequence of unitary evolution operators.

This paper presents the results of a numerical analysis of the system so defined. In previous studies of almost-periodic kicking of the quantum rotator, a linear increase of the energy has been observed [20,21], which amounts to *regular* diffusion. In contrast, the power-law increase of energy with a nonunit exponent defines *anomalous* diffusion.

Energy is just one of the indicators of the growth of the excitation of this quantum system. We shall use here a larger set of indicators: moments, entropy, and the inverse participation ratio. The asymptotic behavior of these quantities is consistent with a power-law delocalization of the motion. As a consequence, an associated *growth exponent* can be defined for each of them, in such a way that the value $\frac{1}{2}$ corresponds to regular diffusion.

This analysis will show that two dynamical regions are present in this system. For large kicking strength, the motion is diffusive: the growth exponents are equal to $\frac{1}{2}$, and a proper diffusion coefficient can be defined. To the contrary, when the kick strength is less than a certain threshold, the motion typically features *anomalous subdiffusion*: growth exponents still exist, but are smaller than $\frac{1}{2}$. It is interesting to note that a similar situation has been found in quasicrystals in two dimensions [22]: in these systems, for large hopping amplitude, the wave-packet spreading over a two-dimensional lattice is diffusive, while in the opposite case, spreading is characterized by anomalous subdiffusion.

The investigation will be further carried on by varying the kick strength: in so doing, a power-law dependence of the growth exponents on this parameter is observed. In our opinion, these phenomena are significant and make it very interesting to try to explain the motion of this variation of the conventional quantum kicked rotor.

This paper is organized as follows. In the next section, we define the model. In Sec. III, we introduce the growth indicators and their power-law exponents to gauge anomalous diffusion in the dynamics of an initially localized wave packet. The numerical algorithm and the numerically computed long-time dynamics of the system are presented in Sec. IV. Finally, we allow a random element in the definition of the dynamical system to verify the robustness of the obtained results and to support the conjecture that anomalous diffusion in this model is induced by the almost periodicity of the sequence of unitary operators. In the final section we confront these results with those of other almost-periodic systems.

II. THE DYNAMICAL SYSTEM

The basis of our investigation is the quantum kicked rotator [1]: this is the unitary evolution generated in the usual Hilbert space $L_2(S_{2\pi})$ by the operators

$$U(k,T)=e^{-ik\cos\theta}\ \ e^{-i(T/2)\hat{n}^2}, \tag{1}$$

which depend on the two parameters k,T and where the momentum operator is $\hat{n}=-i\partial_\theta$. The corresponding classical system is the well-known Chirikov standard map [23]. Semiclassical dynamics is obtained letting k grow and T go to zero while keeping their product fixed. This product defines the classical chaos parameter, $K=kT$. In the classical case, the system becomes globally diffusive for $K>0.9716\ldots$.

It is well known that for large values of K and typical irrational values of $T/2\pi$, classical and quantum dynamics of the kicked rotator are profoundly different: this is the content of the quantum phenomenon of dynamical localization, which has been well described in the literature, and which predicts that the quantum evolution of an initially focused wave function is almost periodic in time and localized on the lattice of the free rotor eigenstates [1].

In this paper, we study a variation on this theme: while in the usual approach the unitary evolution of Eq. (1) is performed repeatedly, at *equally spaced* times $t_n=nT$, in our model the length of the time intervals in between kicks will *not* be constant. Equivalently, we can say that the operators U act now in a Fibonacci sequence to produce the full quantum evolution $\mathcal{U}$:

$$\mathcal{U}(t_n)=U(k,\tau_n)\bigcirc U(k,\tau_{n-1})\bigcirc\cdots\bigcirc U(k,\tau_1), \tag{2}$$

where $t_n=\Sigma_{j=1}^n\tau_j$ is time, and where $\bigcirc$ denotes operator composition. As is immediately observed, operators in Eq. (2) are characterized by the same value of the quantum kick amplitude k and differ solely by the value of T. The sequence $\{\tau_1,\tau_2,\ldots\}$ is determined by allowing τ_j to take one of two possible values, which for convenience we denote by A and B. We choose for A and B the values (in arbitrary a dimensional units, as it is customary to do)

$$A=\frac{2\pi}{\lambda},\quad B=\frac{2\pi}{\lambda^2},$$

where $\lambda=1.3247\ldots$ is an algebraic number that solves the equation $\lambda^3-\lambda-1=0$. In a sense, A,B is the *most irrational* couple of numbers. Choosing a different irrational pair will not alter our results, as we shall see.

We line up A's and B's according to the well-known Fibonacci sequence: this is obtained by an accretion rule on the initial finite pieces of the infinite sequence, $\sigma_1=A$ and $\sigma_2=A,B$. According to this rule, the next finite piece is obtained by joining these two sequences, the first at the end of the second: $\sigma_3=A,B,A$. In general,

$$\sigma_{n+1}=\sigma_n\cup\sigma_{n-1}, \tag{3}$$

where the symbol $\cup$ means joining the right and left sequences *in the specified order*, so that $\sigma_4=A,B,A,A,B$, and so on. It is then easy to see that the length of σ_n is equal to the nth Fibonacci number, F_n. Equation (3) expresses a property that will be instrumental in the following to obtain the quantum evolution up to large times.

III. GROWTH EXPONENTS OF AN INITIALLY LOCALIZED WAVE FUNCTION

We now let $\psi(0)$ be the initial state of the evolution, and we compute the time dynamics $\psi(t_n)=\mathcal{U}(t_n)\psi(0)$. Usually, we choose for the initial state $\psi(0)$ the zero momentum basis state of $L_2(S_{2\pi})$, e_0, the full basis being $e_n=e^{in\theta}/\sqrt{2\pi}$ with $n\in\mathbf{Z}$.

As we have already remarked, the quantum kicked rotator, for *typical* irrational values of the kicking period T, displays the phenomenon of *quantum localization*: energy in the system eventually ceases to grow, and quantum diffusion stops. On the other hand, for sequences with *random* time intervals between kicks, the increase in momentum is diffusive and never comes to an end [24]. Of particular interest then is to study the interval sequence defined in the preceding section, which is neither periodic nor random; it belongs to the family of *almost-periodic* sequences, which, in a sense, is intermediate between the two. We shall see that under these circumstances, dynamical localization results in a delocalization that shares many characteristics with what is taking place in almost-periodic lattice systems.

To study the dynamics of the quantum wave function, it is convenient to introduce a number of quantities suitable to define its spreading: we shall call them collectively *growth indicators*. The first obvious choice are moments. Let $\nu_\alpha(t)$ be the moment of index α of the probability distribution over the lattice $\{e_n\}$ given by $|(e_n,\psi(t))|^2$, where obviously $(\cdot,\cdot)$ denotes the scalar product in $L_2(S_{2\pi})$:

$$\nu_\alpha(t):=\sum_n\ |(e_n,\psi(t))|^2|n|^\alpha. \tag{4}$$

For instance, ν_2 is the usual second moment—the dispersion—which, in the case of regular diffusion, grows *linearly* in time. In general, we cannot expect ν_2 to behave as t, but *superdiffusive* and *subdiffusive* growth will be the rule. For this, we also define the *growth exponent* function β by the asymptotic relation

$$\nu_\alpha(t)\sim t^{\alpha\beta(\alpha)}\quad \text{for}\quad t\to\infty, \tag{5}$$

where the asymptotic behavior for large time is to be understood in a suitable sense. The function $\beta(\alpha)$ is the so-called *quantum intermittency function*, whose name and properties are described in [8–10].

When α tends to zero, the moment ν_α tends to the constant $\nu_0=1$. Nonetheless, one can define the limit of the function β for α tending to zero, $\beta(0)$, a number that also appears in the scaling of the logarithmic moment, $\nu_{\ln}$:

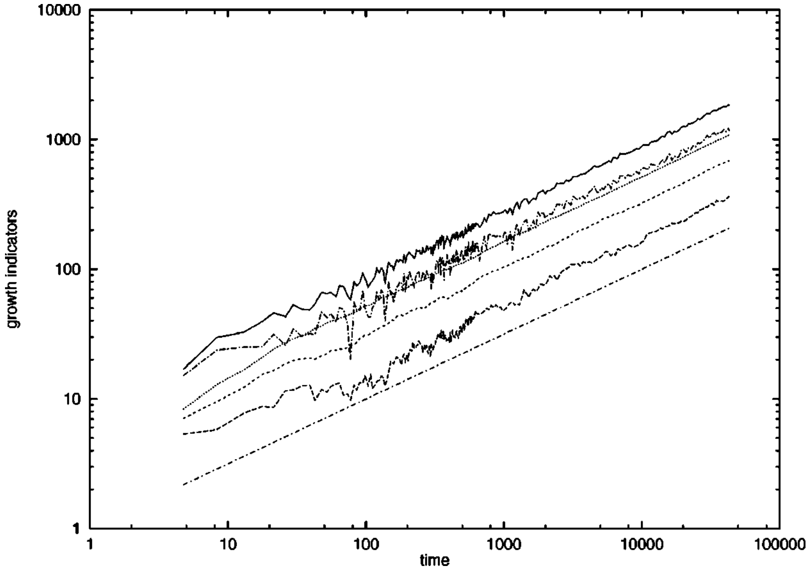

FIG. 1. Regular quantum diffusion. Growth indicators are plotted versus time (in the arbitrary units described in the text) for the case $k=10$. The lowest curve (dots-dashes) is the reference line $\sqrt{t}$. The other curves are (from bottom to top) the exponential of the entropy (long dashes), the square root of the second moment (short dashes), the sixth root of the sixth moment (dots), one over the inverse participation ratio (dots-dashes), and the exponential of the logarithmic moment (continuous line).

$$\nu_{\ln}(t) := \sum_{n\neq 0} |(e_n,\psi(t))|^2 \ln(|n|) \sim \beta(0)\ln(t). \tag{6}$$

In the quantities (5) and (6), the *ordering* of the basis is relevant: a function of n appears in the summations. We can nonetheless define and compute quantities that do not make reference to these characteristics. The first of these is the *entropy* S:

$$S(t) := -\sum_n |(e_n,\psi(t))|^2 \ln(|(e_n,\psi(t))|^2) \sim \gamma + \beta_s \ln(t), \tag{7}$$

whose asymptotic behavior defines the exponent β_s. In the case of localization, β_s is null, and e^{γ} [γ is the constant contribution in Eq. (7)] gives a measure of the localization length. Another basis-independent quantity is the *inverse participation ratio* P,

$$P(t) := \sum_n |(e_n,\psi(t))|^4 \sim t^{-\beta_p}, \tag{8}$$

with its exponent β_p.

We notice that, in principle, the exponents β of all the quantities so far defined are different. Yet, when finite, they are all consistent with the fact that the spreading of the wave packet over the basis takes place in a power-law fashion.

This is indeed the case for our system: in Fig. 1, we plot the $1/\alpha$ power of a set of moments ν_α versus time, in a doubly logarithmic scale, together with one over the inverse participation ratio, and with the exponential of the entropy and of the logarithmic moment. It is immediately apparent that quantum diffusion in this case is regular: these curves are parallel to the reference line $\sqrt{t}$, so that all growth exponents are equal to $\frac{1}{2}$. This case has been computed for $k=10$.

To render even more evident the fact that we are in the presence of regular diffusion, we plot in Fig. 2 a snapshot of the wave packet at a large time, for the same case of Fig. 1: The fitting line is a normal distribution.

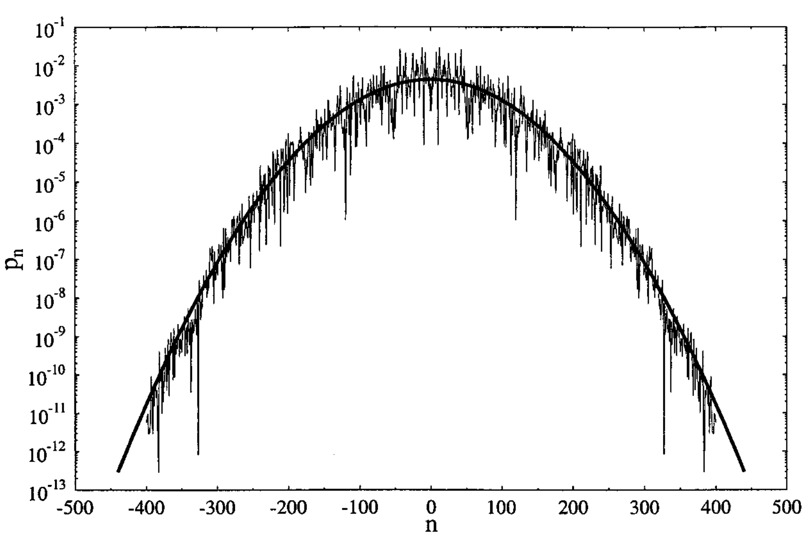

FIG. 2. Wave packet for the case of Fig. 1. Plotted are occupation probabilities $p_n = |(e_n,\psi(t))|^2$ versus site number n at time $t = 1.797\times 10^4$ (thin line). The thick line is an interpolating normal distribution.

IV. DETECTING ANOMALOUS DIFFUSION IN LARGE TIME DYNAMICS

Let us now investigate the effect of varying the kick strength k: in particular, let us decrease it from the value $k=10$ at which we observed regular diffusion. Two difficulties arise when performing this analysis: since we observe a *reduced* delocalization of the wave packet, very large time-scale computations are required to produce reliable information about the asymptotic behavior. Secondly, large *systematic* oscillations in the behavior of the moments (and of the other growth indicators) tend to overwhelm and submerge the power-law increase. The greater the oscillations, the slower is this increase.

Let us first present a solution for the second difficulty: we can introduce a *magnetic flux* ϕ in the evolution (1), which consists merely in replacing the operator $\hat{n}$ with $\hat{n}+\phi$: ϕ is a real number, and for $\phi=0$ we obtain the usual kicked system. By performing the evolution with a set of values of ϕ, and by averaging the results with respect to this sample, we can reduce the systematic fluctuations superimposed to the leading power-law behavior without altering this behavior. Observe also that the classical dynamics is left invariant by the introduction of this flux.

For the first difficulty, a more complex strategy is required. We can push the evolution to extremely large times by resorting to a matrix algorithm. Let $\mathcal{U}^{F_n}$ be the unitary evolution operator acting from time zero up to the time of the kick number F_n, the nth Fibonacci number. Then, Eq. (3) implies that these operators are simply related by

$$\mathcal{U}^{F_{n+1}}=\mathcal{U}^{F_{n-1}}\bigcirc\mathcal{U}^{F_n}, \tag{9}$$

where $\bigcirc$ is the usual operator of multiplication, and where the order of operators is essential, as it was in Eq. (3). This relation must be initialized by setting

$$\mathcal{U}^{F_1}=U(k,A),$$

$$\mathcal{U}^{F_2}=U(k,B)\bigcirc U(k,A), \tag{10}$$

and serves to produce the full sequence $\{\mathcal{U}^{F_n}\}$. In fact, the matrix elements of $U(K,\tau)$ are explicitly known, and Eq. (9) involves then only a matrix-to-matrix multiplication. Of course, the *cost* of this operation scales as the cube of the size of the matrix, and this is certainly more expensive than the usual evolution effected via a sequence of fast Fourier transforms. Nonetheless, this technique becomes advantageous when dealing with extremely large times, for in n steps one reaches up to F_n map iterations, and it is well known that Fibonacci numbers are geometrically increasing in n.

We therefore effect the matrix multiplication for $n=1,\ldots,N$ and then look at the zeroth column of the resulting operator $\mathcal{U}^{F_N}$: this is the evolution of the initial state e_0. In our numerical experiments, the initial state of the evolution will always be e_0. Suitable combinations of the columns can provide the evolution of any arbitrary initial state, if desired.

A question of numerical concern must now be discussed, prior to showing the results of these calculations: when dealing with Eqs. (9) and (10), a finite truncation of the operators involved is necessary. Now, for any practical purpose, $U(k,\tau)$ is a banded matrix, and we must obviously choose the size of the numerical truncation to be much larger than the band size. Yet this is not enough: appropriate boundary conditions at the basis edges must be imposed. Two choices are at hand: Dirichlet and Neumann. In the latter case, we are producing the unitary evolution on the *torus*, while in the first case we end up with a nonunitary evolution.

Having in mind what we want to obtain (i.e., the evolution of the e_0 basis state in the full cylinder space), we have adopted Dirichlet conditions for two reasons. First, when the truncation size is sufficiently large, both Dirichlet and Neumann conditions *must* produce an excellent approximation of the true $\mathcal{U}^{F_N}e_0$. But in addition, choosing Dirichlet, we can gauge numerically two quantities: the normalization of the zeroth column of $\mathcal{U}^{F_N}$ (which is a common technique) and its *effective dimension*, which is less common but more instructive. In fact, this latter is defined as the number of components of $\mathcal{U}^{F_N}e_0$ of larger magnitude than a certain threshold: this quantifies the dimension of the wave packet after F_N kicks, and can be profitably used to control dynamically, as time evolves, the optimal size of the truncation. It must be added that the effective dimension in itself has physical and mathematical relevance [11].

In summary, we have been able to reach easily about 60 iterations of the Fibonacci multiplication, Eq. (10), which correspond to more than 10^{12} usual iterations, with a basis size of the order of thousands. Numerical stability, controlled by various techniques, is the limiting factor here.

A set of typical results is shown in Fig. 3, where we plot the second moment, ν_2, averaged over a sample of phases, as a function of time, in a doubly logarithmic scale, for a set of decreasing values of k. We clearly see that ν_2 decreases with k, but, more importantly, the slope of the curves [i.e., the growth exponent $\beta(2)$] also diminishes in accordance with k. As a consequence, we can safely conclude that quantum diffusion is anomalous in this model for a wide set of parameters [25]. In Fig. 3 we have also plotted the moments for $\phi=0$: it appears that the averaging procedure has succeeded in extracting the leading behavior.

At this point, a comment must be made about intermittency: this is present when the function $\beta(\alpha)$ [see Eq. (5)] is not constant. In the case at hand, the variation of β with α is smaller than the uncertainty with which β is determined, both due to numerical effects and to the superimposed oscillations of the moments ν_α. We can therefore only conclude that intermittency, if present, is low.

This fact is not totally negative: by averaging over α, we can define a more reliable growth exponent β_{av}, which is now a function only of the kick amplitude k. This immediately prompts for the study of this dependence. Figure 4 plots the numerical results obtained by the procedure just exposed. Two regions clearly emerge from the investigation of this picture: for large values of k, we observe the regular diffusive value $\beta_{\text{av}}=\frac{1}{2}$. Anomalous diffusion is observed for smaller values of k: quite interestingly, in this region we find a power-law behavior of the growth exponent of the form

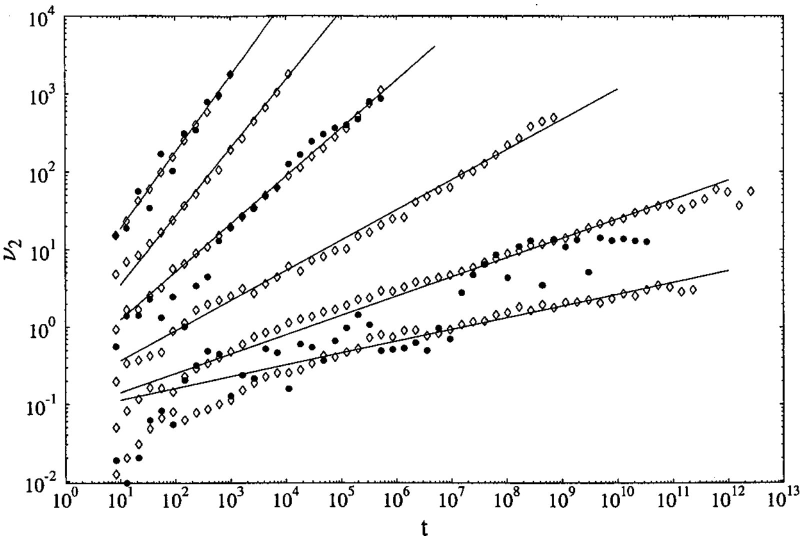

FIG. 3. Second moment ν_2 as a function of time (in the arbitrary units described in the text) for $k=0.125,0.25,0.5,1,2,4$ (bottom to top diamond curves), averaged over a small sample of phases ϕ. For comparison, also shown are unaveraged values at $\phi=0$ for $k=0.25,1,4$ (filled circles). Solid lines are the fitting lines from which the exponent $\beta(2)$ is extracted.

$$\beta_{\mathrm{av}}(k) \sim k^{\eta}, \tag{11}$$

with η very close to the value $\frac{2}{3}$. The transition from anomalous to regular diffusion takes place for $k>k_c$: for the set of values we have chosen, this critical value k_c is approximately equal to 2. Of course, the transition between the two behaviors, as judged from finite-time simulations, does not appear to be sharp.

Let us now investigate the origin of anomalous diffusion in this model. When acting on the basis set $e_n = e^{in\theta}$, the free evolution $e^{i(T/2)\partial_\theta^2}$ produces the phase factor $e^{-i(T/2)n^2}$. The arithmetic nature of $T/2\pi$ is at the root of the spectral properties of the ''conventional'' kicked rotator. The numerical studies presented so far have been carried on for the most irrational pair A,B. We have found similar results for other irrationally related pairs, such as $2\pi/\sqrt{5},2\pi/\sqrt{5}$. To the contrary, the case of rational pairs A,B is subtler (for resonances may set in) and is not considered in this paper. We are thus led to conclude that the nature of the observed unbounded diffusion lies in the almost-periodic arrangement of irrational in-kick intervals A and B.

To substantiate this hypothesis, we have replaced the phases $(T/2)n^2$ by two sequences of equally distributed pseudorandom numbers, one for the operator $U(k,A)$ and one for its companion $U(k,B)$. In so doing, we have obtained quite similar results to those reported above. In Fig. 4, the exponents β_{av} for this experiment are also reported.

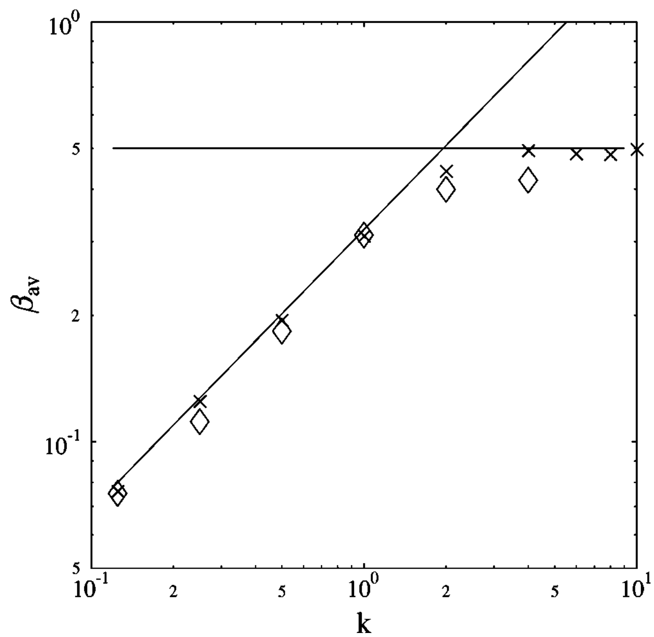

FIG. 4. Growth exponent β_{av} (crosses) obtained as the average of $\beta(\alpha)$, for $\alpha=2,4,6$, versus the kick amplitude k. The horizontal line marks the value $\beta_{\mathrm{av}}=\frac{1}{2}$; the fitting line for small values of k grows proportionally to k^{η}, with $\eta=\frac{2}{3}$. Also reported (diamonds) are the result obtained when random rotation phases replace the irrational phases $Tn^2/2$ (see text).

V. CONCLUSION

We have studied the dynamics of a quantum rotator kicked at discrete times generated by the almost-periodic Fibonacci sequence. Contrary to that of the usual kicked rotator, this dynamics does not show quantum localization. We have introduced and computed various indicators of the spreading of an initially localized wave packet. This has permitted us to show that the dynamics features regular diffusion for large values of the kick amplitude ($k>k_c$) and anomalous subdiffusion for small values ($k<k_c$). In this latter range, the average exponent of this diffusion displays a power-law behavior with the kick amplitude, $\beta_{\mathrm{av}}(k) \sim k^{2/3}$. This relation, although only numerically established, is quite interesting, and deserves further investigation in our opinion.

Similarities and differences between the Fibonacci kicked rotator and other quasiperiodic models are to be noted. We have already observed that anomalous diffusion is typically found in almost-periodic one-dimensional lattice systems. It also appears in the kicked Harper model. To the contrary, the Fibonacci kicked rotator is qualitatively different from the rotator acted upon by equally spaced kicks, with an amplitude k is a quasiperiodic function of time. The case in which this function contains m incommensurate frequencies has been studied in [26–28]. It was shown that this model is a dynamical analog of the Anderson localization in a space of effective dimension $d=m+1$. In this way, the usual kicked rotator $(m=0)$ corresponds to $d=1$, and, of course, is always localized. For $m=1$, the excitation is still always localized, but the localization length grows exponentially with k [26], in analogy with the Anderson localization in $d=2$. Finally, for $m>1$, i.e., $d>2$, a transition from localization to diffusive excitation takes place above some critical kick amplitude, in analogy with the Anderson transition for $d>2$ [27,28]. It is evident that this behavior differs significantly from the one we find in this paper for the Fibonacci kicked rotator.

In our opinion, this difference might be due to the fact that in [26–28], the kick amplitude k is an analytic function of incommensurate phases (frequencies). For this, such a model can be mapped into an effective solid-state hopping model, similar to the Anderson model, with hopping only to a finite number of nearby sites. On the contrary, the almost-periodic sequence of unitary operators of the Fibonacci kicked rotator renders the situation much richer, and gives rise to a transition from regular to anomalous diffusion.

As a matter of fact, the dynamics of the Fibonacci kicked rotator seems more similar to a wave spreading on a two-dimensional quasicrystal lattice. Indeed, studies of quantum diffusion over an octagonal quasiperiodic tiling have shown a similar transition from anomalous to regular diffusion [22]. However, in spite of this initial similarity, more detailed studies are required to establish a quantitative relation between these models and to gain a better theoretical understanding of the results presented in this paper.

[1] G. Casati, B. V. Chirikov, J. Ford, and F. M. Izrailev, Lect. Notes Phys. **93**, 334 (1979); F. M. Izrailev, Phys. Rep. **196**, 299 (1990).

[2] H.L. Cycon, R.G. Froese, W. Kirsch, and B. Simon, *Schöd-inger Operators* (Springer Verlag, Berlin, 1987), and references therein.

[3] B. Simon, Adv. Appl. Math. **3**, 463 (1982), and references therein.

[4] M. Kohmoto and Y. Oono, Phys. Lett. **102A**, 145 (1984).

[5] A. Sütö, J. Stat. Phys. **56**, 525 (1989).

[6] H. Hiramoto and S. Abe, J. Phys. Soc. Jpn. **57**, 230 (1988); Harper Model, *ibid.*, **57**, 1365 (1988).

[7] S. Abe and H. Hiramoto, Phys. Rev. A **36**, 5349 (1987).

[8] I. Guarneri and G. Mantica, Phys. Rev. Lett. **73**, 3379 (1994).

[9] G. Mantica, Physica D **103**, 576 (1997); **109**, 113 (1997).

[10] G. Mantica, J. Phys. IV **8**, 253 (1998).

[11] G. Mantica, Int. Ser. Numer. Math. **131**, 153 (1999).

[12] I. Guarneri, Europhys. Lett. **10**, 95 (1989); **21**, 729 (1993).

[13] I. Guarneri and G. Mantica, Ann. I.H.P. Phys. Theor. **61**, 369 (1994).

[14] F. Piéchon, Phys. Rev. Lett. **76**, 4372 (1996).

[15] R. Artuso, G. Casati, D.L. Shepelyansky, Phys. Rev. Lett. **68**, 3826 (1992); R. Artuso, F. Borgonovi, I. Guarneri, L. Rebuzzini, and G. Casati, *ibid.* **69**, 3302 (1992); R. Artuso, G. Casati, F. Borgonovi, L. Rebuzzini, and I. Guarneri, Int. J. Mod. Phys. B **8**, 207 (1994).

[16] Of course, proper diffusive (chaotic) behavior is not possible in quantum mechanics: we use the term *quantum diffusion* to signify the stable (and time reversible) power-law increase of the moments of the position operator.

[17] J.M. Luck, H. Orland, and U. Smilansky, J. Stat. Phys. **53**, 551 (1988).

[18] A. Crisanti, M. Falcioni, G. Paladin, M. Serva, and A. Vulpiani, Phys. Rev. E **50**, 138 (1994).

[19] I. Guarneri and M. Di Meo, J. Phys. A **28**, 2717 (1995).

[20] M. Combescure, Ann. I.H.P. Phys. Theor. **57**, 68 (1992).

[21] C.R. de Oliveira and G.Q. Pellegrino, Phys. Lett. A **259**, 399 (1999).

[22] V.G. Benza and C. Sire, Phys. Rev. B **44**, 10 343 (1991); B. Passaro, C. Sire, and V. G. Benza, *ibid.* **46**, 13 751 (1992).

[23] B. V. Chirikov, Phys. Rep. **52**, 263 (1979).

[24] I. Guarneri, Lett. Nuovo Cimento Soc. Ital. Fis. **40**, 171 (1984).

[25] Of course, numerical experiments are not proof that anomalous diffusion holds on indefinitely. Yet the large time scale over which this phenomenon is observed in Fig. 3 renders this conclusion plausible.

[26] D. L. Shepelyansky, Physica D **8**, 208 (1983).

[27] G. Casati, I. Guarneri, and D. L. Shepelyansky, Phys. Rev. Lett. **62**, 345 (1989).

[28] F. Borgonovi and D. L. Shepelyansky, Physica D **109**, 24 (1997).