Quantum Computing of Quantum Chaos and Imperfection Effects

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We study numerically the imperfection effects in the quantum computing of the kicked rotator model in the regime of quantum chaos. It is shown that there are two types of physical characteristics: for one of them the quantum computation errors grow exponentially with the number of qubits in the computer, while for the other the growth is polynomial. A certain similarity between classical and quantum computing errors is also discussed.

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A great interest in quantum computers has been generated recently by prominent theoretical results and impressive experimental progress which allowed one to realize operations with a few qubits (see [1] for a review). The most striking theoretical advantage is the enormous parallelism of quantum computing. Using the Shor algorithm [2] the factorization of large numbers can be done exponentially faster on a quantum computer than by any known algorithm on a classical computer. Also a search of an item in a long list is much faster on a quantum computer as shown by Grover [3]. Another very important step is the development of error-correcting codes which show that a certain amount of noise could be tolerable in quantum computations [4]. Physically, one qubit can be viewed as a two level system and controlled coupling between qubits should be included to operate gates in quantum computations. A variety of physical systems have been considered to build a quantum computer including ion traps [5,6], nuclear magnetic resonance systems [7], nuclear spins with interaction controlled electronically [8,9] or by laser pulses [10], electrons in quantum dots [11], Cooper pair boxes [12], optical lattices [13], and electrons floating on liquid helium [14]. At present a two-qubit gate has been experimentally realized with cold ions [15], and the Grover algorithm has been performed for three qubits made from nuclear spins in a molecule [16].

It is clear that in any realistic quantum computer special attention should be paid to the imperfection effects. Indeed, the imperfections are always present and they, in principle, may seriously modify the computation results compared to the algorithms based on ideal qubit operations. At present the imperfection effects have been tested in the numerical simulations of the quantum Fourier transform (QFT) [5] and the Shor algorithm factorization of 15 [17,18]. The obtained results look to be promising for the quantum computing. They indicate that a small amount of noise does not change strongly the computations [5] even if in some cases only a rather low level of noise is tolerable [18]. The developed quantum error-correcting codes ensure tolerance to a certain level of noise [4], but they require the introduction of a lot of redundant qubits and make some assumptions about noise properties. Because of that it is not easy to study numerically the effects of imperfections for a quantum computing of a physical system with many qubits n_q. The effects of static imperfections on the stability of quantum computer hardware were analyzed in [19], but these results cannot be directly applied to quantum algorithms operating in time.

In view of the importance of imperfection effects we analyze in this paper their influence on a quantum computation of quantum chaos evolution in time. The quantum chaos in time-dependent systems was studied intensively during last two decades. It has been understood that the quantum interference can lead to dynamical localization of classical diffusive excitation in a close analogy with the Anderson localization in a random potential [20,21]. The study of such systems should represent a serious test for quantum computing. Indeed, in a classically chaotic system the numerical errors grow exponentially with time due to exponential local instability of motion which leads to chaotic diffusion in the phase space. In the quantum case the error growth is not so strong [22], but still the dynamical localization of quantum chaos remains rather sensitive to external perturbations and noise [22,23]. In the case of quantum computing, our work is relevant to the situation when a problem is faced with iterative quantum gate operations on a quantum computer, e.g., the Grover algorithm [3].

To investigate the imperfection effects on quantum computing we choose the kicked rotator model introduced in [24]. It represents the main features of time-dependent quantum chaos and had been studied extensively in numerical simulations [21] and experiments with cold atoms [25]. The unitary evolution operator $\hat{U}$ over the period T of the perturbation is given by

$$\bar{\psi} = \hat{U}\psi = e^{-ik\cos\hat{\theta}} e^{-iT\hat{n}^2/2}\psi , \qquad (1)$$

where $\hbar = 1$ so that the commutator is $[\hat{n}, \hat{\theta}] = -i$ and the classical limit corresponds to $k \to \infty, T \to 0$, while the classical chaos parameter $K = kT$ remains constant. The operator $\hat{U}$ is given by the product of two unitary operators representing kick $\hat{U}_k = \exp(-ik\cos\hat{\theta})$ and free rotation $\hat{U}_T = \exp(-iT\hat{n}^2/2)$; it acts on N quantum levels with

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periodic boundary conditions. The classical dynamics is described by the Chirikov standard map:

$$\bar{n} = n + k\sin\theta; \qquad \bar{\theta} = \theta + T\bar{n}\,. \tag{2}$$

For $K > 0.9716$ the global chaos sets in with the diffusive growth $n^2 = Dt$, where t is given in the number of kicks and the diffusion rate is $D \approx k^2/2$ for $K > 4.5$ [21]. The quantum interference leads to suppression of this diffusion after a time scale $t^* \approx D$ and exponential localization of the eigenstates of the $\hat{U}$ operator with the localization length $l \approx D/2$ [21].

The most efficient way of numerical simulation of quantum dynamics (1) on a classical computer is based on the fast Fourier transforms (FFT) between θ and n representations. Indeed, the operators $\hat{U}_k$ and $\hat{U}_T$ are diagonal in θ and n representations, respectively, which take $O(N)$ multiplications for their realization. The transition between representations is done by forward and back FFT with $O(N\log_2 N)$ multiplications. Thus the FFT is the most time-consuming part in the classical computations of model (1).

On the contrary the quantum computer requires only $O(\log_2^2 N)$ gate operations to perform QFT (see [2,26]) and makes very easily the forward/back transformations between θ and n representations. Hence, in this part the quantum computer has the exponential gain comparing to the classical one. However, it is not so easy to reach the exponential gain in the multiplication by the diagonal matrices $\hat{U}_k$ and $\hat{U}_T$ in θ and n representations, respectively. Of course, as for the classical computation this can be done in $O(N)$ operations. In this worst case the quantum computer will have $O(\log_2 N)$ gain comparing to the classical one. We suppose that a much better performance can be reached for the above diagonal part of the quantum algorithm with a strong gain increase. However, in this paper we leave this question for future research and assume that the unitary diagonal transformations $\hat{U}_k$ and $\hat{U}_T$ in (1) are performed by some quantum circuit exactly while imperfections are present only in the QFT part. Namely, for the QFT description in [26] [see Eqs. (14)–(21) there] each basic unitary operation A_j (one-qubit) or B_{jk} (two-qubit) is rotated on a small random angle of amplitude $\epsilon \ll 1$. At $\epsilon = 0$ the operation A_j is written as $\hat{n}_0 \cdot \vec{\sigma}$, where $\hat{n}_0 = (1/\sqrt{2}, 0, 1/\sqrt{2})$ and σ_i's are Pauli matrices. With imperfections $A_j = \hat{n}_j \cdot \vec{\sigma}$ is given by a rotation of a unit vector $\hat{n}_j$ on an angle ϵ_j from $\hat{n}_0$. For B_{jk} a random angle of size ϵ_{jk} is added to θ_{jk} in Eq. (18) of [26] ($|\epsilon_j|, |\epsilon_{jk}| \leq \epsilon$). These random ϵ rotations vary in time producing an effective noise in the QFT and quantum computing of the kicked rotator dynamics on $N = 2^{n_q}$ levels with n_q qubits.

The effect of imperfections in the QFT on the second moment $\langle n^2\rangle$, computed from the probability distribution W_n over unperturbed levels n ($\langle n^2\rangle = \sum_n n^2 W_n$), is shown in Fig. 1 for the regime of quantum chaos ($k > K > 1$) and the different number of qubits n_q at $\epsilon = 10^{-4}$ and

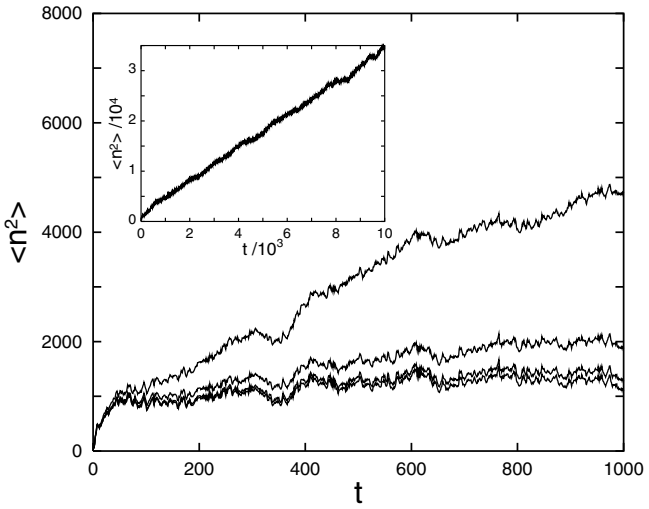

FIG. 1. Dependence of the second moment $\langle n^2\rangle$ on time t for the different imperfection strengths ϵ in quantum computing and the different number of qubits n_q. Curves are for $n_q = 13, 12, 11$ at $\epsilon = 10^{-4}$ from top to bottom and the lowest curve is for $\epsilon = 0$ being the same for $n_q = 13, 12, 11$. Here $k = 10, K = 5$ and at $t = 0$ all probability is at $n = 0$. The inset shows the upper curve up to larger times.

$\epsilon = 0$. The data show that the noise from imperfections produces an effective diffusive growth of the second moment with the rate D_ϵ which grows *exponentially* with the number of qubits. In fact, the data in Fig. 2 show that in the regime $k > K > 1$ this rate is well described by the relation $D_\epsilon \approx 5\epsilon^2 2^{2n_q}$ for different ϵ, n_q, and k [27].

The physical origin of the exponential error growth in $\langle n^2\rangle$ with n_q becomes clear from Fig. 3 which shows

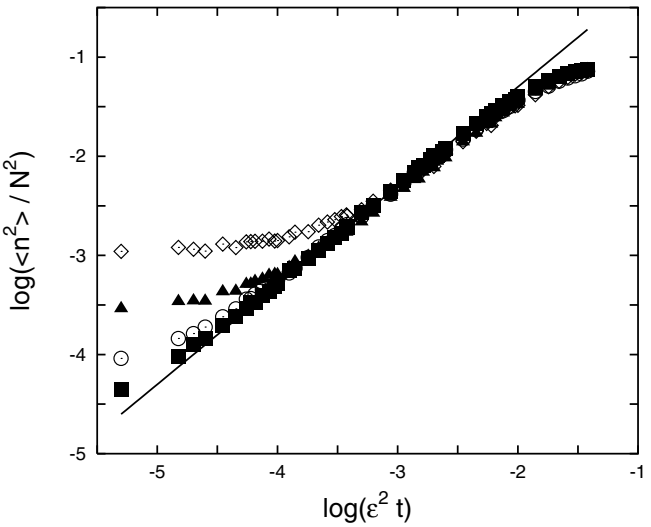

FIG. 2. Scaling of $\langle n^2\rangle/N^2$ is shown for various values of ϵ and $N = 2^{n_q}$: $10^{-4} \leq \epsilon \leq 2 \times 10^{-3}$ and $n_q = 10$ ($\diamond$), 11 (full triangle), 12 ($\circ$), 13 (full square) for $k = 10$ and $K = 5$. Each point represents the averaged value over 10^3 kicks for $t \leq 10^4$ and the straight line represents the scaling given by $\langle n^2\rangle = D_\epsilon t \approx 5\epsilon^2 N^2 t$ for $t_q < t < t_\epsilon$ (see text). Here and in the next figures the logarithms are decimal.

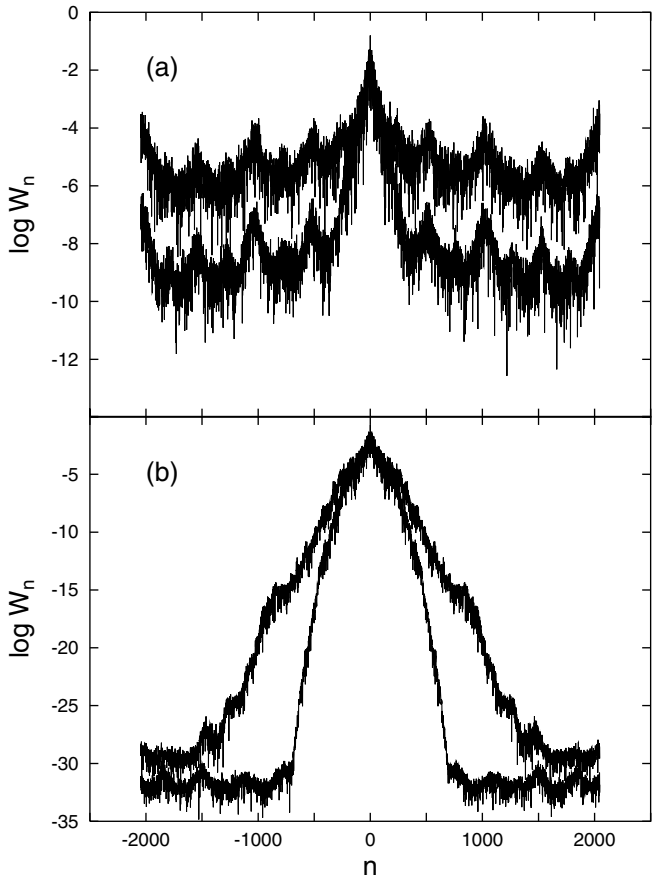

FIG. 3. Probability distribution W_n over unperturbed levels for $n_q = 12$, $k = 10$, and $K = 5$ at two moments of time $t = 100$ (lower curve) and $t = 10^5$ (upper curve): (a) $\epsilon = 10^{-4}$ and (b) $\epsilon = 0$. Initially all probability is at $n = 0$.

the probability distribution W_n at two moments of time. At $\epsilon = 0$ the probability decays exponentially from the initially excited level $n = 0$ due to dynamical localization [see Fig. 3(b)]. This decay continues up to a noise probability level $W_p \sim 10^{-32}$ at $t = 100$ which is determined by the round-off errors in the classical computer being of the order $\epsilon_c \sim 10^{-16}$. In fact, these errors produce an effective diffusive growth so that $W_p \sim \epsilon_c^2 t$. Indeed, W_p is increased approximately by 10^3 when t is changed from 100 to 10^5. The classical errors have certain similarities with the imperfection effects in the quantum case [Fig. 3(a)]. Indeed, the quantum errors also lead to a noise probability level W_{pq} appearing at large n in a form of plateau with peaks. Below this level the quantum imperfections completely modify the quantum probabilities W_n comparing to the ideal case with $\epsilon = 0$ [compare Figs. 3(a) and 3(b)]. From Fig. 3(a) it is clear that the noise probability level grows diffusively with time: $W_{pq} \propto \epsilon^2 t$. It is interesting to note that the classical computer gives the noise level W_p homogeneous at large n while in the quantum case W_{pq} has pronounced peaks located around the levels $n_m = \pm 2^m$ with $m = 1, 2, \ldots, n_q/2$. This property is related to the QFT structure which due to imperfections generates transitions to levels n_m with probability $W_{n_m} \propto \epsilon^2$. At $k \ll 1$ the probability on other levels ($n \neq n_m$) is much smaller than W_{n_m} (data not shown), but for $k \gg 1$ each peak at n_m is strongly broadened. Hence, quantum chaos enhances strongly the effect of imperfections. It also gives relatively strong secondary peaks placed between primary n_m peaks [Fig. 3(a)].

In fact, QFT is performed by $O(n_q^2)$ gate operations [26] with imperfect rotations. This imperfection noise creates n_q peaks with probability $W_{pq} \sim \epsilon^2 n_q t$ in each peak [28] that leads to the diffusive growth $\langle n^2 \rangle \sim N^2 W_{pq} \sim D_\epsilon t$ with $D_\epsilon \approx n_q \epsilon^2 2^{2n_q}/2$. The numerical factor here is taken from the data in Fig. 2. There the variation of n_q by 30% is too small to allow one to distinguish the n_q prefactor in front of the exponential dependence 2^{2n_q} from a constant. Since at $\epsilon = 0$ the second moment is bounded due to quantum localization of chaos and fluctuates around $\langle n^2 \rangle \approx D^2 \approx 4l^2 \approx k^4/4$ [21], the imperfections strongly modify $\langle n^2 \rangle$ after the time scale

$$t_q \approx D^2/D_\epsilon \approx k^4/(\epsilon^2 n_q 2^{2n_q}), \tag{3}$$

which drops exponentially with n_q. Because of the finite system size, the imperfection induced diffusive growth of $\langle n^2 \rangle$ is saturated around the maximal value N^2 after the time $t_\epsilon \approx N^2/D_\epsilon \approx 2/(n_q \epsilon^2)$ (it is seen in Fig. 2 for large $\epsilon^2 t$). The imperfection induced diffusion exists on the large time interval $t_q \ll t \ll t_\epsilon$ (see Fig. 2). From the above estimates for W_{pq} growth with time it follows that the probability in n_q peaks becomes comparable with the probability inside the central peak ($W_{pq} n_q \sim n_q^2 \epsilon^2 t \sim 1$) after time $t_p \sim 1/(n_q \epsilon)^2$. For $t \ll t_p$ the noise level W_{pq} is rather low and some characteristics should remain close to their values in the absence of imperfections. One of them is the inverse participation ratio (IPR) ξ which is often used in the problems with localization and determines the effective number of basis states contributing to the wave function. It is defined as $\sum_n W_n^2 = 1/\xi$. By comparing the value of IPR in the presence of imperfections with its value at $\epsilon = 0$ we determine the time scale t_p by the condition $\xi(\epsilon)/\xi(\epsilon = 0) = 1.5$ for different n_q, ϵ, and k. The dependence of t_p on the parameters is shown in Fig. 4 for $k = 10$. According to these data

$$t_p \approx 0.33/(\epsilon n_q)^2, \tag{4}$$

in agreement with the above estimate. Thus, the dependence of t_p on n_q is polynomial. We also checked that the numerical coefficient $C = 0.33$ in (4) does not vary significantly with k, e.g., $C \approx 0.32$ for $k = 15$ and $C \approx 0.35$ for $k = 20$. In our opinion this is related to the fact that the peaks at W_{n_m} are rather sparse and for large n_m the distance between them is much larger than the localization length $l \approx k^2/4$. The comparison between (4) and (3) is striking: a correct quantum computation of ξ during $t = 10^3$ kicks for $k = 10$ and $n_q = 100$ requires the

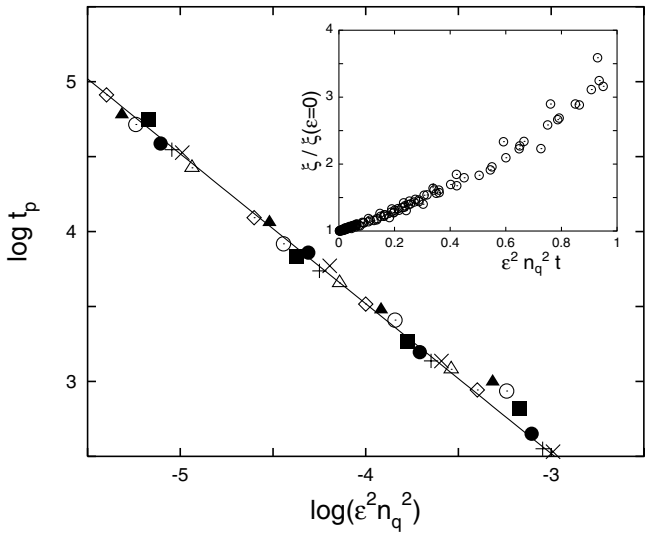

FIG. 4. Dependence of time scale t_p on system parameters for $10^{-4} \leq \epsilon \leq 2 \times 10^{-3}$, and $n_q = 10$ ($\diamond$), 11 (full triangle), 12 ($\circ$), 13 (full square), 14 ($\bullet$), 15 ($+$), 16 ($\times$), and 17 ($\triangle$) for $k = 10$ and $K = 5$. The straight line is given by Eq. (4). The inset shows scaling of the normalized IPR ratio $\xi(\epsilon)/\xi(\epsilon = 0)$.

imperfection amplitude $\epsilon < 2 \times 10^{-4}$, while for the computation of $\langle n^2 \rangle$ one needs $\epsilon < 2 \times 10^{-31}$.

In conclusion, our studies of imperfection effects on quantum computing of the kicked rotator show that for certain characteristics, e.g., the second moment of the probability distribution, the errors grow exponentially with the number of qubits n_q. At the same time there are other characteristics, e.g., IPR ξ, which are much more stable and for which the errors grow with n_q only polynomially. However, such characteristics stable to imperfections are essentially local and are determined only by a small fraction of levels of the whole Hilbert space $N = 2^{n_q}$. In principle, the quantum error-correcting codes [4] allow the reduction of the effective strength of imperfections by introduction of many redundant qubits $n_r \geq 5n_q$. However, a correct quantum simulation of *all* physically important quantities of quantum chaos on N levels requires an exponentially small strength of imperfections $\epsilon \propto 2^{-n_q}$, and it is not obvious if this can be reached with a reasonable number of redundant qubits n_r [29]. Thus the results obtained for a simple quantum chaos model raise an important problem of the necessity of exponentially high accuracy for exponentially fast quantum computing.

Note added.—After the submission of this paper an exponentially efficient algorithm was proposed for (1) by Georgeot and Shepelyansky [30].

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[27] We checked that the data for $k = 15, 20$ and $K = 5$ are also consistent with the expression $D_\epsilon \approx 5\epsilon^2 N^2$.

[28] Noise amplitude is effectively increased by a factor $\sqrt{n_q}$ since $O(n_q)$ gate operations are performed per each peak.

[29] According to [4] the error-correcting codes also correct errors induced by redundant qubits themselves.

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