Dynamical Turbulent Flow on the Galton Board with Friction

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We study numerically and analytically the dynamics of charged particles on the Galton board, a regular lattice of disk scatters, in the presence of constant external force, magnetic field, and friction. It is shown that under certain conditions friction leads to the appearance of a strange chaotic attractor. In this regime the average velocity and direction of particle flow can be effectively affected by electric and magnetic fields. We discuss the applications of these results to the charge transport in antidot superlattices and the stream of suspended particles in a viscous flow through scatters.

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It is well known that dissipation can lead to the appearance of strange chaotic attractors in nonlinear nonautonomous dynamical systems [1,2]. In this case the energy dissipation is compensated by an external energy flow so that stationary chaotic oscillations set in on the attractor. Such an energy flow is absent in the Hamiltonian conservative systems and therefore the introduction of dissipation or friction is expected to drive the system to simple fixed points in the phase space. This rather general expectation is surely true if the system phase space is bounded. However, a much richer situation appears in the case of unbounded space, where unexpectedly a strange attractor can be induced by dissipation in an originally conservative system. To investigate this situation we study the dynamics of particles on the Galton board in the presence of constant external fields and friction. This board, introduced by Galton in 1889 [3], represents a triangular lattice of rigid disks with which particles collide elastically. For the case of free particle motion, the collisions with disks make the dynamics completely chaotic on the energy surface, as was shown by Sinai (see, e.g., [4]). In this paper we study how the dynamics of a charged particle in the presence of electric and magnetic fields is affected by a friction force $F_f = -\gamma v$ directed against particle velocity $v$ (see Fig. 1). Without disks an external in-plane electric field $E$ and a perpendicular magnetic field $B$ create a stationary particle flow with the velocity $v_f = (f + F_{L})/\gamma$. Here $f = eE$ is the effective force, $F_{L} = e v_f \times B$ is the Lorentz force, and $e, m$ are the particle charge and mass, respectively. All perturbations decay to this flow with a rate proportional to $\gamma$ so that this laminar flow can be considered as a simple attractor. The effects of friction inside one cell of the Galton board at $B = 0$ have been studied in Ref. [6] and it has been found that friction leads to the appearance of a nontrivial strange attractor. At present the effects of energy dissipation are actively investigated with the aim to construct equilibrium and nonequilibrium steady states in a deterministic way (see [7,8], and references therein). Here the Nosé-Hoover and Gaussian thermostats with variable friction coefficients lead to a number of interesting results with applications to molecular dynamics and nonequilibrium liquids [7]. In our studies, contrary to [6], we concentrate mainly on the spatial structure of the turbulent chaotic particle flow appearing in the presence of friction. We show that the flow direction can be efficiently affected by a magnetic field. The obtained results describe the electron dynamics in antidot superlattice which has been experimentally realized in semiconductor heterostructures [9]. In such structures the effects of classical chaos play an important role [10] and the effects of friction we discuss here can appear for relatively strong electric fields.

To study numerically the dynamics of this model, we fix the disk radius $a = 1$ and $e = m = 1$ so that the system is characterized only by the distance $R$ between disks packed into equilateral triangles all over the $(x,y)$ plane, the friction coefficient $\gamma$, the external force of strength

![FIG. 1 (color). Chaotic flow on the Galton board. Here the distance between disks is $R = 2.24$, $f = (-0.5, -0.5), B = 2$, and $\gamma = 0.1$. Initially 200 particles are distributed homogeneously along a straight line segment in the upper right corner, their color homogeneously changes from red to green along this segment [5].]
and spreading. This trajectory allows one to follow their interpenetration very large. Surprisingly, for smaller friction, the drift velocity is still visible. The properties of color penetration can be understood from the analysis of single trajectory dynamics. Such a typical example is presented in Fig. 2. It shows that the particle moves with an average constant velocity \( v_f \) under some angle \( \alpha \) to the external force \( \mathbf{f} \), except for \( B = 0 \), where \( \alpha = 0 \). This drift velocity is constant only on average since on a smaller scale the particle moves chaotically between scatters following a strange chaotic attractor. In Fig. 2 for \( \gamma = 0.1 \), the drift velocity is relatively large and the particle does not have enough time to move around many scatters in the direction perpendicular to the flow. As a result the penetration depth for color mixing is not very large. Surprisingly, for smaller friction, the drift velocity becomes smaller and the penetration depth increases so that the particle makes many turns around disks, as is shown in Fig. 2 for \( \gamma = 0.004 \). This dependence is opposite to the case without scatters, where \( v_f \) drops with the growth.

This result can be understood on the basis of the following physical arguments (for \( B = 0 \) see also [6]). In the regime with weak friction the particles start to diffuse among disks in a chaotic manner with the diffusion rate \( D = v l / 2 \), where \( v \) is the particle velocity and \( l \) is the mean-free path [12]. For \( R \ll 1 \) we have \( l \sim R \approx 1 \), while the dependence of \( l \) on \( R \) will be discussed in more detail later. During the dissipative time scale \( \tau_y = m / \gamma \), this diffusion leads to the particle displacement \( \Delta r \sim \sqrt{Dm / \gamma} \) along the direction of the drift velocity \( \mathbf{v}_f \). This gives the change in the potential energy \( U \sim f \Delta r \cos \alpha \sim f_{\text{eff}} \sqrt{Dm / \gamma} \), where \( f_{\text{eff}} = f \cos \alpha \). In the stationary regime at time \( t \gg \tau_y \), this potential energy should be comparable with the kinetic energy of the particle so that \( U \sim mv^2 \), where \( v^2 = \langle u_x^2 + u_y^2 \rangle \) is the average velocity square. Hence

\[
v^2 \sim (f \cos \alpha)^{4/3} (l / \gamma m)^{2/3}.
\]

This relation allows one to determine the drift velocity of the flow \( \mathbf{v}_f \). Indeed during the time \( \tau_c \) between collisions the particle is accelerated by average forces \( \mathbf{f} \) and \( \mathbf{F}_L = e v_f \times \mathbf{B} \) that gives the average drift velocity \( \mathbf{v}_f = (\mathbf{f} + \mathbf{F}_L) \tau_c / m \). Since the dynamics is chaotic, the direction of velocity is changed randomly after each collision so that \( \mathbf{v}_f \) is accumulated only between collisions. The time \( \tau_c \) is determined by the mean-free path \( l \) and the average velocity \( v \): \( \tau_c = l / v = 2D / v^2 \). Thus for the angle \( \alpha \) between \( \mathbf{v}_f \) and \( \mathbf{f} \) we obtain the relation,

\[
\tan \alpha = eB \tau_c / m \sim eBl^{2/3} \gamma^{1/3} / (mf \cos \alpha)^{2/3}.
\]

The amplitude of fluctuations around this direction is \( \Delta r \sim \sqrt{Dm / \gamma} \) which also determines the color mixing depth (see Fig. 1).

From (1) and (2) we obtain the drift velocity amplitude

\[
v_f = f_{\text{eff}} \tau_c / m \sim (l / f_{\text{eff}} / m)^{1/3},
\]

with \( f_{\text{eff}} = f \cos \alpha \). For \( B = 0 \) the particles flow in the \( \mathbf{f} \) direction. In this case their mobility is \( \mu = v_f / f = \tau_c / m = D / (mv^2) / 2 \). This is in fact the Einstein relation according to which the mobility is given by the ratio of the diffusion rate to the average kinetic energy (temperature) [13]. At \( B = 0 \) the relations (1)–(3) are in agreement with those in [6] and with the numerical data shown in Fig. 3. The values of \( v_f \) and \( v^2 \) are obtained from one very long trajectory (with a length of \( 10^5 - 10^6 R \)) or ten shorter trajectories. Within statistical fluctuations this gives the same \( v_f \) and \( v^2 \) independent of their initial values.

The relations (1)–(3) allow one to estimate the value of the Lyapunov exponent \( \lambda \). Indeed, the particle moves with a typical velocity \( v \) and, as in the case of the Sinai

\[
\text{FIG. 2.} \quad \text{Example of a single trajectory for the case of Fig. 1 shown on small (main figure) and large (lower inset) scales.}
\]

\[
\text{The drift velocity of the flow} v_f \approx 0.13 \text{ is directed at angle} \alpha = 0.48 \text{ to} \mathbf{f}. \text{The upper inset shows a single trajectory in the region} (-160 \leq x \leq -140, -190 \leq y \leq -170) \text{ for} \gamma = 0.004 \text{ with} v_f \approx 0.05.
\]
billiard, with \( l \sim R \sim 1 \) we have \( \lambda \sim v/l \). Therefore, in the regime when

\[
\gamma < \gamma_c \sim \sqrt{m f_{\text{eff}}/l},
\]

the value of \( \lambda \) is much larger than the dissipation rate \( \gamma/m \). As a result for \( \gamma \ll \gamma_c \) the strange attractor is fat and its fractal dimension is close to the maximal dimension 4, which is determined by the number of degrees of freedom (we remember that, contrary to the nondissipative case, the energy is not conserved). For \( \gamma \gg \gamma_c \) the dissipation time \( \tau_c \) becomes much shorter than the time between collisions \( \tau_c \). In this case the dissipation dominates chaos and the strange attractor degenerates into a simple attractor. For \( \gamma < \gamma_c \) our numerical simulations performed with high computer accuracy show that trajectories remain chaotic for displacements from the origin being larger than \( 10^3 R \). Also at \( B = 0 \) it can be shown that \( \gamma_c \sim m v l/1 \sim \sqrt{m a/R} \) for \( R \gg a \) and \( \gamma_c \sim \sqrt{m f/a^2} \Delta R \) for \( \Delta R = R - 2a \ll a \).

The above changes in the mean-free path \( l \) at \( B = 0 \) also affect the drift velocity of the flow through the relation (3). Indeed, for \( \Delta R \ll 1 \) we expect \( l \sim \Delta R \) that gives \( v_f \sim \Delta R^{2/3} \). This dependence is close to the numerical data shown in Fig. 4 even if the numerical value of the exponent is approximated better by 0.5. In the other limit \( R \gg 1 \), we have \( l \sim R^2/a \) that gives \( v_f \sim R^{4/3} \). This power dependence is in satisfactory agreement with the data in Fig. 4 although the numerical value of the exponent is closer to 1. We attribute these small deviations in the exponent values to a restricted interval of variation in \( R \). Actually we can not use very large/small values of \( \Delta R \) since in these limits the value of \( \gamma \) becomes comparable with \( \gamma_c \) and the chaotic attractor disappears. We note that according to our data (see Fig. 4) the strange attractor exists even in the case \( R > R_c = 4/\sqrt{3} \) when at

\[
\frac{\ln v^2}{\ln f^{4/3} / \gamma^{2/3}}
\]

\[
\ln v_f
\]

\[
\ln(\Delta R)
\]

\[
\ln(\tan(\alpha))
\]

\[
\ln(B_s)
\]

FIG. 3. Dependences of \( v^2 \) and \( v_f \) (inset) on \( f \) and \( \gamma \) for \( R = 2.24, f/|f| = (-1, -1), B = 0 \), and \( 0.001 \leq \gamma \leq 0.4; 0.5 \leq f \leq 32 \); circles show numerical data and lines show the slopes from (1) (main figure) and (3) (inset).

FIG. 4. Dependence of \( v_f \) on \( \Delta R = R - 2 \) for \( \gamma = 0.1, B = 0 \), and \( f = (-0.5, -0.5) \); the points give numerical data and the lines show the slopes 0.5 and 1. for \( f = 0, \gamma = 0 \) there are straight trajectories crossing the whole plane without collision. Apparently the contribution of these orbits is not significant if \( \gamma > 0 \) and if \( \gamma \) is not directed along these lines.

The introduction of the magnetic field allows one to change efficiently the direction of the flow [14]. The numerical data for the variation of \( \tan \alpha \) with the magnetic field and other system parameters are presented in Fig. 5. The average dependence is in good agreement with Eq. (2) for a large region of parameter variation, where \( \tan \alpha \) changes by 2 orders of magnitude. At the same time, for moderate angles \( \alpha < 1 \), the flow velocity \( v_f \) is weakly affected by \( B \). For example, for the case of Fig. 2, \( v_f \) remains practically the same for \( B = 2 \) and \( B = 0 \). for \( f = 0, \gamma = 0 \) there are straight trajectories crossing the whole plane without collision. Apparently the contribution of these orbits is not significant if \( \gamma > 0 \) and if \( \gamma \) is not directed along these lines.

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For \( \alpha > 1 \) the magnetic field starts to change significantly \( v^2 \) and \( v_f \), in agreement with (1) and (3). While the data in Fig. 5 on average follow the dependence (2) the fluctuations around the average are rather large. Their origin becomes clear from Fig. 6, where only the magnetic field is changed. In this case, \( \tan \alpha \) has a pronounced peak which is located near the value of \( B \), where the cyclotron radius \( r_c = v/B \) is equal to the disk radius. Indeed for \( r_c > a \) a trajectory can make a full turn around a disk that allows one to increase \( \alpha \) and to reach a strong deviation of the global flow from the direction of the electric field. The growth of \( \alpha \) leads to a drop in current (conductivity) in the \( f \) direction and hence to the increase of resistivity. In fact, the peaks in the direction and velocity as a function of magnetic field enables one to determine \( \gamma \) and \( l \) via Eqs. (2) and (3). Our study also represents a certain interest for transport properties of neutral/charged particles suspended in a viscous flow streaming through a system of scatters. Indeed, a laminar stream with the velocity \( v_s \) creates an effective force \( f = v_s \gamma_{\text{eff}} \), where \( \gamma_{\text{eff}} \) is the effective friction created by the viscosity of the liquid. This type of transport can be studied experimentally with viscous liquids and its investigation can contribute to a better understanding of the interplay between dissipation, turbulence, and chaos.

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[11] Dimensional analysis shows that one of these parameters can be fixed, but to trace their physical origin we keep all of them.
[14] Some flow oblique to \( f \) is also possible at \( B = 0 \) but it appears only for fairly strong fields; see the Gaussian thermostat case in J. Lloyd, M. Niemeyer, L. Rondoni, and G. P. Morriss, Chaos 5, 536 (1995).