EFFICIENT QUANTUM COMPUTING INSENSITIVE TO PHASE ERRORS

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We show that certain computational algorithms can be simulated on a quantum computer with exponential efficiency and be insensitive to phase errors. Our explicit algorithm simulates accurately the classical chaotic dynamics for exponentially many orbits even when the quantum fidelity drops to zero. Such phase-insensitive algorithms open new possibilities for computation on realistic quantum computers.

1 Introduction

The problem of quantum computation has attracted recently a great deal of attention [1,2,3]. This interest stems from the fact that the massive parallelism permitted by quantum mechanics enables to reach exponential efficiency of computation in certain quantum algorithms. The most famous example is the Shor algorithm which allows to factor large numbers exponentially faster than any known classical algorithm [4]. Recently, other types of exponentially efficient algorithms has been developed for the simulation of various physical systems [5,6,7]. The exponential gain in computation is related to the exponentially large Hilbert space of the quantum computer which is composed of multi-qubit states operating in parallel (each qubit is a two-level quantum system). Usually the algorithms are constructed for ideal quantum computers operating free of noise and imperfections. In reality, any physical realization of such a computer involves a certain level of imperfections, noise in gate operations and decoherence. First investigations showed that quantum computation can tolerate a sufficiently low level of errors [8,9]. More recently, it has been found that quantum computation is tolerant to quantum errors when simulating classical chaotic dynamics for which classical computer errors grow exponentially with time [7]. However in general the quantum errors grow with the number of gate operations and any realistic quantum computer is faced with this problem. To deal with this problem of fault-tolerant computation, quantum error-correcting codes were recently developed [10,11,12]. They allow to reduce the level of errors in a systematic way, but require the introduction of many redundant qubits and additional gates, which significantly complicates the computational process. The complexity of these codes depends strongly on the type of errors they should correct. Indeed, while simpler classical codes need to correct only bit errors, the quantum ones should in addition simultaneously correct the quantum phase errors. The quantum phase errors seem to be of primary importance since the massive parallelism of quantum computing is related to entanglement in the Hilbert space which is directly based on phase coherence. Therefore, according to this common lore it seems impossible to perform efficient and accurate quantum computations in presence of uncontrolled strong phase errors. In this paper we show on an explicit example that it is not always the case, and that there are algorithms insensitive to phase errors which perform accurate and efficient computation. Our example is based on the simulation of classical chaotic dynamics, which is very hard to simulate accurately on classical computers since this dynamics is unstable and round-off errors grow exponentially with time. In spite of this, our quantum algorithm, including measurement, remains insensitive to phase errors for arbitrary time.

2 The Arnold-Schrödinger cat quantum algorithm

To illustrate this phenomenon, we choose an algorithm which simulates the classical chaotic dynamics of the well-known Arnold cat map [13,14]. It was recently shown [7] that quantum computers can simulate this dynamics with exponential efficiency. In addition it was shown that a small level of quantum errors in the gate operations of order ϵ allows to simulate accurately this map on times of order $O(1/\epsilon^2)$. Thus quantum computers can face classical exponential instability and chaos.

The dynamics of the map we are studying is given by:

$$\bar{y} = y + x \pmod 1 \ , \ \ \bar{x} = y + 2x \pmod 1 \ , \tag{1}$$

where bars denote the new values of the variables after one iteration. This is an area-preserving map, in which x can be considered as the space variable and y as the momentum. A discretized version on a $N \times N$ square lattice is also described by this map. In [7], a quantum algorithm called Arnold-Schrödinger cat map was introduced, and it was shown to simulate this dynamics on the lattice with exponential efficiency. In this paper, we modify this algorithm in such a way that exponential efficiency is preserved and in addition it becomes insensitive to phase errors. This is obtained by the introduction of a new measurement procedure.

The quantum algorithm introduced in [7] simulates the discrete classical dynamics given by (1) and operates with $3n_q - 1$ qubits. These qubits are organized in three registers. Two of them describe the classical phase space with N^2 points and $N = 2^{n_q}$. The third register with $n_q - 1$ qubits is used as workspace. In this way, an initial classical phase space density can be represented by a quantum state $\sum_{i,j} a_{ij}|x_i > |y_j > |0 >$, with $x_i = i/N$, $i = 0, ..., N-1$ and $y_j = j/N$, $j = 0, ..., N-1$, written in binary representation, and we choose initially $a_{ij} = 0$ or $1/\sqrt{N_d}$ where N_d is the number of points in the classical distribution. Then, iterations of the map (1) are performed with the help of additions of integers modulo (N) (modular additions). The quantum algorithm we use for this operation is similar to the one described in [15] (see also [7]). First we compute all the carries of the addition, using two Toffoli gates and one controlled-not (CNOT) gate per qubit. Then we perform the addition starting from the last qubit and erasing the carries by running the inverse of the preceding step. This needs two CNOT gates per qubit addition and the same gates as above to erase the carries. The result is taken modulo (N) by eliminating the last carry. After these operations, the amplitudes $|a_{ij}|$ describe the classical phase space distribution function after iteration of (1). In total, one needs $16n_q - 22$ Toffoli and CNOT gates per map iteration. On the contrary, a classical computer will require $O(2^{2n_q})$ operations per iteration for $N_d = O(N^2)$ orbits.

It is important to stress that during the whole process the classical distribution function is determined only by the probabilities $|a_{ij}|^2$ of the quantum computer wavefunction expanded on the Hilbert space basis of register states $|x_i > |y_j >$ (after each map iteration the third register is always in the state $|0 >$). Thus, the information about the classical distribution function is stored in these probabilities, and is not sensitive to the relative quantum phases of a_{ij}. This suggests that the phase errors accumulated during gate operations will not affect the quantum computer simulation of this classical chaotic dynamics.

3 Effect of phase errors

To study the effects of quantum errors, one usually uses the fidelity [9], defined as: $f(t) = | < \psi_\epsilon(t)|\psi_0(t) > |^2 = |\sum_{i,j} a_{ij}^{(\epsilon)}(t) a_{ij}^{*(0)}(t)|^2$. Here $|\psi_0(t) >= \sum_{i,j} a_{ij}^{(0)}(t)|x_i > |y_j >$ is the quantum state after t perfect iterations, while $|\psi_\epsilon(t) >= \sum_{i,j} a_{ij}^{(\epsilon)}(t)|x_i > |y_j >$ is the quantum state after t imperfect iterations. Obviously, this quantity is very sensitive to the relative phases of a_{ij}. Since the classical phase space density is not sensitive to these phases, we introduce another characteristic which is related only to the amplitudes $|a_{ij}|$. We call it *faithfulness* and define it by: $f_a(t) = (\sum_{i,j} |a_{ij}^{(\epsilon)}(t) a_{ij}^{(0)}(t)|)^2$. This quantity can be considered as a generalization of the usual fidelity. As well as $f(t)$, the faithfulness $f_a(t)$ is always ≤ 1, and it determines the deviation from the exact amplitudes (the value 1 is reached only for $|a_{ij}^{(\epsilon)}(t)| = |a_{ij}^{(0)}(t)|$ for all

i, j). In addition, one has always $f_a(t) \geq f(t)$. Contrary to the usual fidelity, the faithfulness measures only amplitude errors, being insensitive to the quantum phases. We note that its definition is related to the preferential basis chosen initially in the Hilbert space.

To study the dependence of fidelity and faithfulness on phase errors, the nondiagonal part of each Toffoli and CNOT gate used in the algorithm was multiplied by a diagonal matrix with elements $\exp(i\theta_m)$, with random phases θ_m homogeneously distributed in $[-\epsilon_\phi, \epsilon_\phi]$. The results are shown in Fig. 1 (Left). Here the initial state was chosen in the form of a cat's smile (see Fig. 1 of [7] and the coarse-grained version in Fig. 2). In the presence of phase errors only, the fidelity f decreases with the number of map iterations and drops almost to zero for sufficiently strong phase noise. At the same time, the faithfulness f_a is not affected even by the maximal possible phase noise. We also checked that $f_a = 1$ is not affected if each a_{ij} is multiplied after each gate by a random phase $\exp(i\theta_m)$ with θ_m distributed in $[-\pi, \pi]$, although in this case the fidelity is almost zero after one map iteration.

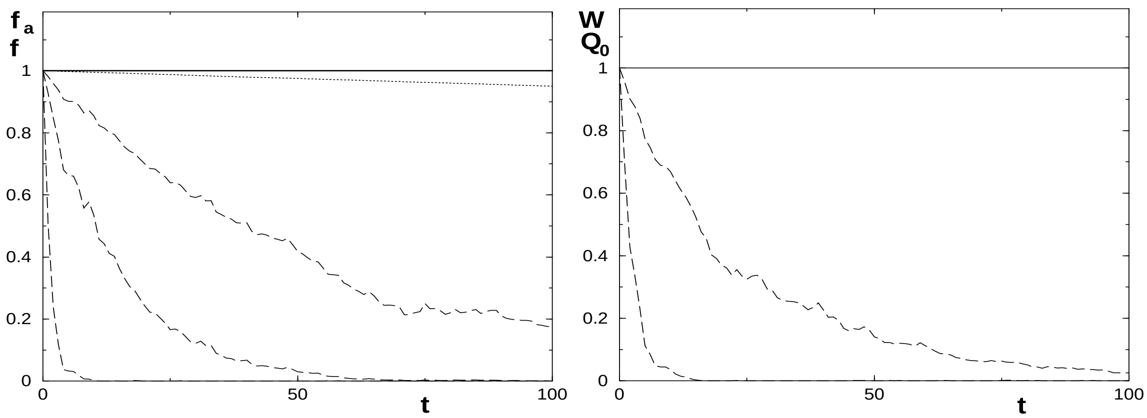

Figure 1: LEFT: Quantum fidelity f of Arnold-Schrödinger cat as a function of time t for quantum phase errors of strength $\epsilon_\phi = 0.05, 0.1, 0.3$ (dashed curves from top to bottom). Faithfulness f_a is shown for the maximal phase errors with $\epsilon_\phi = \pi$ (full line). The dotted line shows $f_a(t)$ when in addition to maximal phase errors there are small amplitudes errors of strength $\epsilon = 0.01$. Initial state is chosen in the form of a cat's smile on a 128×128 lattice (see text and Fig. 2), and $n_q = 7$. In total, 20 qubits are used for the computation.
RIGHT: Zero harmonic Q_0 of Arnold-Schrödinger cat normalized by its value in absence of errors as a function of time t for quantum phase errors of strength $\epsilon_\phi = 0.07, 0.2$ (dashed curves from top to bottom). Full line shows the total probability W inside one cell (i'_g, j'_g) (see text) normalized in the same way for the maximal phase errors with $\epsilon_\phi = \pi$. Initial condition is chosen as for Left, $n_q = 7$ and $n_g = 5$.

To show the difference between phase and amplitude (bit) errors, we computed the faithfulness in presence of a small unitary noise in the gates. For that, in addition to large phase errors, for each gate the nondiagonal part was diagonalized, and each eigenvalue was multiplied by a random phase $\exp(i\eta)$, with $-\epsilon < \eta < \epsilon$. This introduces both phase and amplitude errors, and Fig. 1 (Left) shows that the faithfulness starts to drop slowly with time. Hence despite the presence of strong phase errors the faithfulness is sensitive only to the amplitude errors.

Thus the information stored in the amplitudes is not sensitive to phase decoherence. Still, one should find a way to retrieve a part of this exponentially large information. Usually one performs a quantum Fourier transform (QFT) and measures the maximal harmonics of the distribution, as was suggested in [7]. However, the QFT is extremely sensitive to the quantum phases of a_{ij}, as is illustrated in Fig. 1 (Right). The zero harmonic $Q_0(t) = \sum_{i,j} a_{ij}(t)/N = \sqrt{N_d}/N$ is time-independent in the absence of errors, but drops rapidly with t if phase errors are present. To avoid the effects of phase errors, one can measure only the first n_g qubits from the n_q qubits present in the register $|x>$ and the same for the register $|y>$. This procedure introduces a coarse-graining of the phase space (x, y), with the number of cells $N_g = 2^{2n_g}$. The result of such a measurement is determined by the total probability inside each cell $W_{i_g j_g} = \sum_{<i,j>} |a_{ij}|^2$

where the summation is performed over all (i,j) inside the cell (i_g,j_g). This probability is not sensitive to phase errors, and can be extracted by a number of measurements which is polynomial in N_g. Fig. 1 (Right) shows that the probability in a chosen cell (i'_g, j'_g) is indeed insensitive to phase errors. We note that this coarse-grained probability is a very natural quantity for the dynamical system under investigation. One is not interested in the exponential amount of information present in all a_{ij} since one cannot even store it classically, and therefore it is better to operate with coarse-grained characteristics as is usually done in chaotic dynamical systems. The number of cells N_g can be kept constant while the number of iterated classical orbits increases exponentially with n_q. All these orbits are iterated accurately and with exponential efficiency during quantum computation, and the constant number of cells N_g is used only to extract the essential information generated by this chaotic dynamics.

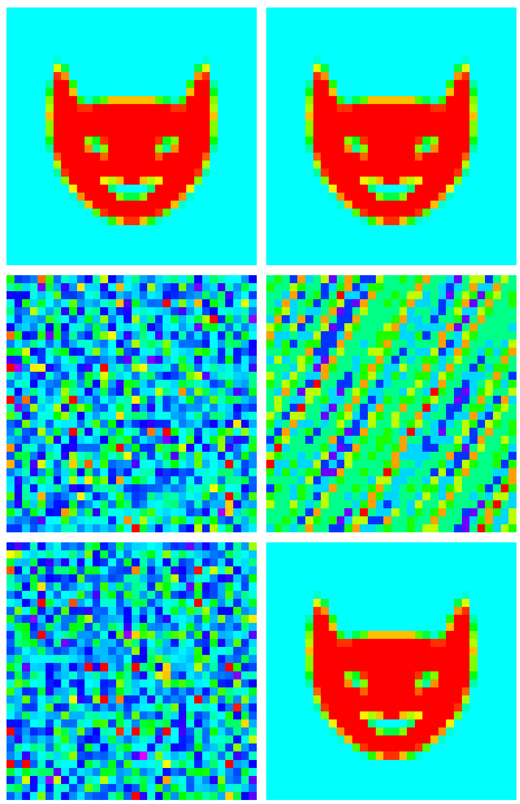

Figure 2: Coarse-grained image of Arnold-Schrödinger cat measured through the probabilities $W_{i_g j_g}$ at different moments of time $t=0$(upper panels), $t=50$ (middle panels), $t=100$ (lower panels). Color is proportional to $W_{i_g j_g}$, from blue (minimum) to red (maximum). Time inversion is done at $t=50$. The quantum computation is done with phase errors of amplitude $\epsilon_\phi = \pi$ for both columns. In addition, for the left column the computation includes also amplitude errors of strength $\epsilon = 0.3$. The initial state is as in Fig.1, and $n_q = 7$. The coarse-graining corresponds to measuring the first five qubits ($n_g = 5$) in the $|x>$ and $|y>$ registers.

Another test of the effects induced by phase and amplitude errors can be performed on the basis of time-inversion. Indeed, the exact map (1) is exactly time-invertible. However, in presence of imperfections this reversibility can be destroyed. In the régime of classical chaos, the classical round-off errors grow exponentially with time and destroys time-reversibility in a logarithmically short time. For quantum simulations, it was shown in [7] that quantum errors grow only polynomially with time. Due to that, time-reversibility is preserved in quantum computation of (1) for relatively small errors. However, it is naturally expected to be destroyed in the case of strong errors. Contrary to this expectation, Fig.2 (Right) shows that time-reversibility is exactly preserved in the presence of phase errors of maximal amplitude, and the classical distribution is exactly reproduced for all t. In the right column, the difference between the two images at $t = 0$ and $t = 100$ is on the level of classical computer precision. On the contrary, strong enough amplitude errors completely destroy time-reversibility, as is shown on Fig.2 (Left).

4 Discussion

Thus, all the data clearly show that our quantum algorithm simulating (1) is insensitive to phase errors. This result can be understood in the following way. All nondiagonal parts of the gates used in the algorithm are represented by the operator σ^x, while the noncommuting part of phase errors is represented by σ^z. Of course, σ^x and σ^z do not commute. However, the action of σ^x, $\sigma^z\sigma^x$ and $\sigma^x\sigma^z$ on a two-component spinor gives the same amplitudes of the components (with different relative phases). Thus any quantity encoded in the amplitudes, in our case the classical distribution function, remains invariant in presence of σ^z (phase) errors. On the contrary, it is sensitive to σ^x (amplitude) errors. Another way of understanding this insensitivity to phase errors is to remark that all used gates belong to a very specific subgroup among unitary transformations of the Hilbert space, that is the group of permutations of the basis formed by the states where each qubit is polarized in the z direction (each qubit is either $|0>$ or $|1>$). The amplitudes in this basis are insensitive to phase errors if only such transformations are present in the algorithm. Indeed, any permutation can be written as a product of transpositions which exchange only two states. By the same argument as for $\sigma^x\sigma^z$ given above, such a transformation is immune to phase errors, and hence any permutation. We stress again that phase errors do affect the final state through the relative phases, but do not affect the measurement which gives the cell probabilities $W_{i_g j_g}$.

The above mathematical argument explains the insensitivity to phase errors. In a more physical way, we can say that the map (1) describes the classical dynamics of the Arnold cat map, which naturally should not be sensitive to quantum phases. Of course, one can imagine other quantum algorithms which will simulate this classical dynamics using both phases and amplitudes of the wave function, and therefore will be sensitive to phase errors. However, on the basis of the Arnold-Schrödinger cat algorithm discussed in this paper, we make the conjecture that classical Hamiltonian dynamics of generic systems can always be simulated on a quantum computer in a way insensitive to phase errors. Indeed, for such a dynamics the classical information can be naturally encoded in the amplitudes only [16]. It is rather likely that such a situation can appear in quantum computations which are not connected with classical mechanics, for example probing the range of values of a function.

The implementation of such algorithms insensitive to phase errors can be enormously simpler than in the case of other algorithms sensitive to quantum phases. Indeed, the necessity to correct both phase and amplitude errors significantly complicates quantum error-correcting codes [10,11,12]. If only amplitude errors are to be corrected, one can use much simpler codes close to the classical ones. Also, in some physical systems phase errors can be naturally much stronger than amplitude ones. For example, recent studies of the emergence of quantum chaos in a quantum computer [17] showed that for sufficiently strong residual inter-qubit interaction, exponentially many states are mixed and amplitude errors become enormously strong. On the contrary, below the quantum chaos border, amplitude errors are very small whereas phase errors are still important. In spite of that, a quantum computer in this régime can efficiently simulate algorithms of the type discussed here.

We note that while the algorithm presented above is exponentially faster than any deterministic classical algorithm iterating the map (1) nevertheless one can try to compete with it with the help of classical Monte Carlo simulation with a polynomial number of trajectories. Such an approach does not produce the exact density distribution with an exponential number of orbits which is hidden in the quantum wavefunction. However, the statistical accuracy of both methods can be comparable since one makes a polynomial number of measurements of the quantum final state. At the same time one should keep in mind that such a Monte Carlo simulation is based on the statistical assumption that a polynomial number of trajectories can correctly describe the fine structure of classical phase space. In contrast, the quantum simulation takes exactly into account the dynamics on *all* scales. We also stress that without large phase errors the QFT gives access to information which is unaccessible even for classical Monte Carlo algorithms.

In conclusion, we have shown the existence of quantum algorithms which can simulate efficiently certain computational problems and at the same time are insensitive to phase errors. Our explicit example is related to the simulation of classical motion and we make the conjecture that classical Hamiltonian dynamics can always be simulated in a way immune to phase decoherence. The existence of such efficient quantum algorithms insensitive to the relative phases shows that contrary to the common lore, the massive parallelism of quantum computing is not necessarily related to quantum interference. Actually, quantum mechanics allows to follow in parallel exponentially many computational paths in a way insensitive to phase decoherence.

Acknowledgments

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