QUANTUM INFORMATION AND DECOHERENCE IN NANOSYSTEMS

edited by

D. Christian Glattli
Marc Sanquer
Jean Trần Thanh Văn
Sponsored by

- CNRS (Centre National de la Recherche Scientifique) - Formation permanente.
- CEA (Commissariat à l'Énergie Atomique) - Direction des sciences de la matière : DRFMC (Grenoble) et DRECAM (Saclay).

---

The Gidi Publishers
Printed in Vietnam
VN-TG-11325.1

XXXIXth Rencontres de Moriond
La Thuile, Aosta Valley, Italy - January 25 - February 1st, 2004

Quantum Information and Decoherence in Nanosystems
Series: Moriond Workshops

© Copyright 2004 by Rencontres de Moriond
All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the Publisher.
Can Quasi-Particles Tunnel Through A Barrier?
Y. Gefen, E. Shopen and Y. Meir
Evidence for many-body behaviour at $0.7 \, 2e^2/h$
J. T. Nicholls
Kondo model for the "0.7 anomaly" in transport through a Quantum Point Contact
Y. Meir, K. Hirose, N. S. Wingreen
A paramagnon theory of the 0.7 anomaly
A. Khaetskii, Y. Tokura
Transport and noise properties of quantum dots coupled to interacting one-dimensional electrodes
A. Komnik and A. O. Gogolin
Luttinger liquid behavior in crossed metal single-wall nanotubes
A. Bachtold, B. Gao, D. C. Glattli, A. Komnik, R. Egger
Transport properties of an interacting quantum wire with an impurity: effects of the finite length
F. Dolcini, B. Trauzettel, I. Safi, H. Grabert
A one-channel conductor in an ohmic environment: mapping to a TLL and full counting statistics
I. Safi, H. Saleur

III Quantum Entanglement and Information Processing with Mesoscopic Systems
Fighting decoherence in a Josephson qubit circuit
E. Collin, G. Ithier, P. Joyez, D. Vion and D. Estève
Josephson chains as quantum channels
A. Romito, C. Bruder, R. Fazio
Macroscopic tunneling of quantum bits
J. Ankerhold and H. Grabert
Nonlinear Effects in Superconducting Stripline Resonators
E. Buk
Determination of the tunnel rates through a few-electron quantum dot
R. Hanson, I. T. Vink, D. P. DiVincenzo, L. M. K. Vandersypen, J. M. Elzerman,
L. H. W. van Beveren and L.P. Kouwenhoven
Concurrence in a simple quantum phase transition
J. Vidal and R. Mosseri
Observation of Multiphoton Absorption and Switching Current Behaviors in Superconducting
Flux-Qubit Readouts
H. Takayanagi, H. Nakano, H. Tanaka, S. Saito, K. Semba, and M. Ueda
Charge detection in Quantum Dots
R. Schleser, E. Ruh, L. Meier, A. Fuhrer, T. Ihn, K. Ensslin, D. D. Driscoll, A. C. Gossard,
and W. Wegscheider
Universal regime of fidelity decay in realistic quantum computation
K. M. Frahm, R. Fleckinger, D. L. Shepelyansky
Coherent charge oscillation in a semiconductor double quantum dot
T. Hayashi, T. Fujiwara, Y. Hirayama
Realistic operation of an electron entangler: a density matrix approach
O. Sauret, D. Feinberg, and T. Martin
Bloch Oscillating Transistor and Coulomb blockade of Cooper pairs
J. Delahaye, J. Hassel, R. Lindell, M. Sillanpää, M. Paalanen, H. Sepp, P. Hakonen
Dynamics of the Inductive Single-Electron Transistor
M. A. Sillanpää, Leif Roschier, and P. J. Hakonen
UNIVERSAL REGIME OF FIDELITY DECAY
IN REALISTIC QUANTUM COMPUTATIONS

KLAUS M. FRAHM, ROBERT FLECKINGER, DIMA L. SHEPELYANSKY
Laboratoire de Physique Théorique, UMR 5152 du CNRS,
Université Paul Sabatier, 31062 Toulouse Cedex 4, France

We determine the universal scaling law for fidelity decay in quantum computations of complex
dynamics in presence of internal static imperfections in a quantum computer. Based on
random matrix theory we show that this decay is governed by an exponential decay with
Fermi’s golden rule decay rate for time scales smaller than the Heisenberg time and a gaussian
decay for larger time scales. The theoretical predictions are tested and confirmed in extensive
numerical simulations of a quantum algorithm for quantum chaos in the dynamical tent map
with up to 18 qubits and with ten orders of magnitude for the relevant scaled fidelity interval.

Recently a great deal of attention has been attracted \(^1\) to the problem of quantum com-
putation. A quantum computer is viewed as a system of qubits. Each qubit can be considered
as a two-level quantum system, e.g. one-half spin in a magnetic field. For \(n_q\) qubits the whole
system is characterized by a finite-dimensional Hilbert space with \(N = 2^{n_q}\) quantum states. It
has been shown that all unitary operations in this space can be realized with certain elementary
quantum gates \(^3\) acting on one or two particular qubits. A quantum computation can be much
faster than a classical one due the massive parallelism of many-body quantum mechanics since
any step of a quantum evolution is a multiplication of a vector by a unitary matrix. There
are computational algorithms that can be represented as a sequence of such elementary gates
involving only a polynomial number (in \(n_q\)) of gates such as the Shor algorithm \(^5\) for factorization
of integers with \(n_q\) digits, the Grover algorithm \(^6\) for a search in an unstructured database
and the quantum Fourier transform \(^1\) (QFT). With the help of QFT the quantum evolution of
certain many-body quantum systems can be performed in a polynomial number of gates \(^7\).
Other examples can be found in the evolution of quantum dynamical systems which are chaotic
in the classical limit \(^9\). Such systems are described by chaotic quantum maps and include
the quantum baker map \(^1\), the quantum kicked rotator \(^1\), the quantum saw-tooth map \(^3\) and
the quantum double-well map \(^14\). We furthermore mention the quantum simulation of the An-
derson metal-insulator transition \(^15\) and the study of classical chaotic dynamics where quantum
computation provides a more efficient access to some new information \(^16\).

For potential experimental implementations of a quantum computer one has to take into ac-
count errors caused either by decoherence induced by unavoidable couplings to external world \(^18\)
or by internal static imperfections inside the quantum computer. These static imperfections
generate residual couplings between qubits and variation of energy level-spacing from one qubit
to another. Such imperfections lead to emergence of many-body quantum chaos in a quantum
computer hardware if a coupling strength exceeds a quantum chaos threshold. \(^19\) To analyze the
effects of errors one may consider the fidelity \(f\) defined as \(f(t) = | < \psi_{err}(t) | \psi(t) > |^2\) where
\(|\psi(t)\rangle\) is the quantum state at time \(t\) computed with perfect (or ideal) gates, while \(|\psi_{err}(t)\rangle\) is
the quantum state at time \(t\) computed with errors. If the fidelity is close to unity then a quantum
computation with imperfections is close to the ideal one while if \(f\) is significantly smaller than
1 then the computation gives, generally, wrong results. The fidelity decay for generic quantum
evolution with different models of dynamical evolution and perturbations has attracted a con-
siderable interest over the last years (see e.g. \([20,21]\) and references therein). In this work we
consider the fidelity decay in quantum computation due to static imperfections a case previously
studied only in a few works. \(^{13,22,21}\) We present analytical results of a random matrix approach
valid for a regime of complex quantum dynamics and a numerical study of a particular quantum
map with 18 qubits.
We consider a kicked rotator whose classical dynamics is governed by the map,
\[
\begin{align*}
\tilde{p} &= p - V'(\theta) \\
\tilde{\theta} &= \theta + \tilde{p} T \mod 2\pi, \\
V(\theta) &= \begin{cases} \\
-\frac{k}{2} \theta (\theta - \pi) & 0 \leq \theta < \pi \\
\frac{k}{2} \theta (\theta - \pi) (\theta - 2\pi) & \pi \leq \theta < 2\pi
\end{cases}
\end{align*}
\] (1)

Here the kick-potential \( V(\theta) \) is composed of two parabolic pieces. The parameter \( k \) determines the kick strength and \( T \) gives the rotation of phase between kicks. This map is similar in structure to the Chirikov standard map. The derivative \( V'(\theta) \) has a tent form and is continuous but not differentiable at \( \theta = 0 \) and \( \theta = \pi \). This is an intermediate case between the standard map with a perfectly smooth kick-potential and the saw-tooth map with a non-continuous potential derivative. The dynamics of the classical tent map (1) depends only on one dimensionless parameter \( K = kT \), its properties have been studied in [25,26]. For small values of \( K \) the dynamics is governed by a KAM-scenario with the Kolmogorov-Arnold-Moser (KAM) invariant curves and a stable island at \( \theta = 3\pi/2 \), \( p = 0 \) and a chaotic layer around separatrix starting from the unstable fixed point (saddle) at \( \theta = \pi/2 \), \( p = 0 \). At \( K = 4/3 \), the last invariant curve is destroyed and one observes a transition to global chaos with a mixed phase space containing big regions with regular dynamics. In the following, we are particularly interested in a typical case \( K = 1.7 \), which exhibits global chaos with quite large stable islands in phase space related to the main and secondary resonances.

The quantized version of the classical map (1) is given by the unitary quantum map,
\[
|\psi(t+1)\rangle = U |\psi(t)\rangle = e^{-iT\tilde{p}^2/2} e^{-iV(\tilde{\theta})} |\psi(t)\rangle.
\] (2)

where the \( |\psi(t)\rangle \) is the quantum state at the (integer) time \( t \) and the variables \( \tilde{p} \) and \( \tilde{\theta} \) are operators with the commutator \([\tilde{p}, \tilde{\theta}] = -i\). They have integer eigenvalues \( p \) for \( \tilde{p} \) and real eigenvalues \( \theta \) in the interval \([0, 2\pi]\) for \( \tilde{\theta} \). Furthermore \( \hbar = 1 \) and the quasiclassical limit correspond to \( T \to 0 \), \( k \to \infty \) with \( K = kT = \text{const} \). For the quantum dynamics we concentrate our studies on the case \( K = kT = 1.7 \) and \( T = 2\pi/N \) that corresponds to the evolution on one classical cell (see Fig. 1) with \( N \) quantum states.

The quantum map (2) can be efficiently simulated on a quantum computer. For this we represent the quantum state \( |\psi(t)\rangle \) by a quantum register with \( n_q \) qubits with a total number of \( N = 2^{n_q} \) different basis states. The eigenstates of \([p, \theta]\) are identified with the quantum register states \( |\alpha_0\rangle \otimes |\alpha_1\rangle \otimes \cdots \otimes |\alpha_{n_q-1}\rangle n_{n_q-1} \) with \( p = \sum_{j=0}^{n_q-1} \alpha_j 2^j \in \{0, \ldots, N-1\} \) and \( \alpha_j \in \{0, 1\} \). Here \( |0\rangle_j \) and \( |1\rangle_j \) correspond to the two basis states of the \( j \)-th qubit.

For quantum computations one typically assumes that the quantum computer can be constructed with quantum gates that manipulate at most two qubits such as the simple phase-shift, controlled phase-shift, controlled-NOT and the Hadamard gate.\(^3,4\) Without entering into the details we mention that it is possible\(^21\) to express the unitary operator \( U \) in the quantum map (2) in terms of \( n_q = \frac{3}{2} n_q^2 - \frac{1}{2} n_q + 4 \) elementary quantum gates. For a moderate number of qubits \((n_q = 10 \sim 20)\) it is possible to test the quantum algorithm on a classical computer.\(^21\) For example in the first and third panel of Fig. 1, we show the density plot of the Husimi function of a quantum state which was obtained by such a quantum computation of the quantum map (2) after a large number of iterations. As an initial state we have used a minimal coherent wave packet placed in the chaotic or integrable component (near the unstable fix point at \( \theta = \pi/2 \), \( p = 0 \) for the first panel and in the main stable island at \( \theta = 5.35 \), \( p = 0 \) for the third panel). In both cases the classical behavior is well reproduced, i. e. in the first case the chaotic region is ergodically filled up while for the regular initial condition the wave packet stays close to the classical invariant curve.

We have investigated the stability of the quantum algorithm for the tent map with respect to errors. For this we note that quantum algorithm of the tent map\(^21\) is given by the expression \( U = \prod_{j=1}^{n_q} U_j \) where \( U_j \) are the unitary operators associated to the elementary gates and \( n_q \) is

168
the number of these gates. The errors are modeled by the replacement $U_j \rightarrow U_j e^{i\delta H_j}$ where $\delta H_j$ is a hermitian operator representing a perturbation. There are two models of imperfections. The first one represents random noise errors in quantum gates fluctuating in time from one gate to another with $\delta H_j \sim \varepsilon$ random and different at each $j$ and $t$. In this case the fidelity decay is clearly exponential $^{21}$ for very long time scales, $f(t) = \exp(-t/t_r)$ with $t_r = 1/(0.095\varepsilon^2 n_\sigma^2)$. The second model describes only static imperfections. $^{19,13,22,15}$ In this case we chose $\delta H_j = \delta H$ independant of $j$ and $t$ with: $\delta H = \sum_{j=0}^{n_q-1} \delta_j \sigma_j^{(z)} + 2 \sum_{j=0}^{n_q-2} J_j \sigma_j^{(z)} \sigma_{j+1}^{(z)}$ where $\sigma_j^{(z)}$ are the Pauli matrices acting on the $j$th qubit and $\delta_j, J_j \in [-\sqrt{3}\varepsilon, \sqrt{3}\varepsilon]$ are random coefficients which are drawn only once at the beginning and kept fixed during the simulation. In the second and fourth panel of Fig. 1 we show the Husimi functions of two quantum states obtained by a quantum computation with such static imperfections.

We now introduce the effective operator $\delta H_{\text{eff}}$ for the full static error at one complete iteration with the quantum map (2) by: $\prod_{j=1}^{n_q} (U_j e^{i\delta H}) = U e^{i\delta H_{\text{eff}}}$. As it was shown by Prosen et al. $^{20}$, in the limit $(1-f) \ll 1$ one can express the fidelity in terms of a correlation function, $^{20,21}$

$$f(t) \approx 1 - \frac{t}{t_c} - \frac{2}{t_c} \sum_{\tau=1}^{t-1} (t-\tau) C(\tau), \quad C(\tau) = t_c \langle U^{-\tau} \delta H_{\text{eff}} U^\tau \delta H_{\text{eff}} \rangle_Q.$$  

(3)

Here $(\cdots)_Q$ denotes the quantum expectation value and the time scale $t_c = 1/((\delta H_{\text{eff}})^2)_Q$ ensures the normalisation $C(0) = 1$ of the correlation function.

For an initial state in the chaotic region we may assume that the unitary quantum map $U$ can be modeled by a random matrix drawn from Dyson’s circular orthogonal ensemble. $^{23}$ Performing the ensemble average with respect to $U$, we obtain the following scaling expression for the fidelity: $^{21}$

$$-(\ln f(t))_U \approx \frac{N}{t_c} \chi \left( \frac{t}{N} \right), \quad \chi(s) \approx s + 2s^2 \Rightarrow f(t) \approx \exp \left( -\frac{t}{t_c} - \frac{2}{\sigma t_c t_H} \frac{t^2}{t_H} \right)$$  

(4)

where $N$ corresponds to the number of “chaotic” states fixing the dimension of the random matrix $U$. In (4) we have replaced $N = \sigma t_1$ where $\sigma$ is the fraction of the chaotic part of phase space ($\sigma \approx 0.65$ for $K = 1.7$). This result gives a clear transition from exponential to gaussian decay at $t \approx t_H$ if $t_c \ll t_H$. If $t_c \gg t_H$ the exponential regime is barely visible and dominated by the gaussian decay. We have numerically verified this scaling behavior for $n_q$ between 6 and 18,
for $\varepsilon$ between $10^{-4}$ and $5 \cdot 10^{-7}$ and for 10 orders of magnitude of the rescaled fidelity (see Fig. 2). We have performed a scaling analysis of the fidelity for the chaotic and regular initial condition.

Figure 2: Left: scaling representation of the fidelity $f$ for the initial condition in the chaotic region. The upper scaling curve shows: $-\ln(f)/\varepsilon/t_H$ as a function of $t/t_H$ with the two theoretical time scales $t_c = (\varepsilon^2 n_q n_g^2)^{-1}$ and $t_H = 2^{n_g}$. The full line in the upper curve corresponds to the scaling curve (4). The lower scaling curve (shifted down by a factor 0.01) correspond to $-\ln(f)/\varepsilon/t_H$ versus $t/t_H$ with the times scales $t_c$ and $t_H$ obtained from the fit $y = x + x^2$ (with $y = -\ln(f)/\varepsilon/t_H$ and $x = t/t_H$) for each value of $n_g$ and $\varepsilon$. Full line is $y = x + x^2$. Middle: $t_H$ versus $t_\varepsilon$ for the chaotic initial condition for $6 \leq n_g \leq 18$, $\varepsilon = 5 \cdot 10^{-7}$ [data points (a)] and $\varepsilon = 10^{-6}$ [data points (b)]. The data points (c) are obtained from an ensemble average over 200 realizations of static imperfections for each value of $n_g = 6, 8, 10, 12$, and $14$ and $\varepsilon = 5 \cdot 10^{-7}$. The full line corresponds to the theoretical expression $t_H = (\sigma/2)t_H$. Right: $t_H$ versus $t_\varepsilon$ for the initial condition in the integrable region.

In particular, we have determined for each value of $n_g$ and $\varepsilon$ two time scales $t_\varepsilon$ and $t_H$ by a numerical least square fit of $\ln(f(t)) = -t/t_\varepsilon - t^2/(t_\varepsilon t_H)$ (with appropriate weight factors). We find that for both types of initial conditions this fits works quite well and that the first time scale $t_\varepsilon$ coincides quite well with its theoretical expression:** $t_\varepsilon = 1/(\varepsilon^2 n_q n_g^2)$. The situation is different for the second time scale $t_H$ which coincides with its theoretical value $(\sigma/2)t_H$ only for the chaotic initial condition (see middle and right part of Fig. 2). This observation is in agreement with a natural expectation that random matrix theory is not applicable to regular dynamics.

The scaling result (4) provides a universal description of the fidelity decay in quantum algorithms simulating complex dynamics on a realistic quantum computer with static imperfections. This decay determines the time scale $t_f$ of reliable quantum computation [defined by $f(t_f) = 0.9$] according to $t_f \approx t_c/10$ for $t_f > t_c$ (corresponding to $\varepsilon > \varepsilon_{ch} = 2^{-n_g/2}/(n_g \sqrt{n_q})$) or $t_f \approx 0.2\sqrt{t_c t_H}$ for $t_f < t_c$ ($\varepsilon < \varepsilon_{ch}$). Therefore the total number of gates which can be performed with fidelity $f > 0.9$ is given by $N_g = t_f n_g \approx 1/(10\varepsilon^2 n_q n_g)$ (for $\varepsilon > \varepsilon_{ch}$) or $N_g \approx 2^{n_g/2}/(5\varepsilon \sqrt{n_q})$ (for $\varepsilon < \varepsilon_{ch}$). The first case compares to the behavior for random errors:** $N_g \approx 5/\varepsilon^2$ with the same dependence on $\varepsilon$ while the second case, which may be dominant for 10-15 qubits, is completely different. This difference should play an important role for the quantum error correction codes.  

This work was supported in part by the EC IST-FET project EDIQIP and the NSA and ARDA under ARO contract No. DAAD19-01-1-0553 and by the French government grant ACI Nanosciences-Nanotechnologies LOGIQUEANT. We thank CalMiP at Toulouse and IDRIS at Orsay for access to their supercomputers.

References

5. P.W. Shor, in Proc. 35th Annual Symposium on Foundation of Computer Science, Ed.
    (2002).