Fractal Weyl law for quantum fractal eigenstates

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The properties of the resonant Gamow states are studied numerically in the semiclassical limit for the quantum Chirikov standard map with absorption. It is shown that the number of such states is described by the fractal Weyl law and their Husimi distributions closely follow the strange repeller set formed by classical orbits nonescaping in future times. For large matrices the distribution of escape rates converges to a fixed shape profile characterized by a spectral gap related to the classical escape rate.

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The Weyl law [1] gives a fundamental link between the properties of quantum eigenstates in closed Hamiltonian systems, the Plank constant $\hbar$ and the classical phase space volume. The number of states in this case is determined by the system dimension d and the situation is now well understood both on mathematical and physical grounds [2, 3]. Surprisingly, only recently it has been realized that the case of nonunitary operators describing open systems in the semiclassical limit has a number of new interesting properties and the concept of the fractal Weyl law has been introduced to describe the dependence of number of resonant Gamow eigenstates on $\hbar$ [4, 5]. The Gamow eigenstates find important applications for decay of radioactive nuclei [6], quantum chemistry reactions [7], chaotic scattering [8] and microlasers with chaotic resonators [9, 10, 11]. Thus the understanding of their properties in the semiclassical limit represents an important challenge.

According to the fractal Weyl law [4, 5] the number of Gamow eigenstates N_γ, which have escape rates γ in a finite band width $0 \leq \gamma \leq \gamma_b$, scales as

$$N_\gamma \propto \hbar^{-(d-1)} \qquad (1)$$

where d is a fractal dimension of a classical strange repeller formed by classical orbits nonescaping in future (or past) times. By numerical simulations it has been shown that the law (1) works for the 3-disk system [5] and quantum chaos maps with absorption [12, 13] at specific values of d. Recent mathematical results for open quantum maps are presented in [14]. The law (1) for open systems with a fractal dimension $d < 2$ leads to a striking consequence: only a relatively small fraction of eigenstates $\mu \sim N_\gamma/N \propto \hbar^{(2-d)}$ have finite values of γ while almost all eigenstates of matrix operator of size $N \propto 1/\hbar$ have infinitely large γ. The later states are associated [12] with classical orbits which escapes from the system after the Ehrenfest time [15]. The former states with finite γ are related to the classical fractal repeller and have been named quantum fractal eigenstates due to a fractal structure of their Husimi distributions closely following the classical fractal [16]. The semiclassical description of probability density for such states has been derived recently in [17].

In view of the recent results described above I study numerically a simple model of the quantum Chirikov standard map (kicked rotator) with absorption introduced in [18] which allows to vary continuously the fractal dimension of the classical strange repeller. In this way the fractal Weyl law (1) is verified in the whole interval $1 \leq d \leq 2$. The model also allows to establish the limiting semiclassical distribution over escape rates γ and find its links with the fractal properties of the classical strange repeller.

The quantum dynamics of the model is described by the evolution matrix:

$$\bar{\psi} = \hat{U}\psi = \hat{P}e^{-iT\hat{n}^2/4}e^{-ik\cos\hat{\theta}}e^{-iT\hat{n}^2/4}\psi, \qquad (2)$$

where $\hat{n} = -i\partial/\partial\theta$ and the operator $\hat{P}$ projects the wave function to the states in the interval $[-N/2, N/2]$. The semiclassical limit corresponds to $k \to \infty$, $T \to 0$ with the chaos parameter $K = kT = const$ and absorption boundary $a = N/k = const$. Thus N is inversely proportional to the effective Plank constant $T = \hbar_{eff}$, it gives the number of quantum eigenstates and the number of quantum cells inside the classical phase space. The classical dynamics is described by the Chirikov standard map [19] in its symmetric form:

$$\bar{n} = n + k\sin\left[\theta + \frac{Tn}{2}\right], \bar{\theta} = \theta + \frac{T}{2}(n + \bar{n}). \qquad (3)$$

In this model all trajectories (and quantum probabilities) leaving the interval $[-N/2, N/2]$ are absorbed and never return back. It is convenient to fix $K = 7$ so that the phase space have no visible stability islands for $a \leq 6$. Then for the classical dynamics the probability $P(t)$ to stay inside decays exponentially with time as $P(t) \sim \exp(-\gamma_c t)$ [18, 20], where γ_c is the classical escape rate and t is measured in the number of map iterations. For large values of a the spreading goes in a diffusive way and $t_c = 1/\gamma_c \sim N^2/D \approx 2a^2$ where $D \approx k^2/2$ is the diffusion rate for $K \gg 1$. The independence of t_c of N implies $a = N/k = const$. The quantum operator (2) can be considered as a simplified model of chaotic microlasers where all rays with orbital momenta below some

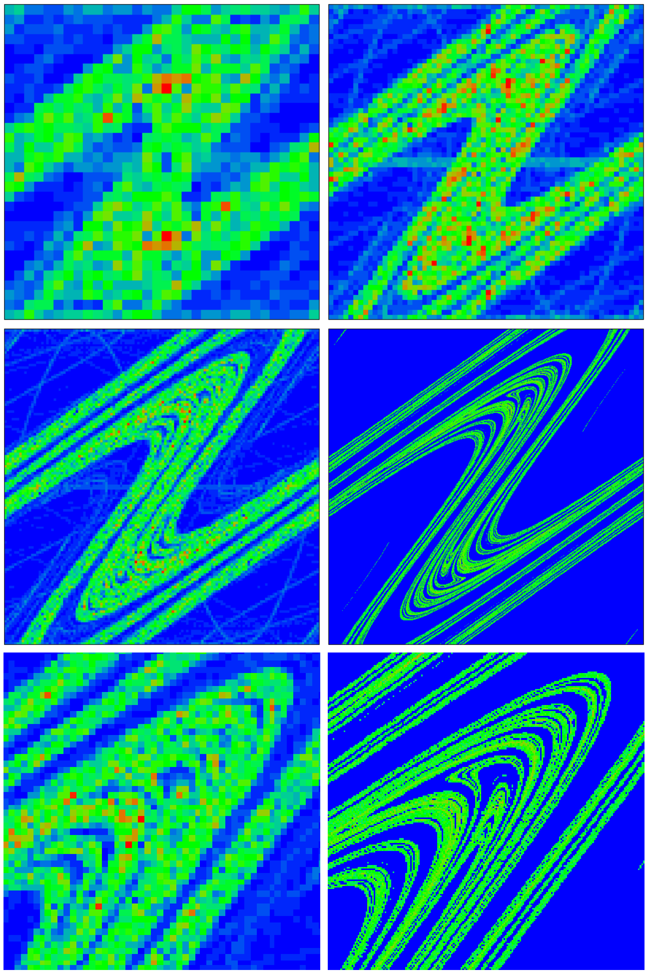

FIG. 1: (color online) Husimi functions of quantum fractal eigenstates with minimal value of γ at $N = 1025$ (top left), $N = 4097$ (top right), $N = 16349$ (middle left) and the density plot of classical strange chaotic repeller formed by orbits nonescaping forward in time (middle right); two bottom panels show zoom for two middle panels respectively. Here $a = N/k = 2$, $K = kT = 7$ and the box counting dimension of the repeller is $d = 1.7230$. In top and middle panels $0 \leq \theta < 2\pi$, $-N/2 \leq n \leq N/2$; density is proportional to color with red/gray for maximal density and blue/black for zero density.

critical value determined by the refraction index escape from a microcavity [9, 10, 11].

The right eigenstates $\psi_n^{(m)}$ and eigenvalues $\lambda_m = \exp(-i\epsilon_m - \gamma_m/2)$ of the evolution operator $\hat{U}$ are determined numerically by direct dioganalization up to a maximal value $N = 22001$ (only states symmetric in n are considered). The Husimi distribution [21], obtained from smoothing the Wigner function on the scale of the Planck constant, is shown in Fig. 1 for eigenstates with minimal γ_m at different values of N at $a = 2$. With the increase of N the Husimi distribution converges to a fractal set which is very similar to the classical strange repeller formed by classical orbits never escaping in the

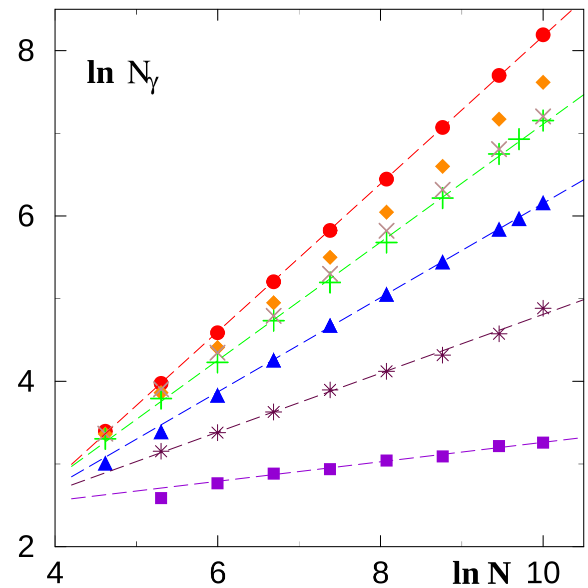

FIG. 2: (color online) Dependence of the integrated number of states N_γ with escape rates $\gamma \leq \gamma_b = 8/a^2$ on matrix size N. Symbols show numerical data for various values of absorption border $a = N/k$: 4 (circles), 2.5 (diamonds), 2 (+), 1.5 (×), 1 (triangles), 0.8 (*), 0.7 (squares). Dashed lines show algebraic fits $N_\gamma \propto N^\nu$ with the fractal Weyl exponent $\nu = 0.8930 \pm 0.0028$ ($a = 4$), $\nu = 0.7129 \pm 0.0073$ ($a = 2$), $\nu = 0.5697 \pm 0.0042$ ($a = 1$), $\nu = 0.3559 \pm 0.0094$ ($a = 0.8$) $\nu = 0.1175 \pm 0.0069$ ($a = 0.7$, here $\gamma_b = 4/a^2$). Logarithms are natural.

future. The classical repeller is obtained by iterating up to 3×10^9 classical trajectories homogeneously distributed in the whole phase space at $t = 0$. The classical remaining probability $P(t)$ decays exponentially with $\gamma_c = 0.2702 \pm 0.0011$ and the computation of the box counting dimension [22, 23] of the strange repeller gives $d = 1.7230 \pm 0.0085$. According to [8, 22, 23] the information dimension d_1 of the repeller can be expressed as $d_1 = 2 - \gamma_c/\Lambda$, where Λ is the Lyapunov exponent. For for large a and small γ_c it can be expressed via its value for the Hamiltonian dynamics on a torus where $\Lambda \approx \ln(K/2) = 1.2527$ (for $K = 7$) [19]. This gives $d_1 = 1.7843$ that is rather close to the numerical value of box counting dimension d (usually these two dimensions are rather close and, contrary to [13], I will not make difference between them). However, for smaller values of a the relation $\Lambda = \ln(K/2)$ is no more valid. To have Λ for all values of a its value is computed numerically following approximation used in [19]: $\Lambda = \langle \ln \mid K\cos(\theta + Tn/2) \mid \rangle$ where the average $\langle ... \rangle$ is done over the orbits on the repeller. In this way Λ varies in the interval $1.913 \leq \Lambda \leq 1.294$ for $0.7 \leq a \leq 6$ ($\Lambda = 1.363$ and $d_1 = 1.801$ at $a = 2$).

To check the validity of the fractal Weyl law (1) for various d the absorption border a is varied in the interval $0.7 \leq a \leq 6$ so that the classical fractal dimension

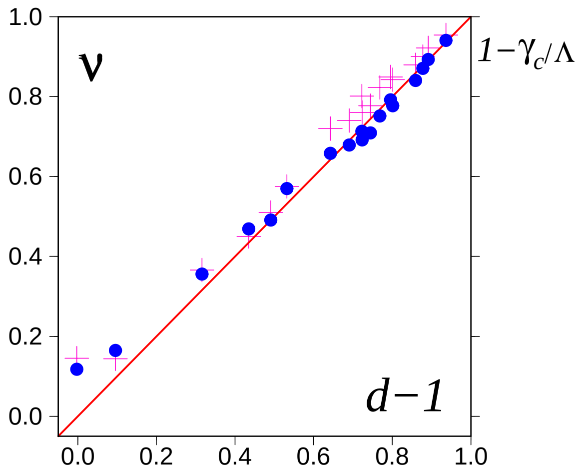

FIG. 3: (color online) Dependence of the fractal Weyl exponent ν on the fractal box counting dimension d: full circles show numerical data, the straight line shows the fractal Weyl law (1) $\nu = d - 1$, pluses show $1 - \gamma_c/\Lambda$ versus $d - 1$ which should follow the relation $d_1 - 1 = 1 - \gamma_c/\Lambda$, where Λ is the Lyaponov exponent computed approximately (see text). Here $0.7 \leq a \leq 6$, $0.1175 \leq \nu \leq 0.9402$, $0.9976 \leq d \leq 1.9367$.

and decay rate vary in the intervals $0.9976 \pm 0.0060 \leq d \leq 1.9367 \pm 0.0067$ and $1.6349 \pm 0.0135 \leq \gamma_c \leq 0.0592 \pm 0.0003$. In the quantum case the number of states N_γ is computed in the band width $0 \leq \gamma \leq \gamma_b$ with $\gamma_b = 8/a^2$. To improve the statistics, N_γ is averaged over N_r cases with slightly different values of $k \pm \delta k$ with $\delta k \leq 2$. Such a small variation of k does not affect the semiclassical properties but allows to improve the statistical accuracy. The number of realizations varied from $N_r = 40$ at $N = 101$ to $N_r = 1$ at $N = 22001$. The dependence of integrated number of states N_γ on $N \propto 1/\hbar$ is shown in Fig. 2. The fit $N_\gamma \propto N^\nu$ allows to determine the exponent ν which according to (1) should satisfy the relation $\nu = d - 1$. It is important to stress that $N_\gamma \ll N/2$ at $a \leq 2$, so that the main part of $(N+1)/2$ eigenvalues has enormously large $\gamma \gg \gamma_b$.

The dependence of ν on d is shown in Fig. 3. The law (1) is well satisfied for fractal dimensions $1 \leq d < 2$. Certain deviations for d close to 1 should be attributed to rather small values of N_γ (*e.g.* $N_\gamma = 26$ at $a = 0.7$ and $N = 22001$) so that even larger N values are required to see the asymptotic behavior. The relation $\nu = 1 - \gamma_c/\Lambda$ also works rather well even if the Lyapunov exponent Λ should be probably computed in a more exact way for small values of $a \sim 0.7$. Thus, the data of Fig. 3 confirms the validity of the fractal Weyl law in the whole available interval of fractal dimensions.

In addition to the integrated characteristics (1) it is interesting to consider the differential distribution $dW/d\gamma$ which determines the number of states in the interval $d\gamma$ at given γ. The evolution of distributions $dW/d\gamma$ with the growth of N is shown in Fig. 4. The data clearly

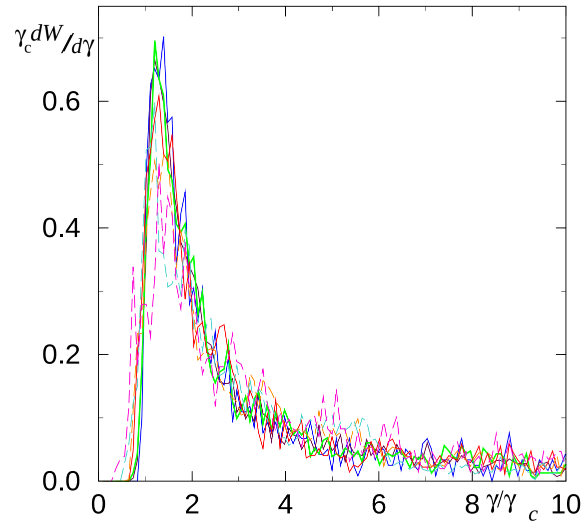

FIG. 4: (color online) Dependence of the distribution $\gamma_c dW/d\gamma$ on the rescaled escape rate γ/γ_c for different values of N at $a = 2$. Here $\gamma_c = 0.2702$, $d = 1.7230$, the probability $dW/d\gamma$ is normalized to unity in the interval γ_b and N is 22001 (blue/black full curve $N_\gamma = 1278$), 12801 (maroon/gray full $N_\gamma = 1022$), 6401 (green/gray full $N_\gamma = 500$), 3201 (red/gray full $N_\gamma = 293$), 1601 (orange/gray dashed $N_\gamma = 181$), 801 (turqse/gray dashed $N_\gamma = 114$), 401 (magenta/gray dashed $N_\gamma = 68.7$).

show that in the semiclassical limit $dW/d\gamma$ converges to a certain limiting distribution independent of N. Such an effect has been noticed already in first studies [18] where mainly the diffusive limit with $a = 10$ and $\gamma_c \ll \Lambda$ has been studied. In such a case the dimension is very close to the integer value $d = 2$ and due to that the fractal dependence (1) has been missed in [16, 18] even if the fractal structure of eigenstates has been clearly detected [16]. In the diffusive case $d \approx 2$ one has $dW/d\gamma \propto 1/\gamma^{3/2}$ for $\gamma > \gamma_c$ that is explained by simple estimates [18] and more rigorous analytical treatment [24]. When the fractal dimension d is noticeably less than 2 than $\gamma_c \sim \Lambda$ and the diffusive approximation is no more valid. A distinctive feature of the distribution in this case is the gap in the distribution $dW/d\gamma$ which is zero for $\gamma < \gamma_c$, sharp peak at $\gamma = \gamma_c$ followed by a smooth drop at $\gamma > \gamma_c$ (this drop is compatible with dependence $1/\gamma^{3/2}$).

These properties of the distribution $dW/d\gamma$ remain essentially the same when γ_c is changed by a factor 3.5 as it is shown in Fig. 5. Indeed, the shape of the distribution varies very little for $1.5 \leq a \leq 4$ and becomes broader only at $a < 1.5$. The later case have however relatively small statistics N_γ and probably larger N should be used to reach a limiting distribution for $a < 1.5$.

In conclusion, the obtained data confirm the validity of the fractal Weyl for all fractal dimensions in the interval $1 \leq d \leq 2$. They show the existence of the limiting distribution of the Gamow resonances $dW/d\gamma$ which has a gap of size γ_c at which the distribution has a sharp

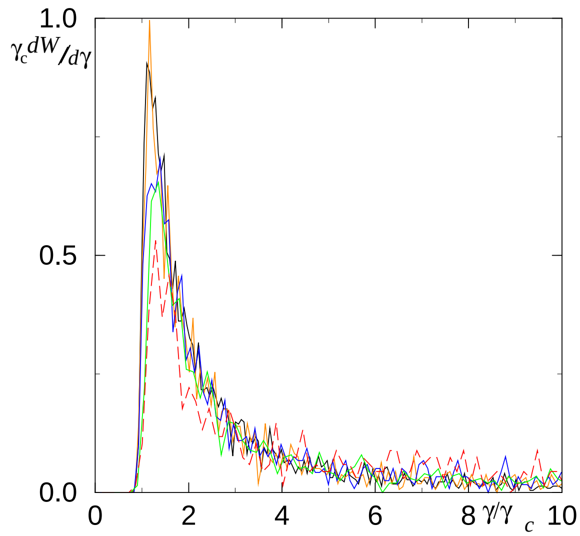

FIG. 5: (color online) Same distribution as in Fig. 4 drown for various absorption boundaries a at $N = 22001$. Here a is 4 ($\gamma_c = 0.1019$, $N_\gamma = 3607$, black curve); 2.5 ($\gamma_c = 0.2063$, $N_\gamma = 2032$, orange/gray curve); 2 ($\gamma_c = 0.2702$, $N_\gamma = 1278$, blue/black curve); 1.5 ($\gamma_c = 0.2961$, $N_\gamma = 1342$, green/gray curve); 1 ($\gamma_c = 0.6967$, $N_\gamma = 472$, red/gray dashed curve).

peak (see Figs. 4,5). Thus the classical decay rate γ_c essentially determines the quantum decay rates on the quantum fractal corresponding to the classical strange repeller with orbits never escaping in future times (Fig. 1). The analytical computation of the limiting distribution $dW/d\gamma$ still remains an open problem. It is possible that the analytical methods pushed forward recently [17] will allow to make progress in this direction. Also, it would be interesting to check the validity of the fractal Weyl law for dimensions $d > 2$. In such a case it is natural to expect that $N_\gamma \propto \hbar^{-\nu}$ with $\nu = d - n_f$ where d is the fractal dimension of the classical strange repeller and n_f is the number of degrees of freedom (in the present model $n_f = 1, d \leq 2$).

At present the properties of large nonunitary matrices find important applications in various areas including search on the Internet [25, 26] and it is possible that the fractal quantum eigenstates may have there certain applications since they give an example of important nontrivially connected fractal sets of small measure.

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