EPJ manuscript No.
(will be inserted by the editor)

Quantum phase estimation algorithm in presence of static imperfections

Ignacio García-Mata and Dima L. Shepelyansky

Laboratoire de Physique Théorique, Université de Toulouse III, CNRS, 31062 Toulouse, France

November 12, 2007

Abstract. We study numerically the effects of static imperfections and residual couplings between qubits for the quantum phase estimation algorithm with two qubits. We show that the success probability of the algorithm is affected significantly more by static imperfections than by random noise errors in quantum gates. An improvement of the algorithm accuracy can be reached by application of the Pauli-random-error-correction method (PAREC).

PACS. 03.67.Lx Quantum computation – 85.25.Cp Josephson devices – 24.10.Cn Many-body theory

1 Introduction

Quantum computers [1] are doing steady progress increasing the number of qubits and accuracy of quantum gates. Among most advanced physical implementations with possible scalable architecture are ion based quantum computers (see e.g. [2] and Refs. therein) and solid state superconducting qubits (see e.g. [3,4,5]). In the present situation when only a few qubits are available and the quantum gate accuracy is limited it is interesting to test a performance of simple quantum algorithms operating at such realistic conditions. One of such algorithms is the Iterative Quantum Phase Estimation Algorithm (IQPEA) proposed recently by Dobšíček *et al.* [6]. According to the results obtained there the IQPEA works reliably even in presence of relatively strong noise in quantum gates. The algorithm [6] is based on the semiclassical Quantum Fourier Transform (QFT) [7] which uses one ancilla qubit, iterative measurements and a classical information feedback. The advantages of the algorithm are the following: it uses only a single ancillary qubit and its theoretical accuracy of the eigenvalue found is limited only by the number of times the algorithm is applied. Due to that the IQPEA can be used as a benchmark algorithm for the maximal accuracy that could be obtained in a given experimental setup [6].

We describe briefly the IQPEA in the case of a two qubit circuit proposed in [6] as a minimal benchmarking circuit. The goal is to measure the eigenphase ϕ of some operator U with precision set to m significant bits. The standard Phase Estimation Algorithm (PEA) [8] requires m ancillary qubits, to get the desired precision, and the possibility to implement efficiently control-U^{2^k} gates. Using the semiclassical implementation of the QFT an alternative algorithm with only one ancillary qubit and measurements can be designed [9]. One further step ahead is done in [6] where a feedback of the measurement result is used to correct the phase. In this way the IQPEA provides a way to compute the phase *theoretically* with arbitrary precision. The proposed scheme can be used as a benchmarking circuit which is tolerant to a rather strong random noise in quantum gates.

In [6] only the case of random noise errors in quantum gates and environment dephasing are considered. At the same time it is known that a presence of static imperfections and residual couplings between qubits may lead to an emergence of quantum chaos in a quantum computer hardware [10]. Such static imperfections affect the accuracy of quantum computation in a significantly stronger way compared to random errors in quantum gates [11,12]. Thus it is interesting to test the effects of static imperfections in IQPEA with a small number of qubits, e.g. two qubits. Indeed, our studies presented in this paper show that the static imperfections lead to a significant drop of the computation accuracy and the algorithm success probability. To correct these quantum errors induced by static imperfections in IQPEA we apply the Pauli-random-error-correction (PAREC) method proposed in [13] and tested in various quantum circuits [14,15]. Our results for IQPEA show that the PAREC allows to improve significantly the accuracy of quantum computation.

The paper is organized as follows. First we briefly describe the IQPEA for a 2-qubit system (Section 2). We then compare the effects of random phase errors in quantum gates and the effects of static imperfections (Section 3). Then we show how the PAREC method corrects the errors induced by static imperfections (Section 4). The summary of the results is given in Section 5.

2 Brief description of IQPEA

The goal of IQPEA is to find the eigenphase of an operator $\hat{U}$. We consider the simplest case with one-qubit unitary diagonal

Correspondence to: http://www.quantware.ups-tlse.fr/dima/

operator in the computational basis

$$\hat{U} = \begin{pmatrix} e^{-\mathrm{i}2\pi\phi} & 0 \\ 0 & e^{\mathrm{i}2\pi\phi} \end{pmatrix}, \tag{1}$$

with $\phi \in [0,1]$. We want to find ϕ with up to m bits of accuracy (or error smaller than 2^{-m}). The IQPEA [6] procedure consists in applying m times the circuit shown in Fig. 1 to the 2-qubit state $|00\rangle$. For simplicity we use the presentation

$$\phi = \sum_{i=1}^{m} \phi_i 2^{-i} \overset{\text{def}}{=} 0.\phi_1\phi_2\ldots\phi_m 000\ldots \tag{2}$$

assuming that the binary expansion of ϕ is finite. For the first step ($i=0$) we take $\omega_i = 0$ so the Z-rotation does not act. After this first run

$$|\phi_0\rangle = \frac{1}{2}\left[(1+e^{\mathrm{i}2\pi\phi})|00\rangle + (1-e^{\mathrm{i}2\pi\phi})|10\rangle\right] \tag{3}$$

and the measurement of the "left" qubit yields $P_0(|0\rangle) = \cos^2(\pi(0.\phi_m))$ which is unity if $\phi_m = 0$ and zero if $\phi_m = 1$. Thus the least significant bit of ϕ is obtained deterministically. The key element is that in the following steps of the algorithm we use the classical information obtained from the measurement to correct the phase by a Z-rotation. Before the last Hadamard gate the phase in the second step is $2\pi(0.\phi_{m-1}\phi_m 00\ldots)$ and after performing a Z-rotation with $\omega_k = -2\pi(0.0\phi_{m-1})$ the probability becomes $P_1(|0\rangle) = \cos^2(\pi(0.\phi_{m-1}))$. Consequently, the result of the first measurement is used as a feedback for the algorithm to obtain the second least significant bit which is obtained deterministically. In theory, following this procedure each bit can be obtained.

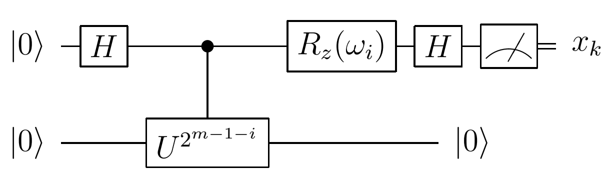

Fig. 1. Step i (where $i = 0,\ldots,m-1$) of the IQPEA of [6]. Here R_Z is a rotation by an angle ω_i around $\hat{z}$ axis.

In reality the phase is

$$\phi = \tilde{\phi} + \delta 2^{-m} \tag{4}$$

where $\tilde{\phi} = 0.\phi_1\phi_2\ldots\phi_m 000$ and $\delta \in [0,1)$ is the reminder. The probability of measuring ϕ_m correctly is thus

$$\left.\begin{array}{l} P_1(|0\rangle) = \cos^2(\pi((0.\phi_m)+\delta/2)) \\ P_1(|1\rangle) = \sin^2(\pi((0.\phi_m)+\delta/2)) \end{array}\right\} = P_1 = \cos^2(\pi\delta/2). \tag{5}$$

The next step gives $P_2 = \cos^2(\pi\delta/2^2)$ and eventually $P_k = \cos^2(\pi\delta/2^k)$ so that the total probability of measuring the phase correctly is given by

$$P_{\text{tot}}(\delta) = \prod_{k=1}^{m} \cos^2(\pi\delta/2^k) = \frac{\sin^2(\pi\delta)}{2^{2m}\sin^2(\pi\delta/2^m)}\,. \tag{6}$$

The success probability in (6) is bounded in the limit $m \to \infty$ by $4/\pi^2$ [8]. In fact, the rounding error permits us to neglect the least significant bit and consider as probability of success the sum $P_{\text{tot}}(\delta) + P_{\text{tot}}(1-\delta) = 8/\pi^2$ when $m \to \infty$ [6]. This lower bound could be raised by repeated measurement of the first few bits and majority vote [8,6].

3 IQPEA and static imperfections

We consider two kinds of circuit imperfections: random phase errors in rotations and static imperfections due to residual couplings between qubits. To model random quantum phase errors we assume that the rotation on angle θ

$$R_{\sigma^{(\nu)}}(\theta) = \exp[-i\sigma^{(\nu)}\theta/2] \tag{7}$$

(with $\sigma^{(\nu)}$ a Pauli operator) is replaced by rotation on angle $\theta(1+\Delta)$ with Δ randomly and uniformly distributed in the interval

$$\Delta \in [-\frac{\varepsilon_1}{2}, \frac{\varepsilon_1}{2}]\,. \tag{8}$$

In other words the original rotation Hamiltonian has now an additional term

$$\delta H_{\text{rnd}} = \frac{\Delta\theta}{2}\sigma^{(\nu)} \tag{9}$$

Each gate in Fig. 1 is implemented with rotations having different random realizations of Δ. This is the case of random noise errors considered in [6] where it was shown that the algorithm is rather robust.

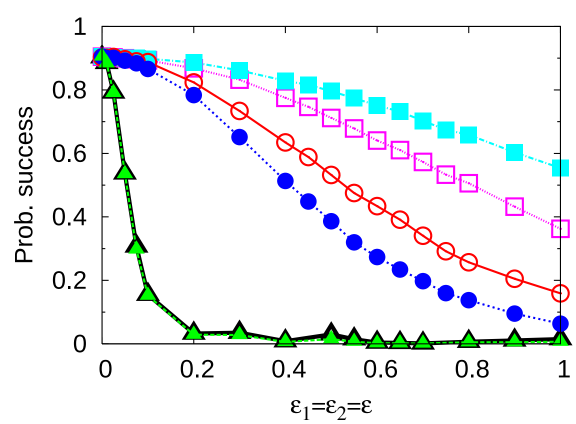

Fig. 2. (Color online) Success probability for the algorithm to determine the phase with a precision of up to 10 bits, as a function of the parameter $\varepsilon = \varepsilon_1 = \varepsilon_2$ characterizing the error strength. Symbols mark: random phase errors in the Hadamard gates(□), the R_z gate (■), the controlled-U^{2^k} (○) and errors in all of the gates (•). The case of static imperfections is shown by (△)/(▲) in absence/presence of random phase errors in the gates. Averaging is done over 2000 randomly chosen phases.

On the other hand, we model the effects of residual static couplings between qubits by an imperfection Hamiltonian of the form used in [10,12]:

$$\delta H_{\text{stat}}(x) = \delta_1\sigma_1^{(z)} + \delta_2\sigma_2^{(z)} + 2J\sigma_1^{(x)}\sigma_2^{(x)} \tag{10}$$

where $\sigma_i^{(\nu)}$ are the Pauli operators acting on the ith qubit and δ_i, J are random coefficients uniformly distributed according to

$$\delta_i,\ J \in [-a\sqrt{3}\varepsilon_2, a\sqrt{3}\varepsilon_2]\ , \tag{11}$$

(with a constant). We suppose that between each gate in the algorithm there is a finite time Δt which remains fixed during the algorithm and that δH acts via the unitary propagator

$$U_{\text{stat}} = e^{i\delta H} \tag{12}$$

where the time Δt has been absorbed into the constants δ and J in (10). The time Δt can be considered as an effective gate duration, a similar scheme is used in [12].

In order to compare the effects of both types of errors we compute $\langle \text{tr}[\delta H^2]\rangle \propto \varepsilon^2$. The value $a \approx 0.37$ is determined so that

$$\langle \text{tr}[\delta H^2_{\text{rnd}}]\rangle \approx \langle \text{tr}[\delta H^2_{\text{stat}}]\rangle \ \text{ if }\ \varepsilon_1 = \varepsilon_2\ , \tag{13}$$

the approximation is done taking $\theta = \pi$ in in Eq. (9).

The static type of imperfections is especially important since generally the errors produced in this case are accumulated coherently that leads to a quadratic term in the decay of fidelity [12] thus limiting considerably the maximum time over which an accurate quantum computation can be performed.

In Fig. 2 we show the success probability of measuring correctly the phase ϕ, for a chosen accuracy of 2^{-10}, as a function of the parameters $\varepsilon_1 = \varepsilon_2 = \varepsilon$. We averaged over 2000 uniformly random phases $\phi \in [0,1]$. For the cases where only random phase errors act we see that the algorithm is rather robust. Indeed, the decay of the success probability is relatively slow even when all the gates involved have errors. On a contrary, a dramatic drop of the accuracy of computation is seen when we include the effects of static imperfections. The decay is much faster than for random phase errors even for comparable values of tr$[\delta H^2]$ corresponding to a typical experimental situation.

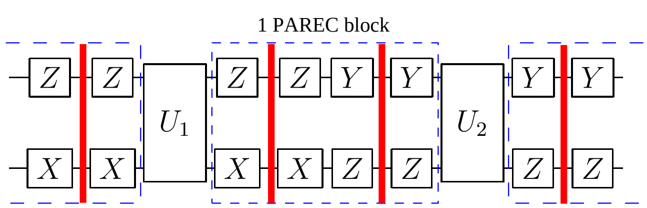

Fig. 3. (Color online) Schematic representation of the PAREC method. The vertical thick (red) lines indicate the place where the static coupling propagator U_{stat} is applied. In order to preserve tr$[\delta H^2]$ we take $\varepsilon/2$ for each propagator. The dashed lines enclose a PAREC block. Repetition of PAREC between two gates means applying repeatedly one PAREC block after another.

4 Accuracy improvement using PAREC

In this Section we address the issue of quantum error correction (QEC) of errors induced by static imperfections during the algorithm. The random errors in gates can be corrected up to a certain reasonable limit by usual QEC schemes which however require a significant increase of the number of qubits [1]. Here we study a different scheme to correct the effects produced by the static imperfections propagator U_{stat}. One possible way to correct errors produced by residual static couplings was introduced in [13]. The idea is simple: contrary to random errors the static imperfections lead to a coherent accumulation of errors [12]. If some randomness is introduced then the effect of the residual couplings changes each time and does not accumulate coherently. The PAREC method [13] profits from the freedom of choice of computational basis and uses this freedom by conveniently changing repeatedly and randomly the computational basis along the computation. In order not to change the algorithm and the desired result a special care must be taken to properly compensate for the changes made renumbering the computational basis.

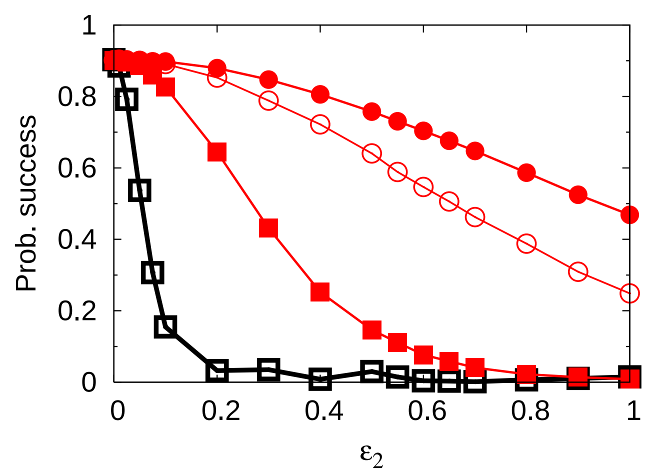

Fig. 4. (Color online) Success probability of IQPEA which determines the phase with a precision of up to 10 bits, in the presence of static imperfections, as a function of the imperfection strength ε_2. The curves show the influence of PAREC and how probability is enhanced with the increase of the number of times PAREC is applied: (□) No PAREC; (■) 1 time; (○)5 times; (●) 10 times. No random imperfections are present ($\varepsilon_1 = 0$). Averaging is done over 2000 randomly chosen phases.

The procedure is represented schematically in Fig. 3. To change the computational basis we first pick randomly from the set of Pauli matrices and identity matrix $\{X_i, Y_i, Z_i, I_i\}$ (where $i = 1,2$), and apply them to each qubit. We suppose that the time it takes to apply the Pauli operators is much shorter than any other time scales. We keep the information of this first choice, X_1, Z_2 in the Figure, and implement the suitably transformed gate $(Z_1 \otimes X_2)U_1(Z_1 \otimes X_2)$. After that to come back to the original basis, the operator $(Z_1 \otimes X_2)$ is applied again. The places where U_{stat} has acted are represented in Fig. 3 by thick vertical gray (red) lines. This procedure is repeated before and after each gate, but of course the key is that a new random sequence of Pauli operators is drawn, in the figure $(Y_1 \otimes Z_2)$. So it is clear that between U_1 and U_2 the imperfection propagator U_{stat} has acted on different bases. We have called the complete sequence of Pauli operators that act between two gates one PAREC block.

The effect of PAREC can be seen in Fig. 4. The black solid curve (with □ symbols) shows the success probability when static (but not random) imperfections act. The gray (red) line (with ■ symbols) show the result when one PAREC block is applied between each gate of the algorithm that already gives a considerable gain.

If instead of one PAREC block we introduce many of them (N_{PAREC}) keeping Δt fixed, and supposing that the time to implement the Pauli gates is negligible, then the imperfection Hamiltonian can be described with the help of the transformation

$$\delta_i \to \frac{\delta_i}{\sqrt{2N_{\text{PAREC}}}} \; ; \; J \to \frac{J}{\sqrt{2N_{\text{PAREC}}}}. \tag{14}$$

As a consequence, the coherent effect of static imperfections is suppressed. This can be seen in Fig. 4. The gray (red) curves show the success probability as a function of ε_2 for different values of N_{PAREC} (up to $N_{\text{PAREC}} = 10$). As N_{PAREC} grows the probability grows accordingly. In the ideal limit of infinitely many PAREC blocks between gates (with a fixed gate-to-gate time) the success probability tends to constant maximum value for all ε, a result which reminds us of a Zeno-like effect [16]. This is also illustrated in Fig. 5 (top), where the dark region in the density plot of the success probability indicates the limiting value attained for large values of N_{PAREC}. The maximum value is the ideal value with perfect gates which is only limited by the value of the reminder δ defined in (4). Nevertheless, the limit can be attained only theoretically because the time between IQPEA gates cannot be fixed if we add (ideally) infinitely many PAREC gates, no matter how fast we can implement them.

Up to now we have considered PAREC with perfect Pauli gates while now we turn to a more realistic situation. With this aim we also consider the possibility of random phase errors of IQPEA gates to be also present in the PAREC Pauli gates. Therefore we expect that in the presence of random imperfections, both in the IQPEA and in PAREC, there will be an optimal value of N_{PAREC} after which the presence of too many faulty Pauli gates yields PAREC useless. This is demonstrated in Fig. 5 (bottom). For this a further consideration must be made. In Fig. 2, for illustration reasons only, we took $\varepsilon_1 = \varepsilon_2$ such that the strength of both effects is comparable. However, we expect that in experiments the effect of random phase errors can be reduced to a minimum, so that in fact we have to assume $\varepsilon_2 > \varepsilon_1$. For the plot in Fig. 5 we took $\varepsilon_2 = 5\varepsilon_1$. As a result the maximum of the success probability occurs approximately at $N_{\text{PAREC}} = 5$ after which the PAREC method looses its efficiency. The position of the peak as well as its height depends on the ratio $\varepsilon_2/\varepsilon_1$. The obtained data show that the IQPEA with PAREC can operate reliably even in presence of relatively strong static imperfections.

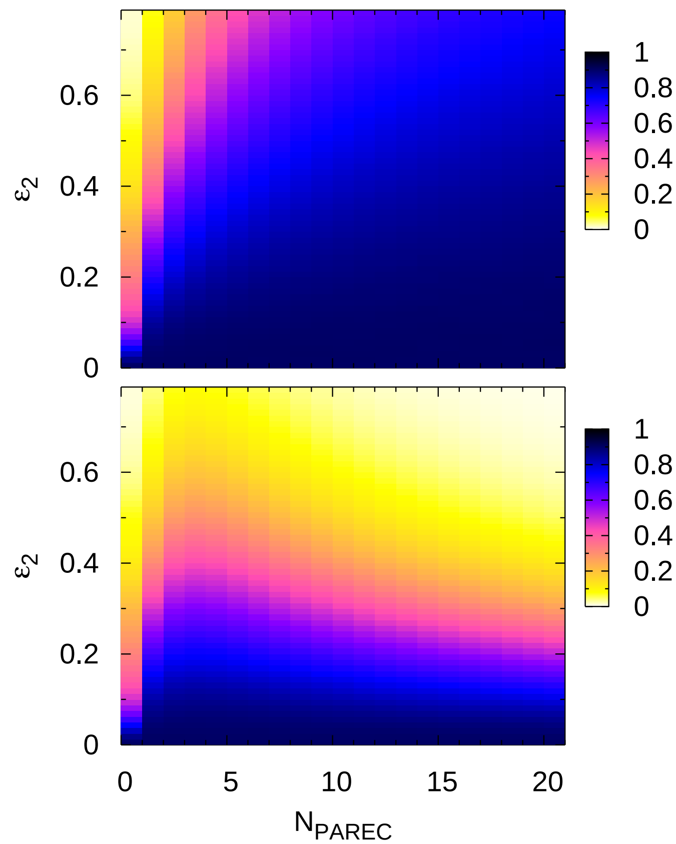

Fig. 5. (Color online) Success probability (shown by color) of the IQPEA which determines the phase with a precision of up to 2^{-10} as a function of the number of times N_{PAREC} PAREC is applied and of the static imperfections strength ε_2. Top: the case where only static imperfections are considered ($\varepsilon_1 = 0$). Bottom: both static and random imperfections are present (including in PAREC gates); here random errors in gates are also present, their strength is taken as $\varepsilon_1 = \varepsilon_2/5$. Averaging is done over 2000 randomly chosen phases.

5 Summary

To summarize, we tested the effects of static imperfections in the IQPEA [6]. Due to its simplicity this algorithm can be used as a benchmarking circuit for quantum computers with two qubits. We have shown that static imperfections produce a dramatic drop of success probability even for algorithms involving a rather small number of gates. In this context we have tested the PAREC method [13] and shown that it improves significantly the computation accuracy, even if the method is more suited to algorithms with a larger gate sequence involved. We also present results with repetitions of the PAREC method that produces a Zeno-like effect in preservation of probability. Even though, the realistic scenario would suggest a small N_{PAREC} (may be even $N_{\text{PAREC}} = 1$), the results obtained demonstrate a convincing improvement of the algorithm success probability induced by PAREC. The extension of the IQPEA circuits to a larger number of qubits is straightforward, as well as the PAREC implementation.

6 Acknowledgments

This work was supported in part by the EC IST-FET project EuroSQIP. For numerical simulations we used the codes of Quantware Library [17].

References

1. M. A. Nielsen, and I. L. Chuang, *Quantum computation and quantum information*, Cambridge Univ. Press, Cambridge (2000).
2. H. Häffner, W. Hänsel, C. F. Roos, J. Benhelm, D. Chek-al-kar, M. hwalla, T. Krber, U. D. Rapol, M. Riebe, P. O. Schmidt, C. Becher, O. Gühne, W. Dür and R. Blatt, Nature **438**, 643 (2005).
3. D. Vion, A. Aassime, A. Cottet, P. Joyez, H. Pothier, C. Urbina, D. Esteve, and M.H. Devoret, Science **296**, 285 (2002).
4. J.H. Plantenberg, P.C. de Groot, C.J.P.M. Harmans, and J.E. Mooij, Nature **447**, 836 (2007).
5. J. Majer, J.M. Cow, J.M. Gambetta, J. Koch, B.R. Johnson, J.A. Schreier, L. Frunzio, D.I. Schuster, A.A. Houck, A. Wallraff, A. Blais, M.H. Devoret, S.M. Girvin, and R.J. Schoelkopf, Nature **449**, 443 (2007).
6. M. Dobšíček, G. Johansson, V. Shumeiko, and G. Wendin, Phys. Rev. A **76**, 030306(R) (2007) .
7. R. B. Griffiths and C.-S. Niu, Phys. Rev. Lett. **76**, 3228 (1996).
8. R. Cleve, A. Ekert, C. Macchiavello and M. Mosca, Proc. R. Soc. Lond. A **454**, 339 (1998).
9. A. Yu. Kitaev, Electron. Coll. Comput. Complex. 3 (1996) (arXiv:quant-ph/9511026 (2005)).
10. B.Georgeot and D.L.Shepelyansky, Phys. Rev. E **62**, 3504 (2000); **ibid. 62**, 6366 (2000).
11. G. Benenti, G. Casati, S. Montangero and D. L. Shepelyansky, Phys. Rev. Lett. **87**, 227901 (2001).
12. K. M. Frahm, R. Fleckinger and D. L. Shepelyansky, Eur. Phys. J. D **29**, 139 (2004).
13. O. Kern, G. Alber, and D. L. Shepelyansky, Eur. Phys. J. D **32**, 153 (2005).
14. O. Kern and G. Alber, Phys. Rev. Lett. **95**, 250501 (2005).
15. L. Viola and L.F. Santos, J. Mod. Optics, **53**, 2559 (2006).
16. B. Misra and E. C. G. Sudarshan, J. Math. Phys. **18**, 756 (1977); Wayne M. Itano, D. J. Heinzen, J. J. Bollinger, and D. J. Wineland, Phys. Rev. A **41**, 2295 (1990); P. Facchi and S. Pascazio, Phys. Rev. Lett. **89**, 080401 (2002).
17. K. M. Frahm and D. L. Shepelyansky (Eds.), *Quantware Library: Quantum Numerical Recipes*, http://www.quantware.ups-tlse.fr/QWLIB/ .