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**Abstract.** We study effects of weak nonlinearity on localization of waves in disordered Stark ladder corresponding to propagation in presence of disorder and a static field. Our numerical results show that nonlinearity leads to delocalization with subdiffusive spreading along the ladder. The exponent of spreading remains close to its value in absence of the static field. The delocalization implies the existence of statistical entanglement between far away parts of the spreading wave packet indicating importance of long-range effects.

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Anderson localization leads to suppression of diffusive propagation of linear waves in systems with disorder [1]. In one and two dimensions all states are exponentially localized [2,3]. For classical waves nonlinearity is naturally present and it is important to understand how it affects localization in a random media [4]. At first glance it seems that a spreading in space leads to an effective decrease of nonlinearity and hence persistence of localization [5]. On the other hand it was argued that nonlinear resonances remain overlapped and localization is destroyed by a moderate nonlinearity which leads to a subdiffusive spreading in space at asymptotically large times [6].

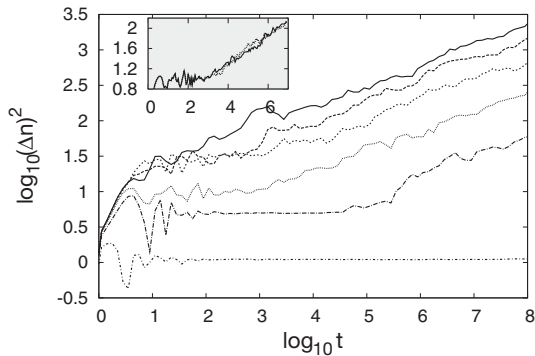
Recent experimental progress with nonlinear photonic lattices [7,8] and Bose-Einstein condensates (BECs) in optical lattices [9,10] with disorder generated a renewal of significant theoretical interest to this problem (see e.g. [11–18] and references therein). Even if for BECs in [9,10] the interactions are weak, it is possible to have experimental situations with strong interactions between atoms (see e.g. [19,20]) where the nonlinear effects start to play an important role. Nonlinear effects can also play an important role for BEC dynamics in kicked optical lattices with a dynamical localization of quantum chaos realized in experiments [21,22]. Thus the investigations of effects of nonlinearity on localized states is timely and opens new interesting problems of nonlinear dynamics. Similar type of problems appear also for energy propagation in disordered molecular chains [13,23] that enlarge a field of possible applications. In addition the problem of interplay

of localization and nonlinearity represents an interesting mathematical problem of stability of pure point spectrum with respect to nonlinear perturbations which led to recent mathematical studies [24,25]. Finally, we note that there are also studies of wave packet spreading on the nonlinear Stark ladder without disorder reported in the literature [26,27]. However, we are interested in the case of relatively weak or moderate nonlinearity and static field and in this case the presence of disorder gives a qualitatively different situation.

The numerical studies are mainly done for the discrete Anderson nonlinear Schrödinger equation (DANSE) showing that the wave packet width  $\Delta n$  spreads at large times  $t$  in a subdiffusive way with  $(\Delta n)^2 \propto t^\alpha$  and an exponent  $\alpha \approx 0.3 - 0.4$  for system dimension  $d = 1$  [6,14,17,18] and  $\alpha \approx 0.25$  for  $d = 2$  [18]. The theoretical estimates give  $\alpha = 2/5$  [6] and  $\alpha = 1/4$  [18] respectively. A noticeable difference between the theory estimates and numerical value  $\alpha \approx 0.3$  for  $d = 1$  is argued to be related with a specific properties of 1d Anderson model [18] but further clarifications of this point are required (see e.g. [17,18]). Indeed, for the 1d DANSE model the numerical data give values  $\alpha \approx 0.33$ , but the 1d kicked nonlinear rotator model gives  $\alpha \approx 0.4$ .

In this work we address a new type of question for the DANSE model: how a static field force affects the properties of nonlinear delocalization? Such a force is experienced by BECs in a gravitational field or effectively in a magnetic field gradient. It can be also effectively created by an acceleration of the optical lattice as a whole. This creates an effective Stark ladder which already has been

<sup>a</sup> URL: <http://www.quantware.ups-tlse.fr/dima>



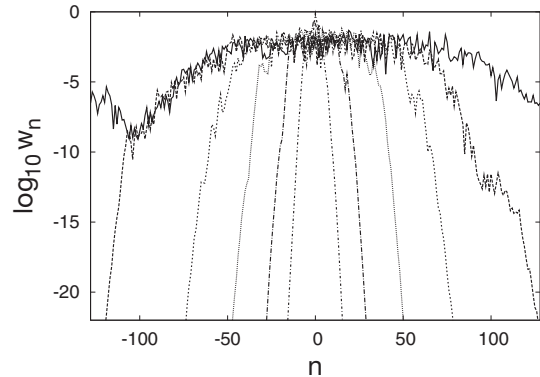
**Fig. 1.** Dependence of the second moment  $(\Delta n)^2$  of probability distribution on time  $t$  for various values of static field  $f$  and one fixed disorder realisation. Curves from top to bottom at  $t = 10^8$  are for  $f = 0., 0.1, 0.25, 0.5, 1., 2.$  and  $\beta = 1, W = 4, N = 256, \Delta t = 0.1$ . Inset shows data for  $f = 0.5$  obtained with integration steps  $\Delta t = 0.1$  (solid curve) and  $\Delta t = 0.01$  (dotted curve). To suppress fluctuations time average was made on logarithmic scale inside intervals  $\Delta(\log_{10} t) = 0.1$ . Initial state is one lattice site  $n = 0$  with energy in the middle of the band.

realized in experiments with cold atoms [28]. In absence of nonlinearity a weak static field does not significantly affect the localization while at strong fields the localization length is significantly reduced since less states are energetically available for hopping over the ladder (see e.g. recent studies [29] and references therein). It is not so obvious what are the effects of nonlinearity in such a system since the nonlinear term is local and is small compared to an energy variation for large displacements along the ladder.

To answer the above question we study numerically the Stark DANSE model described by the equation

$$i\hbar \frac{\partial \psi_n}{\partial t} = (fn + E_n)\psi_n + \beta |\psi_n|^2 \psi_n + V(\psi_{n+1} + \psi_{n-1}), \quad (1)$$

where  $f$  is the strength of the Stark field. At  $f = 0$  the model is reduced to the usual DANSE studied recently in [13,14,16,17]. We fix the units as  $V = \hbar = 1$  and choose a typical set of parameters used here as  $W = 4, \beta = 1$ . On-site energies  $E_n$  are randomly and homogeneously distributed in the interval  $-W/2 < E_n < W/2$ . Then the localization length in the middle of energy band is  $\ell \approx 6$  at  $f = 0$ . The numerical integration was done by the split operator scheme described in [18]. Such a symplectic integration with an integration time step  $\Delta t = 0.1$  and  $0.01$  gives the energy conservation with accuracy 3% and 1% for  $t \leq 10^7$  in a presence of strong field  $f \leq 2$ . The total number of states was fixed at  $N = 256$ , we used averaging over  $N_d = 15$  disorder realisations. The finite value of  $\Delta t$  generates high frequency equidistant harmonics with frequency spacing  $2\pi/\Delta t$ . At  $f = 0$  these frequencies are located outside of energy band while at  $f > 0$  they, in principle, may give resonant transitions. However, at small  $\Delta t$  the distance between such resonant states is much larger than the localization length  $\ell$  and the matrix elements in such cases are exponentially small and do not affect the behavior of the system with variation of  $\Delta t$  (see Fig. 1,

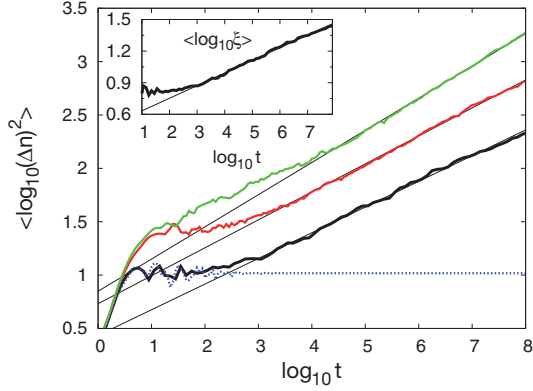


**Fig. 2.** Probability distribution  $w_n = |\psi_n|^2$  over ladder site  $n$  at time  $t = 10^8$  for  $f = 0., 0.1, 0.25, 0.5, 1., 2.$  (curves from small (inside) to large (outside) values of  $|n|$ ). Parameters are the same as in Figure 1.

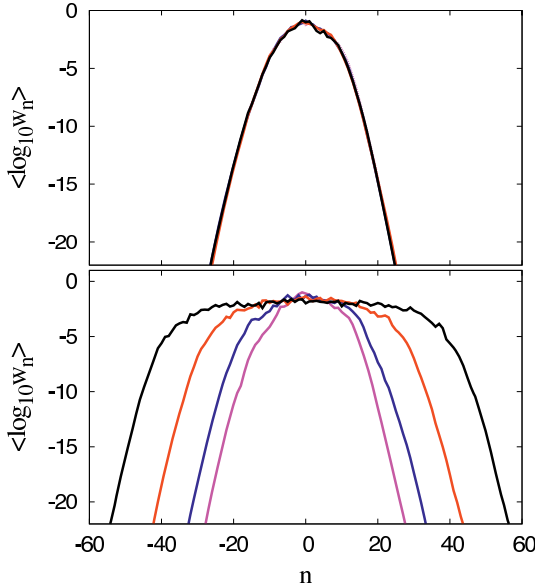
inset). The integration scheme conserves exactly the total probability.

A typical dependence of the second moment  $(\Delta n)^2$  of probability distribution  $w_n$  on time  $t$  is shown in Figure 1 for various values of  $f$ . The distribution of on-site probabilities  $w_n$  at a final time  $t = 10^8$  is shown in Figure 2. The fits of data of Figure 1 show an unlimited subdiffusive growth  $(\Delta n)^2 \propto t^\alpha$  with the exponent  $\alpha = 0.275 \pm 0.006, 0.291 \pm 0.005, 0.276 \pm 0.006, 0.272 \pm 0.004, 0.269 \pm 0.016$  for  $f = 0., 0.1, 0.25, 0.5, 1.$  respectively. The fits are done in the interval  $5 \leq \log_{10} t \leq 8$  for the last case and  $3 \leq \log_{10} t \leq 8$  for all other cases. The data show no significant variation of  $\alpha$  with  $f$  even if one realization at finite time may have noticeable fluctuations of  $\alpha$  being larger than a formal statistical error (see below). At large values of  $f = 1., 2.$  the localization length  $\ell$  becomes rather small (it can be estimated as  $\ell \approx \sqrt{(\Delta n(t = 1000))^2}$ ) and during a long time interval there is not spreading over the ladder. For  $f = 1.$  the growth appears at  $t \geq 10^5$ . There is no visible growth for all computational times for  $f = 2.$  We interpret this as very low transition rates over localized states in the case of small localization length  $\ell \approx 1$  at  $f = 2$ . It remains unclear if localization persists or disappears for such small  $\ell$  at very large times. For the cases with clear delocalization at  $f = 0., 0.1, 0.25, 0.5$  the distribution of  $w_n$  over  $n$  have a form of homogeneous “chapeau” centered near the initial state  $n = 0$ ; its width grows with time, approximately in agreement with the second moment growth (see Fig. 2).

To obtain more statistics we perform averaging over  $N_d$  disorder realisations. The data are presented in Figure 3 for the second moment and in Figure 4 for the probability distribution for  $\beta = 0; 1$ . At  $f = 0$  we obtain the exponent  $\alpha = 0.302$  which is comparable with the values  $\alpha \approx 0.33$  found in previous studies [14,17,18]. The value of  $\alpha$  decreases by about 10% when the static field is increased up to  $f = 0.5$ . This decrease is well visible even if it is not very large and is comparable to the statistical variations of  $\alpha$  at  $f = 0$  discussed above. We also computed the dependence on time for the participation ratio  $\xi = 1/\langle \sum_n w_n^2 \rangle$ . It can be characterized by the dependence



**Fig. 3.** (Color online) Dependence of the second moment  $(\Delta n)^2$  on time  $t$ , average is done over  $N_d = 15$  disorder realisations. The curves from top to bottom at  $t = 10^8$  are for  $f = 0, 0.25, 0.5$  at  $\beta = 1$  and  $f = 0.5$  at  $\beta = 0$ . The fits, shown by thin straight lines, give the exponent of growth at  $\beta = 1$ :  $\alpha = 0.302 \pm 0.001 (f = 0)$ ,  $0.262 \pm 0.001 (f = 0.25)$ ,  $0.241 \pm 0.002 (f = 0.5)$ . Inset shows the dependence of participation ratio  $\xi$  on time for  $\beta = 1, f = 0.5$  (thick curve), the straight line shows the fit dependence with the exponent  $\nu = 0.120 \pm 0.001$ . Other parameters are as in Figure 1.



**Fig. 4.** (Color online) Averaged probability distribution  $w_n$  as a function of the lattice site  $n$  for the cases of Figure 3 at  $f = 0.5$  (average is done over the same 15 disorder realisations). Top panel is for  $\beta = 0$  and bottom panel is for  $\beta = 1$ . Curves are drawn from inside (small  $|n|$ ) to outside (large  $|n|$ ) for  $t = 10^2$  (magenta),  $10^4$  (blue),  $10^6$  (red) and  $10^8$  (black). For  $\beta = 0$  (top panel) the state is almost the same for all times from  $t = 10^2$  to  $t = 10^8$ .

$\xi \propto t^\nu$  with the exponent  $\nu = 0.120 \pm 0.001 (f = 0.5)$ ,  $0.131 \pm 0.001 (f = 0.25)$ ,  $0.159 \pm 0.002 (f = 0)$ . These values are compatible with the usual relation  $\nu = \alpha/2$ .

The averaged probability distributions for the cases of Figure 3 at  $f = 0.5$  are shown in Figure 4 at different moments of time. For  $\beta = 0$  the distribution is localized being

frozen in time. It drops faster than exponential due to the presence of static field showing a qualitative difference between Stark localization and usual exponential Anderson localization. For  $\beta = 1$  the probability spreads over the whole lattice forming a homogeneous plateau in the center. The interesting property of this distribution is its approximate symmetry with respect to the initial state  $n = 0$ . It is clear that the conservation of energy imposes such a symmetric spreading. Indeed, the width of the plateau is approximately  $\delta n \approx 80$  and the energy variation on such a distance is  $\delta E \approx f \delta n \approx 40$  that is much larger than the energy band  $B \approx 6$  at  $f = 0$ . Due to energy conservation at  $|f| > 0$  the spreading can continue unlimitedly only in approximately symmetric way. This excludes the possibility of a compact packet which moves over a lattice on larger and larger distances in some stochastic way (as discussed in [13]). Indeed, since such a packet is compact (its size  $\xi_p$  composed of finite number of populated sites), its displacement on a distance  $r_p \gg \xi_p$  from the origin is forbidden by the total energy conservation at  $|f| > 0$ . Hence, such a packet cannot move diffusively along the lattice. A quasi-symmetric spreading seen in Figure 4 looks rather natural in view of total energy conservation at  $|f| > 0$ . However, it raises an interesting problem of *statistical entanglement* of probabilities  $w_n$  on opposite ends of the plateau. Indeed, on such a distance the probabilities seems to be uncorrelated since they are separated by many localization lengths  $\ell$  of the linear problem. Nevertheless the propagation on these far ends goes in a correlated way since the total energy  $E_{tot} \approx \sum_n f n w_n$  is exactly conserved.

In [6,14,18] it was argued that an infinite spreading is possible since the nonlinear frequency shift  $\delta\omega \sim \beta/\Delta n$  remains comparable with the frequency spacing  $\Delta\omega \sim 1/\Delta n$  between frequencies of excited  $\Delta n$  modes. On the Stark ladder this condition seems to be violated at large  $\Delta n$  since  $\Delta\omega \sim f \gg \delta\omega \sim \beta/\Delta n$ . However, the situation is more subtle. Indeed, we can write DANSE (1) in the basis of linear eigenmodes. The time evolution amplitudes  $C_m$  in this basis is described by equation (see e.g. [6,18]):

$$i \frac{\partial C_m}{\partial t} = (f m + \epsilon_m) C_m + \beta \sum_{m' m_1 m'_1} V_{m m' m_1 m'_1} C_{m'} C_{m_1}^* C_{m'_1} \quad (2)$$

where  $m$  marks the center of eigenmode inside the ladder and eigenenergies  $\epsilon_m$  are randomly distributed inside the energy band width of approximately the same size  $B \sim 6$  as at  $f = 0$ ;  $f m + \epsilon_m$  are eigenvalues of the linear problem. The transition matrix elements induced by nonlinearity are of the order of  $V_{m m' m_1 m'_1} \sim l^{-3/2}$  (see a derivation given in [6,18]). From this equation it is clear that 4-waves resonance conditions are satisfied if the frequency detuning  $\Delta\omega_4$  of these 4-waves remains small:

$$\Delta\omega_4 = f(m + m_1 - m' - m'_1) + \epsilon_m + \epsilon_{m_1} - \epsilon_{m'} - \epsilon_{m'_1} < \delta\omega. \quad (3)$$

Thus the spreading over the ladder can proceed only over such modes where  $m + m_1 - m' - m'_1 = 0$  and thus  $\Delta\omega_4 = \epsilon_m + \epsilon_{m_1} - \epsilon_{m'} - \epsilon_{m'_1}$ . Since all  $\epsilon_m$  are inside the frequency band  $B$  it is possible to have  $\Delta\omega_4 \sim 1/\Delta n$  so

that the resonant detunings will remain small compared to nonlinear shift  $\delta\omega \sim \beta/\Delta n$  even at large  $\Delta n$  values.

The rate of spreading is still determined by the same estimates as in [6,14,18] since we still have  $dC/dt \sim \beta C^3$  and the theoretical exponent is  $\alpha = 2/5$  being independent of  $f$  if  $f\ell < 1$  so that the local transition rates remain the same as at  $f = 0$ . The numerical results presented here show weak dependence of  $\alpha$  on  $f$  for  $f < 1/\ell$  being in a satisfactory agreement with this theoretical estimate. The deviation of  $\alpha \approx 0.3$  from the theoretical value  $2/5$  still should be better clarified both for  $f = 0$  and  $|f| > 0$ .

In summary, we demonstrated that, in presence of a static field applied to a lattice with disorder, a nonlinearity still produces complete delocalization with a subdiffusive spreading over the ladder. The exponent of the spreading remains close to the value without the force. The spreading forms a homogeneous distribution of probability inside a certain plateau. Due to the conservation of total probability and energy the far away parts of this plateau remain statistically entangled even being a large distance apart from each other. The obtained results can be tested in experiments with nonlinear photonic lattices and BEC atoms in optical lattices with a static field.

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