Nonlinear delocalization on disordered Stark ladder

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Abstract. - We study effects of weak nonlineary on localization of waves in disordered Stark ladder corresponding to propagation in presence of disorder and a static field. Our numerical results show that nonlinearity leads to delocalization with subdiffusive spreading along the ladder. The exponent of spreading remains close to its value in absence of the static field.

Anderson localization leads to suppression of diffusive propagation of linear waves in systems with disorder [1]. In one and two dimensions all states are exponentially localized [2,3]. For classical waves nonlinearity is naturally present and it is important to understand how it affects localization in a random media [4]. At first glance it seems that a spreading in space leads to an effective decrease of nonlinearity and hence persistence of localization [5]. On the other hand it was argued that nonlinear resonances remain overlapped and localization is destroyed by a moderate nonlinearity which leads to a subdiffusive spreading in space at asymptotically large times [6].

Recent experimental progress with nonlinear photonic lattices [7,8] and Bose-Einstein condensates (BECs) in optical lattices [9,10] with disorder generated a renewal of significant theoretical interest to this problem (see [11–18] and Refs. therein). Similar type of problems appear also for energy propagation in disordered molecular chains [13,19] that enlarge a field of possible applications. In addition the problem of interplay of localization and nonlinearity represents an interesting mathematical problem of stability of pure point spectrum with respect to nonlinear perturbations which led to recent mathematical studies [20,21].

The numerical studies are mainly done for the discrete Anderson nonlinear Schrödinger equation (DANSE) showing that the wave packet width Δn spreads at large times t in a subdiffusive way with $(\Delta n)^2 \propto t^\alpha$ and an exponent $\alpha \approx 0.3 - 0.4$ for system dimension $d = 1$ [6,14,17] and $\alpha \approx 0.25$ for $d = 2$ [18]. The theoretical estimates give $\alpha = 2/5$ [6] and $\alpha = 1/4$ [18] respectively. A noticeable difference between the theory estimates and numerical value $\alpha \approx 0.3$ for $d = 1$ is argued to be related with a specific properties of 1d Anderson model [18] but further clarifications of this point are required (see e.g. [17]).

In this work we address a new type of question for the DANSE model: how a static field force affects the properties of nonlinear delocalization? Such a force is experienced by BECs in a gravitational field or effectively in a magnetic field gradient. It can be also effectively created by an acceleration of the optical lattice as a whole. This creates an effective Stark ladder which already has been realized in experiments with cold atoms [22]. In absence of nonlinearity a weak static field does not significantly affect the localization while at strong fields the localization length is significantly reduced since less states are energetically available for hopping over the ladder (see e.g. recent studies [23] and Refs. therein). It is not so obvious what are the effects of nonlinearity in such a system since the nonlinear term is local and is small compared to an energy variation for large displacements along the ladder.

To answer the above question we study numerically the

Stark DANSE model described by the equation

$$i\hbar\frac{\partial\psi_n}{\partial t} = (fn + E_n)\psi_n + \beta|\psi_n|^2\psi_n + V(\psi_{n+1} + \psi_{n-1}),\tag{1}$$

where f is the strength of the Stark field. At $f = 0$ the model is reduced to the usual DANSE studied recently in [13,14,16,17]. We fix the units as $V = \hbar = 1$ and choose a typical set of parameters used here as $W = 4, \beta = 1$. On-site energies E_n are randomly and homogeneously distributed in the interval $-W/2 < E_n < W/2$. Then the localization length in the middle of energy band is $\ell \approx 6$ at $f = 0$. The numerical integration was done by the split operator scheme described in [18]. Such a symplectic integration with an integration time step $\Delta t = 0.1$ and 0.01 gives the energy conservation with accuracy 3% and 1% for $t \leq 10^7$ in a presence of strong field $f \leq 2$. The total number of states was fixed at $N = 256$, we used averaging over $N_d = 15$ disorder realisations. The finite value of Δt generates high frequency equidistant harmonics with frequency spacing $2\pi/\Delta t$. At $f = 0$ these frequencies are located outside of energy band while at $f > 0$ they, in principle, may give resonant transitions. However, at small Δt the distance between such resonant states is much larger than the localization length ℓ and the matrix elements in such cases are exponentially small and do not affect the behavior of the system with variation of Δt (see Fig.1, inset). The integration scheme conserves exactly the total probability.
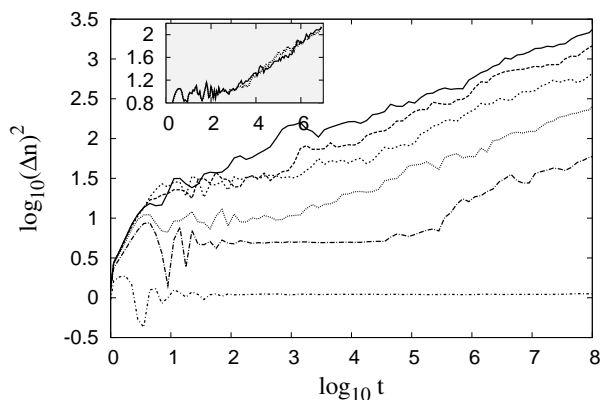

Fig. 1: Dependence of the second moment $(\Delta n)^2$ of probability distribution on time t for various values of static field f and one fixed disorder realisation. Curves from top to bottom at $t = 10^8$ are for $f = 0., 0.1, 0.25, 0.5, 1., 2.$ and $\beta = 1, W = 4, N = 256, \Delta t = 0.1$. Inset shows data for $f = 0.5$ obtained with integration steps $\Delta t = 0.1$ (solid curve) and $\Delta t = 0.01$ (dotted curve). To suppress fluctuations time average was made on logarithmic scale inside intervals $\Delta(\log_{10} t) = 0.1$. Initial state is one lattice site $n = 0$ with energy in the middle of the band.

A typical dependence of the second moment $(\Delta n)^2$ of probability distribution w_n on time t is shown in Fig. 1
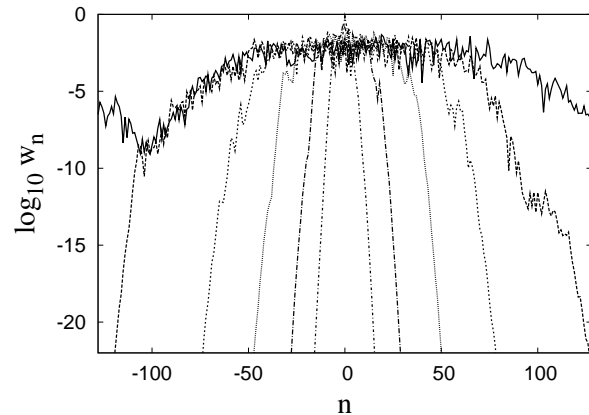

Fig. 2: Probability distribution $w_n = |\psi_n|^2$ over ladder site n at time $t = 10^8$ for $f = 0., 0.1, 0.25, 0.5, 1., 2.$ (curves from small (inside) to large (outside) values of $|n|$). Parameters are the same as in Fig. 1.

for various values of f. The distribution of on-site probabilities w_n at a final time $t = 10^8$ is shown in Fig. 2. The fits of data of Fig. 1 show an unlimited subdiffusive growth $(\Delta n)^2 \propto t^\alpha$ with the exponent $\alpha = 0.275 \pm 0.006$, $0.291 \pm 0.005, 0.276 \pm 0.006, 0.272 \pm 0.004, 0.269 \pm 0.016$ for $f = 0., 0.1, 0.25, 0.5.1$. respectively. The fits are done in the interval $5 \leq \log_{10} t \leq 8$ for the last case and $3 \leq \log_{10} t \leq 8$ for all other cases. The data show no significant variation of α with f even if one realization at finite time may have noticeable fluctuations of α being larger than a formal statistical error (see below). At large values of $f = 1., 2.$ the localization length ℓ becomes rather small (it can be estimated as $\ell \approx \sqrt{(\Delta n(t = 1000))^2}$) and during a long time interval there is not spreading over the ladder. For $f = 1.$ the growth appears at $t \geq 10^5$. There is no visible growth for all computational times for $f = 2$. We interpret this as very low transition rates over localized states in the case of small localization length $\ell \approx 1$ at $f = 2$. It remains unclear if localization persists or disappears for such small ℓ at very large times. For the cases with clear delocalization at $f = 0., 0.1, 0.25, 0.5$ the distribution of w_n over n have a form of homogeneous "chapeau" centered near the initial state $n = 0$; its width grows with time, approximately in agreement with the second moment growth (see Fig. 2).

To obtain more statistics we perform averaging over N_d disorder realisations. The data are presented in Fig. 3 for the second moment and in Fig. 4 for the probability distribution for $\beta = 0; 1$. At $f = 0$ we obtain the exponent $\alpha = 0.302$ which is comparable with the values $\alpha \approx 0.33$ found in previous studies [14,17,18]. The value of α decreases by about 10% when the static field is increased up to $f = 0.5$. This decrease is well visible even if it is not very large and is comparable to the statistical variations of α at $f = 0$ discussed above. We also computed the dependence on time for the participation ratio

$\xi = 1/ < \sum_n w_n^2 >$. It can be characterized by the dependence $\xi \propto t^\nu$ with the exponent $\nu = 0.120\pm0.001(f = 0.5)$, $0.131\pm0.001(f = 0.25)$, $0.159\pm0.002(f = 0)$. These values are compatible with the usual relation $\nu = \alpha/2$.
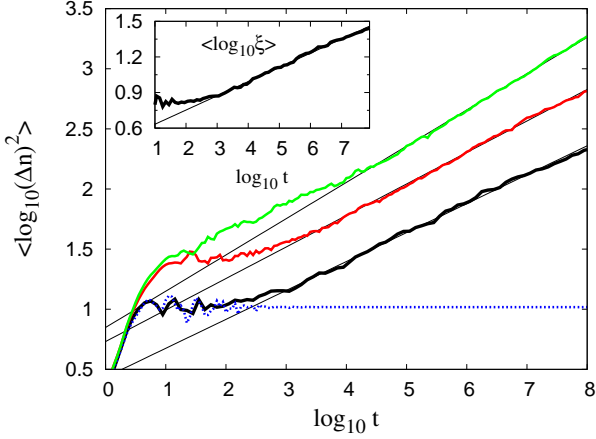

Fig. 3: (Color online) Dependence of the second moment $(\Delta n)^2$ on time t, average is done over $N_d = 15$ disorder realisations. The curves from top to bottom at $t = 10^8$ are for $f = 0., 0.25, 0.5$ at $\beta = 1$ and $f = 0.5$ at $\beta = 0$. The fits, shown by thin straight lines, give the exponent of growth at $\beta = 1$: $\alpha = 0.302 \pm 0.001(f = 0)$, $0.262 \pm 0.001(f = 0.25)$, $0.241 \pm 0.002(f = 0.5)$. Inset shows the dependence of participation ratio ξ on time for $\beta = 1, f = 0.5$ (thick curve), the straight line shows the fit dependence with the exponent $\nu = 0.120 \pm 0.001$. Other parameters are as in Fig. 1.

The averaged probability distributions for the cases of Fig. 3 at $f = 0.5$ are shown in Fig. 4 at different moments of time. For $\beta = 0$ the distribution is localized being frozen in time. It drops faster than exponential due to the presence of static field showing a qualitative difference between Stark localization and usual exponential Anderson localization. For $\beta = 1$ the probability spreads over the whole lattice forming a homogeneous plateau in the center. The interesting property of this distribution is its approximate symmetry with respect to the initial state $n = 0$. It is clear that the conservation of energy imposes such a symmetric spreading. Indeed, the width of the plateau is approximately $\delta n \approx 80$ and the energy variation on such a distance is $\delta E \approx f\delta n \approx 40$ that is much larger than the energy band $B \approx 6$ at $f = 0$. Due to energy conservation at $|f| > 0$ the spreading can continue unlimitedly only in approximately symmetric way. This excludes the possibility of a compact packet which moves over a lattice on larger and larger distances in some stochastic way (as discussed in [13]). A quasi-symmetric spreading seen in Fig. 4 looks rather natural in view of total energy conservation at $|f| > 0$. However, it raises an interesting problem of *statistical entanglement* of probabilities w_n on opposite ends of the plateau. Indeed, on such a distance the probabilities seems to be uncorrelated

since they are separated by many localization lengths ℓ of the linear problem. Nevertheless the propagation on these far ends goes in a correlated way since the total energy $E_{tot} \approx \sum_n fnw_n$ is exactly conserved.
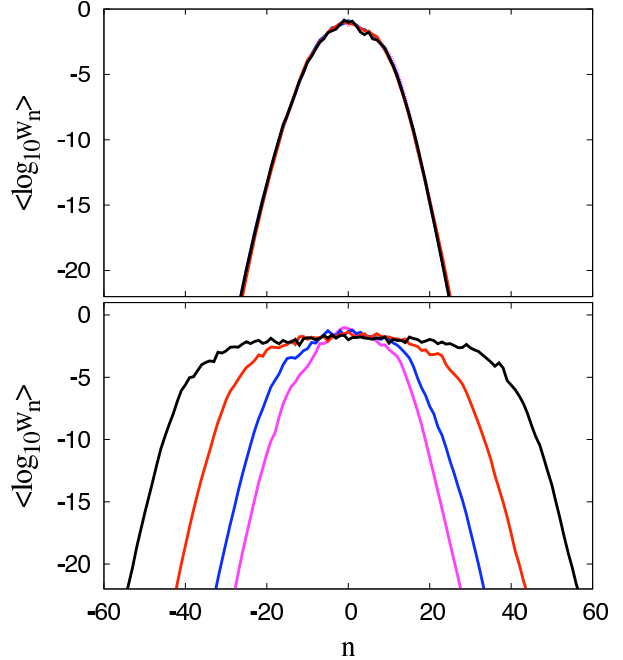

Fig. 4: (Color online) Averaged probability distribution w_n as a function of the lattice site n for the cases of Fig. 3 at $f = 0.5$ (average is done over the same 15 disorder realisations). Top panel is for $\beta = 0$ and bottom panel is for $\beta = 1$. Curves are drown from inside (small $|n|$) to outside (large $|n|$) for $t = 10^2$ (magenta), 10^4 (blue), 10^6 (red) and 10^8 (black). For $\beta = 0$ (top panel) the state is almost the same for all times from $t = 10^2$ to $t = 10^8$.

In [6, 14, 18] it was argued that an infinite spreading is possible since the nonlinear frequency shift $\delta\omega \sim \beta/\Delta n$ remains comparable with the frequency spacing $\Delta\omega \sim 1/\Delta n$ between frequencies of excited Δn modes. On the Stark ladder this condition seems to be violated at large Δn since $\Delta\omega \sim f \gg \delta\omega \sim \beta/\Delta n$. However, the situation is more subtle. Indeed, we can write DANSE (1) in the basis of linear eigenmodes. The time evolution amplitudes C_m in this basis is described by equation (see e.g. [6, 18]):

$$i\frac{\partial C_m}{\partial t} = (fm+\epsilon_m)C_m + \beta \sum_{m'm_1m_1'} V_{mm'm_1m_1'} C_{m'} C_{m_1}^* C_{m_1'}$$
$$(2)$$

where m marks the center of eigenmode inside the ladder and eigenenergies ϵ_m are randomly distributed inside the energy band width of approximately the same size $B \sim 4$ as at $f = 0$. $V_{mm'm_1m_1'} \sim l^{-3/2}$ are the transition matrix elements induced by nonlinearity. From this equation it is clear that 4-waves resonance conditions are satisfied if the

frequency detuning $\Delta\omega_4$ of these 4-waves remains small:

$$\Delta\omega_4 = f(m + m_1 - m' - m'_1) + \epsilon_m + \epsilon_{m_1} - \epsilon_{m'} - \epsilon_{m'_1} < \delta\omega \tag{3}$$

Thus the spreading over the ladder can proceed only over such modes where $m + m_1 - m' - m'_1 = 0$ and thus $\Delta\omega_4 = \epsilon_m + \epsilon_{m_1} - \epsilon_{m'} - \epsilon_{m'_1}$. Since all ϵ_m are inside the frequency band B it is possible to have $\Delta\omega_4 \sim 1/\Delta n$ so that the resonant detunings will remain small compared to nonlinear shift $\delta\omega \sim \beta/\Delta n$ even at large Δn values.

The rate of spreading is still determined by the same estimates as in [6, 14, 18] since we still have $dC/dt \sim \beta C^3$ and the theoretical exponent is $\alpha = 2/5$ being independent of f if $f\ell < 1$ so that the local transition rates remain the same as at $f = 0$. The numerical results presented here show weak dependence of α on f for $f < 1/\ell$ being in a satisfactory agreement with this theoretical estimate. The deviation of $\alpha \approx 0.3$ from the theoretical value $2/5$ still should be better clarified both for $f = 0$ and $|f| > 0$.

In summary, we demonstrated that, in presence of a static field applied to a lattice with disorder, a nonlinearity still produces complete delocalization with a subdiffusive spreading over the ladder. The exponent of the spreading remains close to the value without the force. The spreading forms a homogeneous distribution of probability inside a certain plateau. Due to the conservation of total probability and energy the far away parts of this plateau remain statistically entangled even being a large distance apart from each other. The obtained results can be tested in experiments with nonlinear photonic lattices and BEC atoms in optical lattices with a static field.

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