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Opinion formation driven by PageRank node influence on directed networks

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HIGHLIGHTS

- Opinion formation driven by heterogeneous node influence on real networks.
- PageRank and its sublinear power are considered as node influence measures.
- The more heterogeneous influence distribution, the shorter relaxation time.
- The more heterogeneous influence distribution, the more totalitarian opinion state.
- A group of influential nodes can impose their own opinion on significant number of nodes.

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ABSTRACT

We study a two states opinion formation model driven by PageRank node influence and report an extensive numerical study on how PageRank affects collective opinion formations in large-scale empirical directed networks. In our model the opinion of a node can be updated by the sum of its neighbor nodes' opinions weighted by the node influence of the neighbor nodes at each step. We consider PageRank probability and its sublinear power as node influence measures and investigate evolution of opinion under various conditions. First, we observe that all networks reach steady state opinion after a certain relaxation time. This time scale is decreasing with the heterogeneity of node influence in the networks. Second, we find that our model shows consensus and non-consensus behavior in steady state depending on types of networks: Web graph, citation network of physics articles, and LiveJournal social network show non-consensus behavior while Wikipedia article network shows consensus behavior. Third, we find that a more heterogeneous influence distribution leads to a more uniform opinion state in the cases of Web graph, Wikipedia, and Livejournal. However, the opposite behavior is observed in the citation network. Finally we identify that a small number of influential nodes can impose their own opinion on significant fraction of other nodes in all considered networks. Our study shows that the effects of heterogeneity of node influence on opinion formation can be significant and suggests further investigations on the interplay between node influence and collective opinion in networks.

1. Introduction

Each individual has her/his own opinion about political, social, and economical issues based on her/his own belief, information, and perspective. Individuals also exchange, discuss, and reconcile their opinions with others through social contacts

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or networks. Through these interactions, collective opinions emerge from our society. The recent advent of social media such as Twitter or Facebook accelerates the emergence of collective opinions on global scale. Understanding how collective opinions are formed on various types of social networks has critical importance in the era of information technology.

Statistical physics community has provided quantitative tools to reveal the underlying mechanisms that govern the collective opinion formation through social interactions [1]. Various opinion formation models (see Refs. [1,2] for details) on networks including voter models [3–6], majority rule model [7], bounded confidence model [8], and Sznajd model [9] were suggested and extensively studied. These models have given us analysis tools of how network structure affects opinion dynamics and have provided us mathematical understanding of collective opinion formation.

In order to expand our understanding of collective opinion formation on networks further we can consider the following two directions. First we can consider opinion formation on real social networks rather than on artifact network models such as regular lattices or small-world networks which are mainly considered in previous studies [1,2] and far from real networks. Second, in most of real situations, there are opinion leaders or elites who have strong influence and lead collective opinions in social systems. The roles of these leaders or elites on opinion formation is still elusive. In short, it is necessary to understand how heterogeneous individual influence affects on collective opinion formation on real networks.

In a recent study [10], PageRank is proposed as a node influence measure in an opinion formation model on large-scale real networks such as Web graphs and social media including LiveJournal and Twitter. The PageRank opinion formation (PROF) model, introduced in Ref. [10], takes into account a node influence in the process of opinion formation. In the PROF model, the opinion of a node is updated by the weighted sum of neighbor nodes' opinions and the weight of the neighbor nodes are given by their PageRank (see the next section for details). It is found that a group of top influential elites in the networks (i.e., nodes with high PageRank) can impose their own opinion on a significant fraction of the considered networks [10]. The PROF model is also considered on Ulam networks [11], generated by the intermittency map and the Chirikov typical map, showing a similar behavior with the case of World Wide Web (WWW).

In the present work we consider how heterogeneous node influence affects the collective opinion formation using the modified PageRank opinion formation (PROF) model to go beyond previous works [10,11]. Our goal is to examine how the PROF model behaves on real directed networks if we adjust the heterogeneity of node influence (i.e., the PageRank of nodes). The original PROF model considered only linear case of PageRank as a node influence, it is necessary to consider opinion formation driven by node influence under more general conditions. To do this we modified the PROF model considering sublinear PageRank of nodes such that the influence of node i is given by P_i^g where P_i is the PageRank of node i and $0 \le g \le 1$. Extensive numerical study of the model shows various features of considered opinion formation. First we observed that all networks reach a steady state opinion and the relaxation time to this state is decreasing with the heterogeneity of node influence in the networks. Second we found our model shows consensus and non-consensus behavior in steady state depending on types of networks: Web graph, citation network of physics articles, and LiveJournal social network show non-consensus behavior while Wikipedia article network shows consensus behavior. Third we found that the more heterogeneous distribution of node influence the network has (i.e., higher g), the more uniform opinion state we can observe in Web graph, Wikipedia, and Livejournal. However, in the citation network, the more heterogeneous distribution of node influence leads to the less uniform opinion. Finally we observed that a small number of influential nodes can impose their own opinion on significant fraction of other nodes in all considered networks.

The paper is organized as follows. The modified PROF model is described in Section 2. The description of considered empirical directed networks is given in Section 3. The extensive numerical studies on empirical networks are presented in Section 4. A discussion of the result is given in Section 5.

2. Opinion formation by the modified PROF model

We consider a directed network $G(N, L)$ with N nodes and nodes in the network are connected by L directed links. Based on the network structure, the PageRank probability $P_i(t)$ of node i at iteration time t is given by

$$P_i(t) = (1 - \alpha)/N + \alpha \sum_j A_{ij} P_j(t-1)/k_{out}(j), \tag{1}$$

where A_{ij} is the adjacency matrix of the network G and $A_{ij} = 1$ if there is a directed link from node i to j, $k_{out}(j)$ is the out-degree of node j (i.e., number of out-links from node j), and α is the damping factor [12]. In this study, we used the conventional value $\alpha = 0.85$ [12]. We take the stationary state $P(i)$ of $P(i, t)$ as the PageRank of node i.

PageRank is a widely used node centrality to quantify influence of nodes in a given directed network. Originally PageRank was introduced for Google web search engine to rank web pages in World Wide Web based on the idea of academic citations [13]. Currently PageRank is used to rank nodes in various types of directed networks including citation networks of scientific papers [14,15], social network services [16], world trade network [17], biological systems [18], Wikipedia [19–21], scientists [22], and tennis players [23].

In this work each node i has a binary opinion $\sigma_i \in \{-1, +1\}$ and has PageRank P_i as a node influence based on network structure and Eq. (1). At each opinion update, a node i is randomly chosen and its opinion is updated considering its neighbor nodes' opinions. Each time step consists of N updates. Thus one time step corresponds to one opinion update for each node on average. The opinion updating rule considers node influence of each neighbor node. Adopted from the original PageRank opinion formation (PROF) model [10,11], the update rule reads: if the following function $H(i)$ for the chosen node i is positive, then $\sigma_i = +1$ otherwise $\sigma_i = -1$. The function $H(i)$ is given by:

$$H(i) = a \sum_{j \in \Lambda_{i,in}} \sigma_j P_j^{g} + b \sum_{j \in \Lambda_{i,out}} \sigma_j P_j^{g}, \quad a + b = 1 \tag{2}$$

where $\Lambda_{i,in}$ is the group of in-neighbor nodes of node i (i.e., the nodes have out-links to node i) and $\Lambda_{i,out}$ is the group of out-neighbor nodes of node i (i.e., the nodes have out-links from node i), respectively. The parameter g quantifies the heterogeneity of node influence. If $g = 0$ then every node in the network has same node influence. If $g = 1.0$ then every node in the network can influence other nodes' opinion as much as its PageRank and thus this case is reduced to the original PROF model [10]. Thus, $H(i)$ is the weighted summation of opinions of node i's neighbor nodes. In this study we use $a = b = 0.5$ for simplicity of analysis.

3. Empirical networks

We consider the following four empirical directed networks. (1) *Web graph*: we consider Web graph of University of Cambridge [24,25]; here each node corresponds to a Web page and a link is hyper-link between the Web pages in the domain of University of Cambridge. (2) *Citation network*: we consider Physical Review citation network [15]; here a node corresponds to an article published in Physical Review journal of American Physical Society from 1897 to 2009 and the links correspond to the citation relations between the articles. (3) *Wikipedia*: we consider the network of articles in French Wikipedia [21]; the nodes correspond to articles in French Wikipedia (fr.wikipedia.org) and the links are the inter-articles hyper-links between the articles. (4) *LiveJournal*: we consider the social network of LiveJounral (livejournal.com) users; here the nodes are users of LiveJournal and the links are social relationship between the users; a more detail information on the network data is given in Ref. [26].

Statistical properties of the considered empirical networks are represented in Table 1. It is notable that unlike typical networks such as regular lattices or small-world networks considered in opinion formation models, all considered networks in this work have complex structural properties including broad degree distributions and broad distributions of PageRank [10,15,21,24].

4. Results

With the modified PROF model on described empirical networks, we investigate dynamics of collective opinion formation. First we consider evolution of the fractions of $(+1)$ opinion, $f(t, +1)$, by time t to investigate whether considered networks can reach the steady state or not and whether they reach consensus opinion or not if the networks can reach the steady state. For simplicity, we represent $f(t) = f(t, +1)$. By definition, we can consider the fraction of (-1) opinion $f(t, -1) = 1 - f(t)$ easily. Starting with same initial fraction of two opinions (i.e., $f(0, +1) = f(0, -1) = 0.5$), we numerically investigate how fractions of each opinion state evolve by time t. As shown in Fig. 1, all considered networks have reached the steady states. Sub-figures located in the bottom row of Fig. 1 represent the evolution of the fraction of $(+1)$ opinion nodes $f(t)$ along with time t and $g = 1$ (10 realizations for each network). For Wikipedia case (the third column of Fig. 1), we can observe "consensus" behavior (i.e., most of nodes have single major opinion whether $(+1)$ or (-1)). However, we observed that Web graph (the first column of Fig. 1), Citation network (the second column of Fig. 1), and LiveJournal social network (the fourth column of Fig. 1) show non-consensus behavior (i.e., two finite values of opinion co-exist in the steady states). Here we define that if a given network have reached either $f_s > 0.95$ or $f_s < 0.05$, the network shows consensus behavior where f_s is the fraction of $(+1)$ opinion in the steady state. We find that Web graph and Wikipedia relax to the steady state (either consensus or non-consensus) in short time ($t < 30$) as shown in Fig. 1 while more longer times ($t > 40$) are necessary to reach the steady states in cases of Citation and LiveJournal networks. Sub-linear g values cases (figures from the first to fourth row) show similar behaviors of reaching steady state with the linear cases. But it is notable that for Web graph and Wikipedia, the differences between each steady state fractions of $(+1)$ opinions are bigger with growing g. We can consider this observation as a sign of growing polarization of steady state opinion. However, other networks give no clear signs. A further more quantitative analysis for these gaps between the fraction of steady state opinions are required.

To quantify the effects of g value on the relaxation time to the steady state of the collective opinion, first we define $\langle f(t)\rangle_{10}$ as an average fraction of $(+)$ state for 10 consecutive time steps from time t to $t + 9$ as follows.

$$\langle f(t)\rangle_{10} = \frac{1}{10} \sum_{t}^{t+9} f(t). \tag{3}$$

We define time T_c of reaching the steady state for each network such that the standard deviation $\sigma(10)$ of above ten consecutive fraction $f(t)$ of $(+1)$ opinion nodes from time $t = T_c$ to $t = T_c + 9$ is less than 0.0002. (i.e., $\sigma(10) < 0.0002$). Fig. 2 represents the relation between steady state relaxation time T_c and the influence exponent g. We can observe a clear tendency that bigger g (more heterogeneous influence the network has) leads to shorter time to reach the steady states for all networks. As Fig. 1 implies, Web graph and Wikipedia have shorter relaxation times $T_c < 30$ for various g while Citation and LiveJournal networks have significantly longer $40 < T_c < 110$ and effects of g variation are more pronounced.

Table 1
Basic statistics of empirical directed networks, N gives the total number of nodes and L gives the total number of links.

Network	N	L
Web graph	212 710	1 831 542
Citation	463 349	4 690 897
Wikipedia	1 352 825	34 431 943
LiveJournal	3 577 166	44 913 072

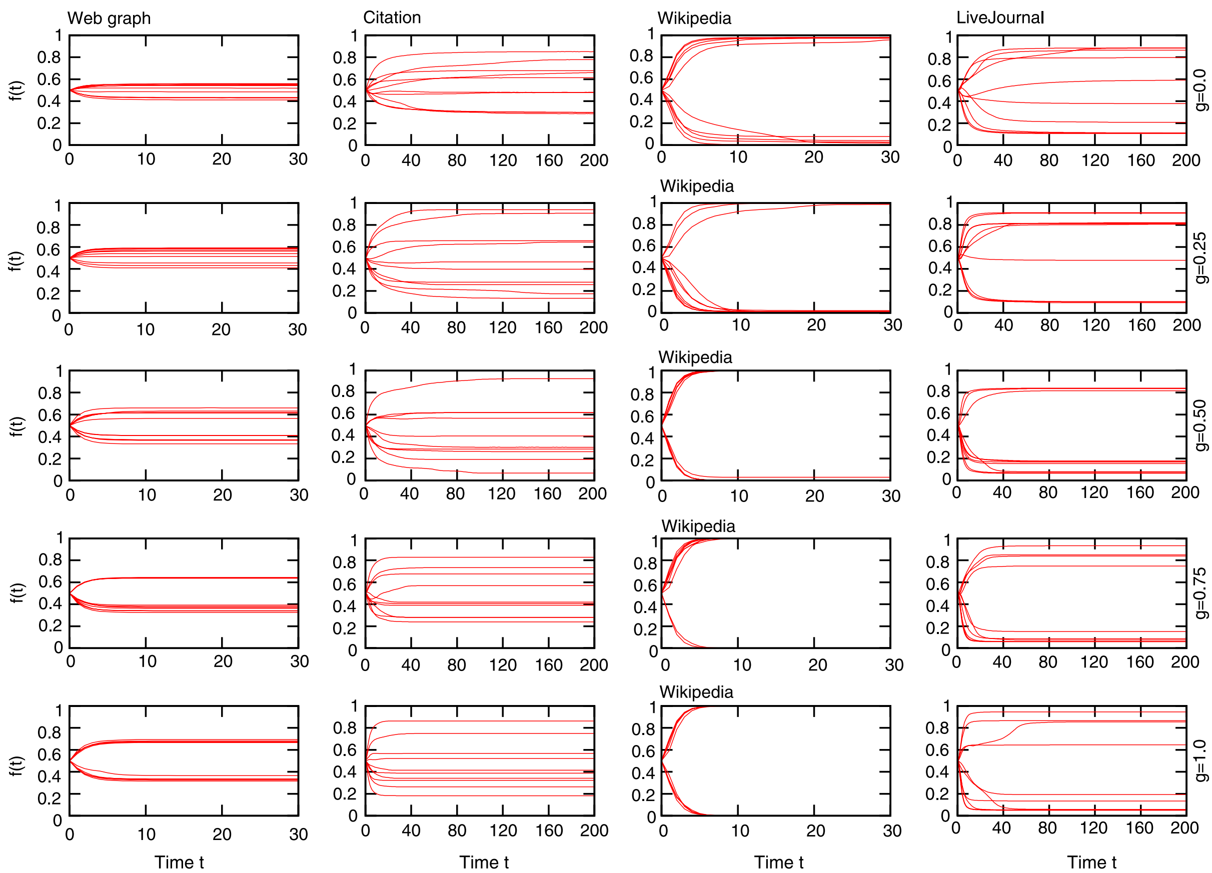

Fig. 1. Evolution of the fractions of (+1) opinion $f(t)$ in time t. Here 10 realizations per each network and each value of g are represented. Each column corresponds to the network and each row corresponds to g.

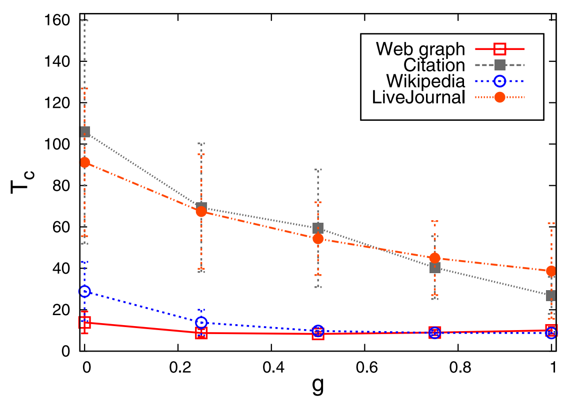

Fig. 2. Dependence of the relaxation time T_c to a steady state on the influence exponent g for considered networks.

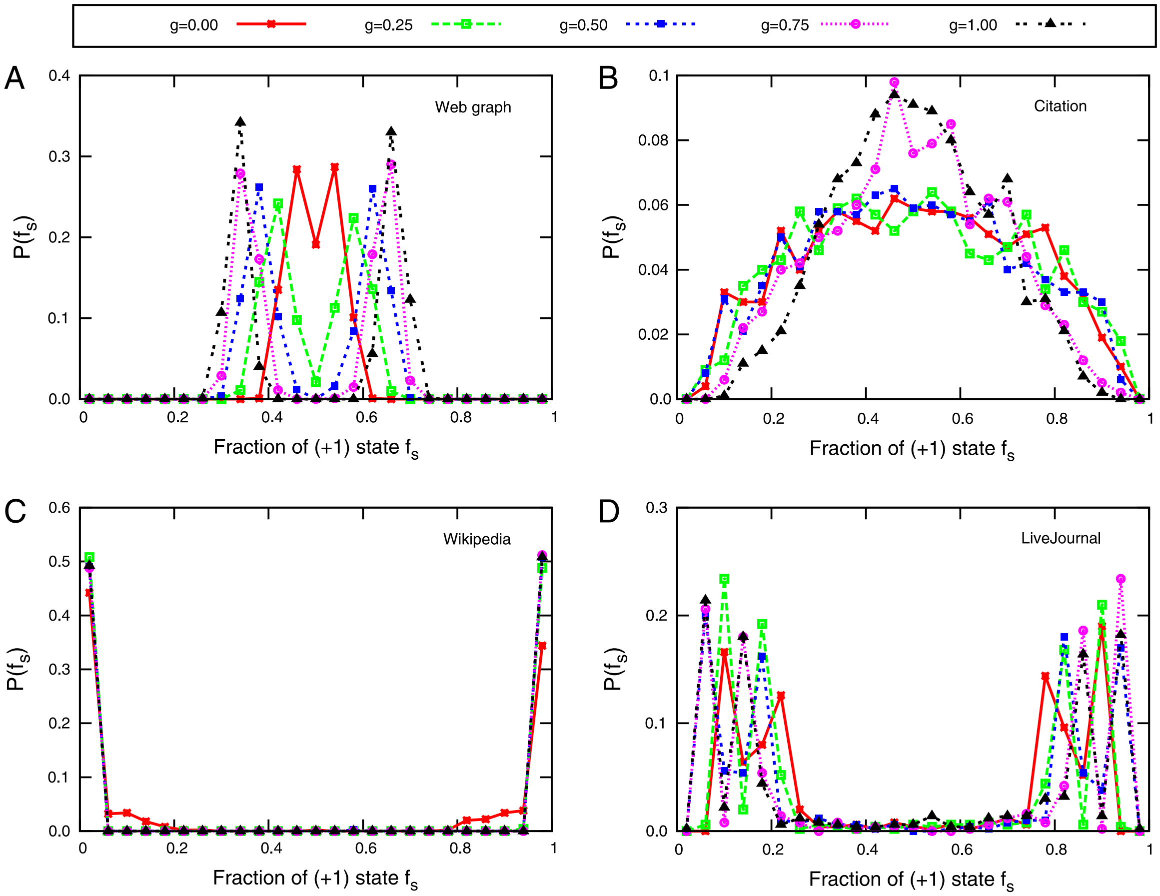

Fig. 3. Distributions of (+1) opinion fraction in the steady state for each empirical network. Here f_s is the fraction of (+1) opinion in steady state and $P(f_s)$ is the probability distribution function of f_s. All the cases start with initial fraction of $f(+1, 0) = f(-1, 0) = 0.5$ with 1000 realizations for Web graph and Citation networks and 500 realizations for Wikipedia and LiveJournal.

In order to analyze opinion formation in the steady states and study polarization of steady state opinions, we investigate distributions of fraction of (+1) opinion f_s in steady state for each network. Fig. 3 represents the distributions of fraction of (+1) opinion in the steady states for each case of empirical network starting with $f(0, +1) = f(0, -1) = 0.5$. For the cases of Web graph, Wikipedia, and LiveJournal, increasing g resulted in more uniform opinion states (i.e., the fractions of majority opinion state whether (−1) or (+1) are getting higher with g). However, the fraction of majority opinion might not be increasing monotonously as a function of g. This indicates that a more heterogeneous node influence distribution in networks may lead to a more "totalitarian" society. However, the Citation network shows the opposite pattern. It is notable that the Citation network has different structural property from other directed networks. Unlike the other considered networks, reciprocal links (i.e., bi-directed links connecting from node i to node j and from node j to i) are very rare in the citation networks due to time-ordering of citation relationships between scientific articles (i.e., it is practically not possible to cite publications in future). Thus this distinctive structure might affect behaviors of collective opinion on the network.

So far we considered only evolution of opinion states starting from the same fractions of initial opinion states (i.e., $f(0, +1) = f(0, -1) = 0.5$). If initial fraction of two opinions are different, then how collective opinions on networks are formed? In order to find out how the steady state fraction f_s of nodes with (+1) opinion depends on its initial fraction $f_i = f(0, +1)$, we investigate opinion formation with varying initial fraction of (+1) opinion and varying g. Fig. 4 represents a fraction of (+1) opinion in the steady state f_s versus an initial fraction of (+1) opinion f_i for each empirical network. Each row in Fig. 4 represents each network and each column represents each value of g.

In the case of Web graph, we can observe the emergence of bistability as g is increasing. Here bistability means there exist two steady state fractions of (+1) opinion. The bistability of Web graphs is also observed in Ref. [10] in the case of University of Cambridge and Oxford Web graph with original PROF model (i.e., $g = 1.0$). When g is small ($g \leq 0.25$), the fraction of (+1) opinion f_s in the steady state reached single value of fraction with some fluctuations. Meanwhile, when $g \geq 0.5$, there are two values of f_s in the steady state. For LiveJournal network, there are signs of multiple steady state fractions of (+1) opinion as shown in Fig. 3(D). This phenomenon is also observed in Fig. 4 but only for $f_i = 0.5$. If $f_i \neq 0.5$, we cannot observe such multistability in the steady state. On the other hand, there is no such bistability for the case of Citation network and

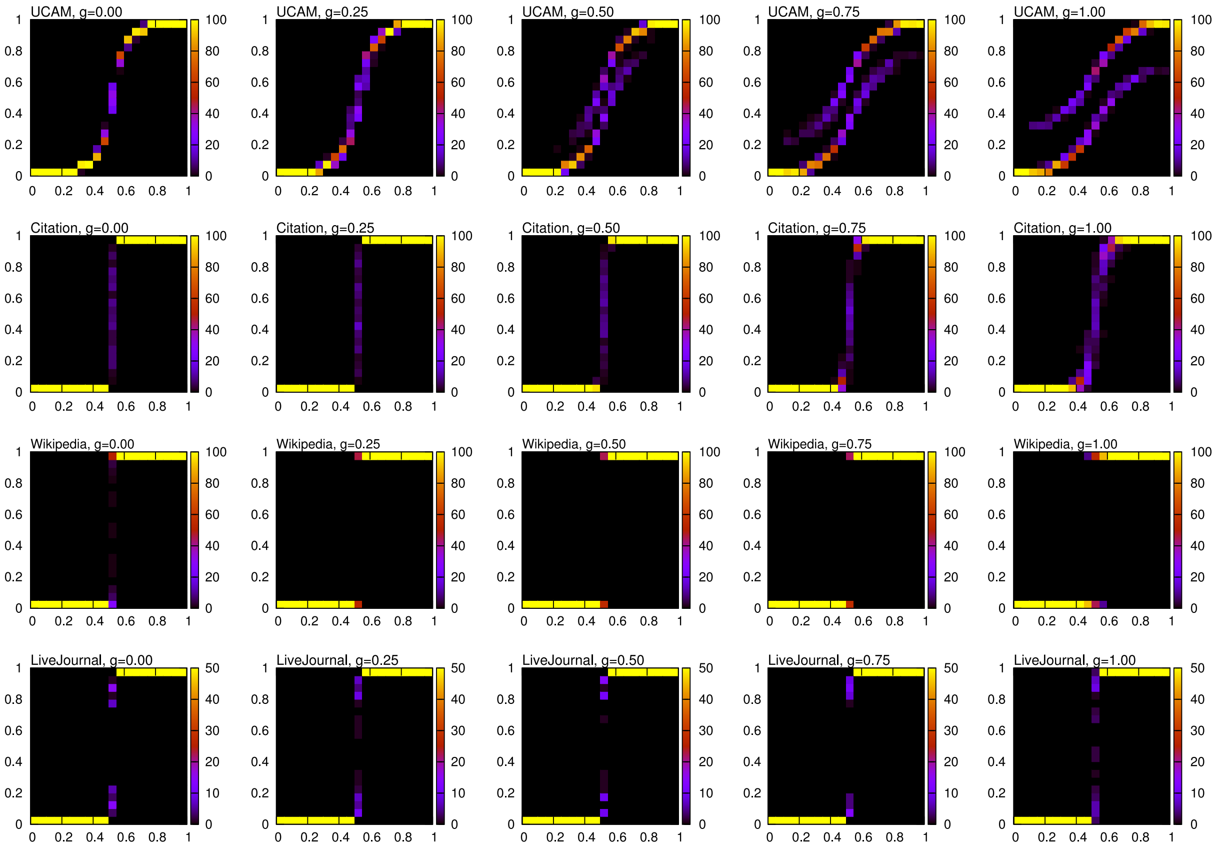

Fig. 4. Fraction of (+1) opinion states f_s (y-axis) in the steady state as function of initial fraction f_i (x-axis) of (+1) opinion state for given network and g. Each row corresponds to each network and each column corresponds to the value of g. Here there are 100 realizations for Web graph, Citation networks, and Wikipedia and 50 realizations for LiveJournal. Here the color marks the relative number of cases obtained for given values (f_i, f_s), the color changes from black (zero) to red (maximal number of cases). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Wikipedia. In particular the Wikipedia network shows if the initial fraction of (+) opinion is less (more) than 0.45 (0.55), the final fraction is always less (more) than 0.05 (0.95). Based on the observation, the initial fraction of the opinion states can be critical for opinion formation in these networks but the detail behaviors can be different depending on the types of networks.

To characterize the effects of influential nodes on opinion formation, we investigate how a group of selected nodes with a fixed opinion can impose their own opinion on the entire network. We compare two opinion implanting strategies of n seed nodes with a fixed opinion.

In the *random implanting strategy*, we choose n nodes as seed nodes from a given network randomly and assign (+1) opinion to them. The opinions of seed nodes are fixed. We assign (−1) opinion to the rest of nodes (i.e., non-seed nodes) in the networks. The opinions of the non-seed nodes are flexible thus their opinions can be changed by the modified PROF rule at each update. Meanwhile in the *targeted implanting strategy*, we choose n nodes as seed nodes in order of PageRank of the nodes and assign (+1) opinion to them. The opinions of seed nodes are also fixed. We assign (−1) opinion to the rest of nodes in the network and update the opinions of non-seed nodes by modified PROF rule as in the random implanting strategy at each update.

Fig. 5 compares the fraction of (+1) opinion nodes in the steady state by two implanting strategies. Regardless of networks and value of g, targeted implanting cases are much more effective to lead collective opinion states of the networks to (+1) opinion. Even when $g = 0.0$ (i.e., every node has the same node influence), targeted implanting is more effective than random implanting strategy to change the nodes in the networks to (+1) opinion. The tendency is getting stronger with g. For the Citation, Wikipedia, and LiveJournal networks, even a very small fraction of top influential nodes with fixed (+1) opinion (i.e., $f(0) \leq 0.01$) can lead to the significant fraction of (+1) opinion in the steady state on the networks. For the Web graph, the tendency is weaker partially due to the "bistability" we observed above. In Ref. [10], it was observed that imposing (+1) opinion on small initial fraction ($\sim$1% of nodes) of top PageRank nodes can lead 40% of (+) opinion states. Our analysis indicates this "elite" effect can exist even when every node has the same influence but the elite effect can be much stronger when node influence are heterogeneously distributed with a larger value of g.

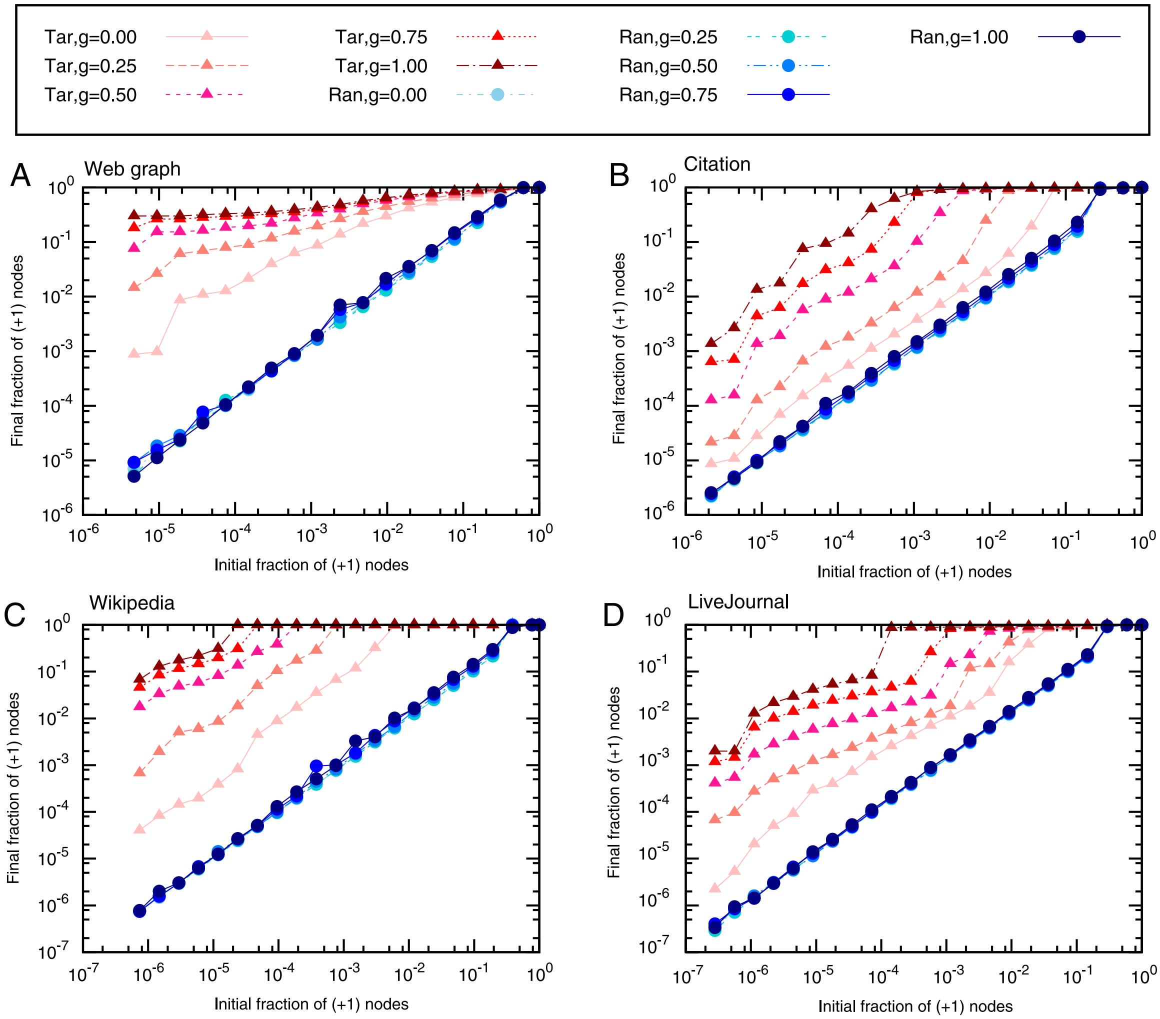

Fig. 5. Comparisons between the targeted implanting strategy and random implanting strategies. "Tar" represents the targeted implanting strategy and "Ran" represents the random implanting strategy. For targeted implanting strategy (filled triangles), pink, salmon, dark-pink, red, and dark-red colors represent $g = 0.0$, $g = 0.25$, $g = 0.5$, $g = 0.75$, and $g = 1.00$, respectively. For random implanting strategy (filled circles), skyblue, dark-turquoise, web-blue, blue, and navy represent $g = 0.0$, $g = 0.25$, $g = 0.5$, $g = 0.75$, and $g = 1.00$, respectively. Here there are 100 realizations for Web graph and Citation networks and 50 realizations for Wikipedia and 25 realizations for LiveJournal. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

It would be also interesting to consider targeted implanting strategies based on other centrality measures. We consider two additional targeted implanting strategies based on in-degree and betweenness centralities for Web graph since it is not feasible to get betweenness for other networks due to their large sizes. As shown in Fig. 6, the performances of three targeted strategies based on in-degree, betweenness, and PageRank are quite similar with each other. We can expect similar results for the other networks since PageRank is known to be positively correlated with in-degree and betweenness centralities. The actual correlation between PageRank and in-degree in Web graph is 0.886 and the correlation between PageRank and betweenness in Web graph is 0.706.

5. Discussion

Opinion formation in social systems is mediated by social interactions between the individuals in the systems and at the same time it is affected by influence of interacting nodes. Thus understanding this interplay between individuals' influence and network structure of social interactions is a salient issue. In this study we used the modified PageRank opinion formation (PROF) model to consider how heterogeneous node influence affects collective opinion formation on real networks and analyzed effects of heterogeneity of node influence on opinion formation. We found that the relaxation time to reach the steady state is decreasing with the heterogeneity of node influence in the networks. We also identified that a small number of influential nodes can impose their opinion on significant fraction of nodes, and the impact of these social elites on collective opinion is growing with the heterogeneity of node influence.

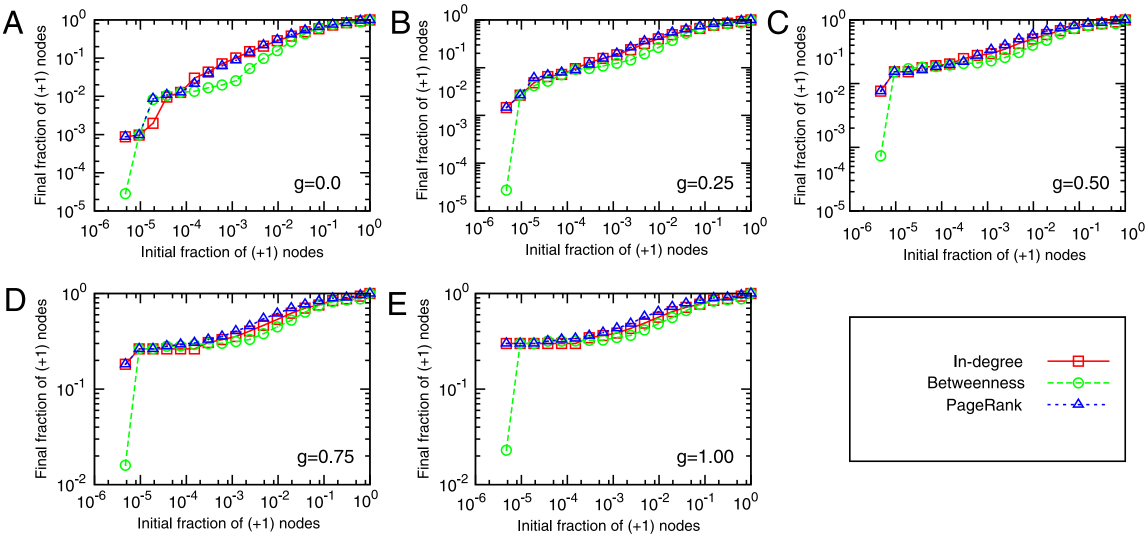

Fig. 6. Comparisons between the degree, betweenness, and PageRank targeted implanting strategy. Here there are 100 realizations for Web graph.

All of considered networks reach a steady opinion state. However, it is not clear why only Wikipedia shows consensus and the other networks do not. Since we considered directed networks, asymmetric nature of links could be the obstacle to reach consensus. To check the effect of the asymmetric nature of links, we considered undirected version of empirical networks but observed the same non-consensus behaviors. Thus we can rule out this explanation. On the other hand, a strong local structure such as communities or modules [27,28] can prohibit to reach the consensus opinion state. Since communities in networks are typical composed of a group of tightly connected nodes, such a densely connected group of nodes may persist the influence from other parts of the networks. It would be interesting to study an interplay between influential nodes and community structure. The Citation network also displays the opposite behaviors from the other networks such that the other networks show more uniform opinions states with growing g while Citation network shows less uniform steady state opinion. It will be interesting to check if other citation networks show similar behaviors with our Citation network.

In this study we used PageRank and its sub-linear power as node influence. However, other node centralities on directed network can be considered as node influence including in-degree, betweenness centrality [29], CheiRank [30], 2DRank [31], or non-structural node attributes. Since PageRank is positively correlated with in-degree, the study of considering node influence which is positively correlated with in-degree can be interesting. As described above, community or core–periphery structures may also significantly affect the collective opinion formation with a local structure-based influence measure.

Due to the advent of information technology and growing usage of social media, the problem of collective opinion formation is getting more and more complicated going to a global scale. A quantitative understanding of opinion formation on large-scale networks becomes of crucial importance. Our study sheds a new light on how the node influence and network structure together affect the collective opinion in directed networks.

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