

Wikipedia mining of hidden links between political leaders

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Abstract. We describe a new method of reduced Google matrix which allows to establish direct and hidden links between a subset of nodes of a large directed network. This approach uses parallels with quantum scattering theory, developed for processes in nuclear and mesoscopic physics and quantum chaos. The method is applied to the Wikipedia networks in different language editions analyzing several groups of political leaders of USA, UK, Germany, France, Russia and G20. We demonstrate that this approach allows to recover reliably direct and hidden links among political leaders. We argue that the reduced Google matrix method can form the mathematical basis for studies in social and political sciences analyzing Leader-Members eXchange (LMX).

PACS. 89.75.Fb Structures and organization in complex systems – 89.75.Hc Networks and genealogical trees – 89.20.Hh World Wide Web, Internet

1 Introduction

At present a free online encyclopaedia Wikipedia [1] becomes the largest open source of knowledge being close to Encyclopaedia Britannica [2] by an accuracy of scientific entries [3] and overcoming the latter by an enormous amount of available information. A detailed analysis of strong and weak features of Wikipedia is given at [4,5].

Since Wikipedia articles make citations to each other they generate a larger directed network with a rather clear meaning of nodes defined by article titles. Due to these reasons it is interesting to apply algorithms developed for search engines of World Wide Web (WWW), those like the PageRank algorithm [6] (see also [7]), to analyze the ranking properties and relations between Wikipedia articles. The clear meaning of Wikipedia nodes allows also to use its network as a test bed for machine learning algorithms computing semantic relatedness [8].

It is convenient to describe the network of N Wikipedia articles by the Google matrix G constructed from the adjacency matrix A_{ij} with elements 1 if article (node) j points to article (node) i and zero otherwise. Then the matrix elements of the Google matrix take the standard form [6, 7]

$$G_{ij} = \alpha S_{ij} + (1 - \alpha)/N, \quad (1)$$

where S is the matrix of Markov transitions with elements $S_{ij} = A_{ij}/k_{out}(j)$, $k_{out}(j) = \sum_{i=1}^N A_{ij} \neq 0$ being the node j out-degree (number of outgoing links) and with $S_{ij} = 1/N$ if j has no outgoing links (dangling node). Here $0 < \alpha < 1$ is the damping factor which for a random surfer determines the probability $(1 - \alpha)$ to jump to any node. The properties of spectrum and eigenstates of

G have been discussed in detail for Wikipedia and other directed networks (see e. g. [9,10]).

The right eigenvectors $\psi_i(j)$ of G are determined by the equation:

$$\sum_{j'} G_{jj'} \psi_i(j') = \lambda_i \psi_i(j). \quad (2)$$

The PageRank eigenvector $P(j) = \psi_{i=0}(j)$ corresponds to the largest eigenvalue $\lambda_{i=0} = 1$ [6,7]. It has positive elements which give the probability to find a random surfer on a given node in the stationary long time limit of the Markov process. All nodes can be ordered by a monotonically decreasing probability $P(K_i)$ with the highest probability at $K = 1$. The index K is the PageRank index. Left eigenvectors are biorthogonal to right eigenvectors of different eigenvalues. The left eigenvector for $\lambda = 1$ has identical (unit) entries due to the column sum normalization of G . One can show that the damping factor α in (1) only affects the PageRank vector (or other eigenvectors for $\lambda = 1$ of S in case of a degeneracy) while other eigenvectors are independent of α due to their orthogonality to the left unit eigenvector for $\lambda = 1$ [7]. Thus all eigenvalues, except $\lambda = 1$, are multiplied by a factor α when replacing S by G . In the following we use the notations ψ_L^T and ψ_R for left and right eigenvectors respectively (here T means vector or matrix transposition).

In many real networks the number of nonzero elements in a column of S is significantly smaller than the whole matrix size N that allows to find efficiently the PageRank vector by the PageRank algorithm of power iterations [7]. Also a certain number of largest eigenvalues (in modulus)

and related eigenvectors can be efficiently computed by the Arnoldi algorithm [10].

For various language editions of Wikipedia it was shown that the PageRank vector produces a reliable ranking of historical figures over 35 centuries of human history [11,12,13,14] and Wikipedia ranking of world universities (WRWU) [11,15]. Thus the Wikipedia ranking of historical figures is in a good agreement with the well-known Hart ranking [16], while the WRWU is in a good agreement with the Shanghai Academic ranking of world universities [17]. At the same time Wikipedia ranking produces some new additional insight as compared to these classifications.

In addition to the matrix G it is useful to introduce a Google matrix G^* constructed from the adjacency matrix of the same network but with inverted direction of all links. The statistical properties of the eigenvector P^* of G^* with the largest eigenvalue $\lambda = 1$ have been studied first for the Linux Kernel network [18] showing that there are nontrivial correlations between P and P^* vectors of the network. More detailed studies have been done for Wikipedia and other networks [11,10]. The vector $P^*(K^*)$ is called the CheiRank vector and the index numbering nodes in order of monotonic decrease of probability P^* is noted as CheiRank index K^* . Thus, nodes with many incoming (or outgoing) links have small values of $K = 1, 2, 3, \dots$ (or of $K^* = 1, 2, 3, \dots$) [7,10]. Examples of density distributions (in the $(\ln K, \ln K^*)$ plane) for Wikipedia editions EN, DE, FR, RU from the year 2013 (see network data in [14]) are shown in Fig. 1.

Other eigenvectors of G have $|\lambda| \leq \alpha$ [7,10]. For Wikipedia it was shown that the eigenvectors with a large modulus of λ select some specific communities of Wikipedia network [9,10]. However, a priori it is not possible to know what the meanings of these communities are. Thus other methods are required to determine effective interactions between N_r nodes of a specific subset (group) of the global network of a large size N .

Recently, the method of reduced Google matrix has been proposed for analysis of effective interactions between nodes of a selected subset embedded into a large size network [19]. This approach uses parallels with the quantum scattering theory, developed for processes in nuclear and mesoscopic physics and quantum chaos. In this work we apply this method to subsets (groups) of Wikipedia articles about political leaders (politicians) considering English, French, German and Russian Wikipedia editions and politicians of USA (US), UK, Germany (DE), France (FR) and Russia (RU). The total number of nodes for these Wikipedia networks is $N = 4212493$ (EN), 1532978 (DE), 1352835 (FR), 966284 (RU) [14]. We also analyze interactions between political leaders of the G20 Los Cabos summit in 2012 [20]. The selected subsets have networks of 20 or 40 nodes and are well suited for analysis of direct and hidden links between politicians. In our analysis we use the Wikipedia networks collected in 2013 and described in [14]. The location of selected nodes on the PageRank-CheiRank plane $(\ln K, \ln K^*)$ is shown in Fig. 1. The obtained results allow to determine interest-

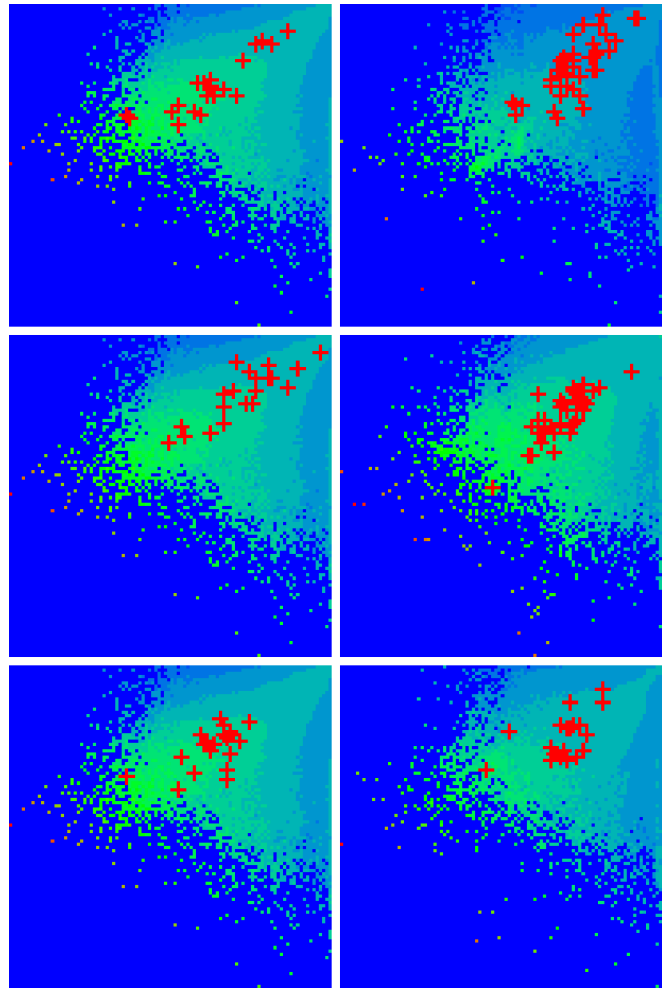


Fig. 1. Density of nodes $W(K, K^*)$ on the PageRank-CheiRank plane averaged over 100×100 logarithmic equidistant grids for $0 \leq \ln K, \ln K^* \leq \ln N$. The density is averaged over all nodes inside each cell of the grid with the x -axis corresponding to $\ln K$ and the y -axis to $\ln K^*$. The panels of the left column correspond all to the English (Enwiki) and panels of the right column to the German (Dewiki, top), French (Frwiki, center) and Russian (Ruwiki, bottom) Wikipedia editions of 2013 collected in [14]. The colors represent maximum (red), intermediate (green) and minimum (blue) values. The red crosses show the positions of the selected reduced network nodes for 20 US (top left), 20 UK (center left), 40 German (DE, top right), 40 French (FR, center right), 20 Russian (RU, bottom right) politicians and 20 state leaders of G20 states (bottom left) with the list of names given in Tables 2-7 and in Figs. 2-7.

ing direct and hidden relations between political leaders of the selected countries.

We should note that the analysis of interactions and relations between political leaders represents a hot topic in social and political sciences [21]. Thus the interactions between leader and group members, known as Leader-Members eXchange (LMX), attracts at present active investigations of researchers in social and political sciences

[22,23,24]. However, only very recently the methods of complex networks [25] started to be used by in the LMX analysis [26]. In this work we argue that the approach to determine the reduced Google matrix G_R represents a useful and efficient tool for the LMX analysis of interactions inside a group of people. Thus for a group of politicians (a group of their articles at Wikipedia) we find that those at the top of PageRank index K are the dominant leaders being usually country presidents or prime-ministers. It turns out that the obtained G_R matrix, describing the interactions between group members, is composed of three matrix components. These components describe: the direct interactions G_{rr} between group members, a projector part G_{pr} which is mainly imposed by the PageRank of group members given by the global G matrix and a component G_{qr} from hidden interactions between members which appear due to indirect links via the global network. Thus the reduced matrix $G_R = G_{rr} + G_{pr} + G_{qr}$ allows to obtain precise information about the group members taking into account their environment given by the global Wikipedia network. We think that this G_R matrix approach provides mathematical grounds for the LMX studies.

The paper is composed as follows: Section 2 shortly describes the method of reduced Google matrix, Section 3 presents distributions of selected subsets on the PageRank-CheiRank plane of global and reduced networks, Sections 4, 5, 6, 7, 8, and 9 describe the results of the reduced Google matrix analysis for politicians of USA, UK, Germany, France, Russia and G20 respectively. Section 10 provides a particular analysis for the group of French politicians in terms of effective networks of strongest friends or followers using either the matrix G_R or the hidden interactions given by the component G_{qr} . The discussion of the obtained results is given in Section 11. All numerical data of the reduced Google matrix of groups of political leaders considered here are publicly available at the web site [27].

2 Reduced Google matrix

The concept of reduced Google matrix G_R was introduced in [19] on the basis of the following observation. At present directed networks of real systems can be very large (about 4.2 million articles for the English Wikipedia edition in 2013 [10] or 3.5 billion web pages for a publicly accessible web crawl that was gathered by the Common Crawl Foundation in 2012 [28]). In certain cases one may be interested in the particular interactions among a small reduced subset of N_r nodes with $N_r \ll N$ instead of the interactions in the entire network. However, the interactions between these N_r nodes should be correctly determined taking into account that there are many indirect links between the N_r nodes via all other $N_s = N - N_r$ nodes of the network. This leads to the problem of the reduced Google matrix G_R with N_r nodes which describes the interactions of a subset of N_r nodes.

In a certain sense we can trace parallels with the problem of quantum scattering appearing in nuclear and mesoscopic physics [29,30,31,32,33] and quantum chaotic scattering [34]. Indeed, in the scattering problem there are

effective interactions between open channels to localized basis states in a well confined scattering domain where a particle can spend a certain time before it escapes by open channels. Having this analogy in mind we construct the reduced Google matrix G_R which describes interactions between selected N_r nodes and satisfies the standard requirements of the Google matrix.

Let G be a typical Google matrix of Perron-Frobenius type for a network with N nodes such that $G_{ij} \geq 0$ and the column sum normalization $\sum_{i=1}^N G_{ij} = 1$ are verified. We consider a sub-network with $N_r < N$ nodes, called “reduced network”. In this case we can write G in a block form :

$$G = \begin{pmatrix} G_{rr} & G_{rs} \\ G_{sr} & G_{ss} \end{pmatrix} \quad (3)$$

where the index “ r ” refers to the nodes of the reduced network and “ s ” to the other $N_s = N - N_r$ nodes which form a complementary network which we will call “scattering network”.

We denote the PageRank vector of the full network as

$$P = \begin{pmatrix} P_r \\ P_s \end{pmatrix} \quad (4)$$

which satisfies the equation $GP = P$ or in other words P is the right eigenvector of G for the unit eigenvalue. This eigenvalue equation reads in block notations:

$$(\mathbf{1} - G_{rr})P_r - G_{rs}P_s = 0, \quad (5)$$

$$-G_{sr}P_r + (\mathbf{1} - G_{ss})P_s = 0. \quad (6)$$

Here $\mathbf{1}$ is the unit matrix of corresponding size N_r or N_s . Assuming that the matrix $\mathbf{1} - G_{ss}$ is not singular, i.e. all eigenvalues G_{ss} are strictly smaller than unity (in modulus), we obtain from (6) that

$$P_s = (\mathbf{1} - G_{ss})^{-1}G_{sr}P_r \quad (7)$$

which gives together with (5):

$$G_R P_r = P_r \quad , \quad G_R = G_{rr} + G_{rs}(\mathbf{1} - G_{ss})^{-1}G_{sr} \quad (8)$$

where the matrix G_R of size $N_r \times N_r$, defined for the reduced network, can be viewed as an effective reduced Google matrix. Here the contribution of G_{rr} accounts for direct links in the reduced network and the second term with the matrix inverse corresponds to all contributions of indirect links of arbitrary order. We note that in mesoscopic scattering problems one typically uses an expression of the scattering matrix which has a similar structure where the scattering channels correspond to the reduced network and the states inside the scattering domain to the scattering network [31].

The matrix elements of G_R are non-negative since the matrix inverse in (8) can be expanded as:

$$(\mathbf{1} - G_{ss})^{-1} = \sum_{l=0}^{\infty} G_{ss}^l \quad . \quad (9)$$

In (9) the integer l represents the order of indirect links, i. e. the number of indirect links which are used to connect

indirectly two nodes of the reduced network. The matrix inverse corresponds to an exact resummation of all orders of indirect links. According to (9) the matrix $(\mathbf{1} - G_{ss})^{-1}$ and therefore also G_R have non-negative matrix elements. It remains to show that G_R also fulfills the condition of column sum normalization being unity. For this let us denote by $E^T = (1, \dots, 1)$ the line vector of size N with unit entries and by E_r^T (or E_s^T) the corresponding vectors for the reduced (or scattering) network with N_r (or N_s) unit entries such that $E^T = (E_r^T, E_s^T)$. The column sum normalization for the full Google matrix G implies that $E^T G = E^T$, i. e. E^T is the left eigenvector of G with eigenvalue 1. This equation becomes in block notation:

$$E_r^T (\mathbf{1} - G_{rr}) - E_s^T G_{sr} = 0, \quad (10)$$

$$-E_r^T G_{rs} + E_s^T (\mathbf{1} - G_{ss}) = 0. \quad (11)$$

From (11) we find that $E_s^T = E_r^T G_{rs} (\mathbf{1} - G_{ss})^{-1}$ which implies together with (10) that $E_r^T G_R = E_r^T$ using G_R as in (8). This shows that the column sum normalization condition is indeed verified for G_R justifying that this matrix is indeed an effective Google matrix for the reduced network.

We can question how to evaluate practically the expression (8) of G_R for a particular sparse and quite large network with a typical situation when $N_r \sim 10^2$ - 10^3 is small compared to N and $N_s \approx N \gg N_r$. If N_s is too large (e. g. $N_s > 10^5$) a direct naive evaluation of the matrix inverse $(\mathbf{1} - G_{ss})^{-1}$ in (8) by Gauss algorithm is not possible. In this case we can try the expansion (9) provided it converges sufficiently fast with a modest number of terms. However, this is most likely not the case for typical applications since G_{ss} is likely to have at least one eigenvalue very close to unity.

Therefore, we consider the situation where the full Google matrix has a well defined gap between the leading unit eigenvalue and the second largest eigenvalue (in modulus). For example if G is defined using a damping factor α in the standard way, as in (1), the gap is at least $1 - \alpha$ which is 0.15 for the standard choice $\alpha = 0.85$ [7]. For such a situation we expect that the matrix G_{ss} has a leading real eigenvalue close to unity (but still different from unity so that $\mathbf{1} - G_{ss}$ is not singular) while the other eigenvalues are clearly below this leading eigenvalue with a gap comparable to the gap of the full Google matrix G . In order to evaluate the expansion (9) efficiently, we need to take out analytically the contribution of the leading eigenvalue close to unity which is responsible for the slow convergence.

Below we denote by λ_c this leading eigenvalue and by ψ_R (ψ_L^T) the corresponding right (left) eigenvector such that $G_{ss}\psi_R = \lambda_c\psi_R$ (or $\psi_L^T G_{ss} = \lambda_c\psi_L^T$). Both left and right eigenvectors as well as λ_c can be efficiently computed by the power iteration method in a similar way as the standard PageRank method. We note that one can easily show that λ_c must be real and that both left/right eigenvectors can be chosen with positive elements. Concerning the normalization for ψ_R we choose $E_s^T \psi_R = 1$ and for ψ_L we choose $\psi_L^T \psi_R = 1$. It is well known (and easy to show) that ψ_L^T is orthogonal to all other right eigenvec-

Table 1. Values of $1 - \lambda_c$ and of Σ_P for the five groups of politicians and four cases of G20 subnetworks with λ_c being the leading eigenvalue of the matrix G_{ss} and $\Sigma_P = \|\mathcal{P}_r\|_1$ being the 1-norm of the projected PageRank representing the relative PageRank weight of the subset in the full network.

Network	Group	$1 - \lambda_c$	Σ_P
Enwiki	Politicians US	0.0003680	0.0003787
Enwiki	Politicians UK	0.0001041	0.0001068
Dewiki	Politicians DE	0.0002705	0.0002802
Frwiki	Politicians FR	0.0003810	0.0004005
Ruwiki	Politicians RU	0.0003136	0.0003247
Enwiki	G20 EN	0.0002465	0.0002508
Dewiki	G20 DE	0.0001492	0.0001513
Frwiki	G20 FR	0.0002131	0.0002159
Ruwiki	G20 RU	0.0002707	0.0002734

tors (and ψ_R is orthogonal to all other left eigenvectors) of G_{ss} with eigenvalues different from λ_c . We introduce the operator $\mathcal{P}_c = \psi_R \psi_L^T$ which is the projector onto the eigenspace of λ_c and we denote by $\mathcal{Q}_c = \mathbf{1} - \mathcal{P}_c$ the complementary projector. One verifies directly that both projectors commute with the matrix G_{ss} and in particular $\mathcal{P}_c G_{ss} = G_{ss} \mathcal{P}_c = \lambda_c \mathcal{P}_c$. Therefore we can write:

$$(\mathbf{1} - G_{ss})^{-1} = (\mathcal{P}_c + \mathcal{Q}_c)(\mathbf{1} - G_{ss})^{-1}(\mathcal{P}_c + \mathcal{Q}_c) \quad (12)$$

$$= \mathcal{P}_c \frac{1}{1 - \lambda_c} + \mathcal{Q}_c (\mathbf{1} - G_{ss})^{-1} \mathcal{Q}_c \quad (13)$$

$$= \mathcal{P}_c \frac{1}{1 - \lambda_c} + (\mathbf{1} - \bar{G}_{ss})^{-1} \mathcal{Q}_c \quad (14)$$

$$= \mathcal{P}_c \frac{1}{1 - \lambda_c} + \mathcal{Q}_c \sum_{l=0}^{\infty} \bar{G}_{ss}^l \quad (15)$$

with $\bar{G}_{ss} = \mathcal{Q}_c G_{ss} \mathcal{Q}_c$ and using the standard identity $\mathcal{P}_c \mathcal{Q}_c = 0$ for complementary projectors. The expansion in (15) has the advantage that it converges rapidly since $\bar{G}_{ss}^l \sim |\lambda_{c,2}|^l$ with $\lambda_{c,2}$ being the second largest eigenvalue which is significantly lower than unity (e. g. $|\lambda_{c,2}| \approx \alpha = 0.85$ for the case with a damping factor). The first contribution due to the leading eigenvalue λ_c close to unity is taken out analytically once the left and right eigenvectors, and therefore also the projector \mathcal{P}_c , are known.

The combination of (8) and (15) provides an explicit algorithm feasible for a numerical implementation for the case of modest values of N_r , large values of N_s and of course for sparse matrices G , G_{ss} , etc. The described method can also be modified to take out analytically the contributions of several leading eigenvalues close to unity as described in [19]. We note that the numerical methods described in [35] allow to determine the eigenvalues λ_c (and corresponding eigenvectors) which are exponentially close to unity (e. g. $1 - \lambda_c \sim 10^{-16}$) so that the expression (12) can be efficiently computed numerically.

On the basis of the above equations (8)-(12) the reduced Google matrix can be presented as a sum of three

components

$$G_R = G_{rr} + G_{pr} + G_{qr}, \quad (16)$$

with the first component G_{rr} given by direct matrix elements of G among the selected N_r nodes. The second projector component G_{pr} is given by

$$G_{pr} = G_{rs} \mathcal{P}_c G_{sr} / (1 - \lambda_c), \quad \mathcal{P}_c = \psi_R \psi_L^T. \quad (17)$$

We mention that this contribution is of the form $G_{pr} = \tilde{\psi}_R \tilde{\psi}_L^T / (1 - \lambda_c)$ with $\tilde{\psi}_R = G_{rs} \psi_R$ and $\tilde{\psi}_L^T = \psi_L^T G_{sr}$ being two small vectors defined on the reduced space of dimension N_r . Therefore G_{pr} is indeed a (small) matrix of rank one which is also confirmed by a numerical diagonalization of this matrix. The third component G_{qr} of indirect or hidden links is given by

$$G_{qr} = G_{rs} [\mathcal{Q}_c \sum_{l=0}^{\infty} \bar{G}_{ss}^l] G_{sr}, \quad \mathcal{Q}_c = \mathbf{1} - \mathcal{P}_c, \quad \bar{G}_{ss} = \mathcal{Q}_c G_{ss} \mathcal{Q}_c. \quad (18)$$

Even though the decomposition (16) is at first motivated by the numerical efficiency to evaluate the matrix inverse, it is equally important concerning the interpretation of the different terms and especially the last contribution (18) which is typically rather small as compared to (17) but plays in an important role as we will see below.

Concerning the numerical algorithm to evaluate all contributions in (16), we mention that we first determine by the power iteration method the leading left ψ_L and right eigenvector ψ_R of the matrix G_{ss} which also provides an accurate value of the corresponding eigenvalue λ_c or better of $1 - \lambda_c$ (by taking the norm of the projection of $G\psi_R$ on the reduced space which is highly accurate even for λ_c close to 1). These two vectors provide directly G_{pr} by (17) and allow to numerically apply the projector \mathcal{Q}_c to an arbitrary vector (with $\sim N$ operations). The most expensive part is the evaluation of the last contribution according to (18). For this we apply successively $\bar{G}_{ss} = \mathcal{Q}_c G_{ss} \mathcal{Q}_c$ to an arbitrary column of G_{sr} which can be done by a sparse matrix vector multiplication or the efficient application of the projector.

Therefore, we can calculate in parallel, for each column j of G_{sr} , the following product $\mathcal{Q}_c \sum_{l=0}^{\infty} \bar{G}_{ss}^l G_{sj}$. This computation can be performed using the power iterations algorithm of PageRank which converges after about ~ 200 – 250 terms. Indeed, the contribution of the leading eigenvalue (of G_{ss}) has been taken out and the eigenvalues of \bar{G}_{ss} are roughly below the damping factor $\alpha = 0.85$. In the end the resulting vector is multiplied with the matrix G_{rs} which provides one column of G_{qr} . This procedure has to be repeated for each of the N_r columns but the number N_r is typically very modest (20 or 40 in this work) and the computation of the different columns can actually be done in parallel on typical multicore machines.

Concerning the choice of the reduced space we use 5 groups of 20 or 40 political leaders of 5 countries (US, UK, DE, FR, RU) for 4 Wikipedia editions (EN, DE, FR, RU with EN-Wikipedia for both US and UK politicians). We also consider the group of G20 state leaders for which we use all 4 of these Wikipedia editions even though here we

concentrate on the G20 data obtained for EN-Wikipedia. A detailed description of these subsets is given in Section 3. For the data sets of politicians considered in this work we find that typically $1 - \lambda_c \sim 10^{-4}$ and the right eigenvector ψ_R of G_{ss} is rather close to the full PageRank of G (for the leading nodes in the full PageRank not belonging to the reduced space). Furthermore, we find that an approximate relation holds: $1 - \lambda_c \approx \Sigma_P = \|P_r\|_1$ where Σ_P is the PageRank probability of the global network concentrated on the subset of N_r selected nodes. The data of Table 1 show that this relation works with an accuracy of a couple of percent. To understand this result mathematically, we replace in (7) the matrix inverse by the first term of (15) which gives $P_s \approx \psi_R \psi_L^T G_{sr} P_r / (1 - \lambda_c)$ (for $N_s \gg N_r$ and $1 - \lambda_c \ll 1$). Furthermore we note that $\psi_L \approx E_s$ (for most nodes) and $E_s^T G_{sr} P_r \approx E_r^T P_r = \|P_r\|_1$ such $P_s \approx \psi_R \|P_r\|_1 / (1 - \lambda_c)$. Since $\|P_s\|_1 \approx \|\psi_R\|_1 = 1$ we find indeed that $\|P_r\|_1 \approx 1 - \lambda_c$.

The numerical computations show that the vectors $\tilde{\psi}_R$ and $\tilde{\psi}_L^T$ introduced below equation (17) are approximately given by $\tilde{\psi}_R = P_r$ and $\tilde{\psi}_L^T = E_r^T$ such that $G_{pr} \approx P_r E_r^T / (1 - \lambda_c)$ is rather close to a rank one Google matrix (since $\|P_r\|_1 \approx 1 - \lambda_c$) and with identical columns given by the normalized vector $P_r / (1 - \lambda_c)$. More precisely, we will indeed see in Sections 4-9, that the overall column sums of G_{pr} account for ~ 95 - 97% of the total column sum of G_R . In other words, in terms of probability the contribution of G_{pr} is dominant in G_R but it is also kind of trivial with nearly identical columns. Therefore the two small contributions of G_{rr} and G_{qr} are indeed very important for the interpretation even though they only contribute weakly to the overall column sum normalization.

The meaning of G_{rr} is rather clear since it gives direct links between the selected nodes. In contrast, the meaning of G_{qr} is significantly more interesting since it generates indirect links between the N_r nodes due to their interactions with the global network environment. We note that G_{qr} is composed of two parts $G_{qr} = G_{qrd} + G_{qrnd}$ where the first diagonal term G_{qrd} represents a probability to stay on the same node during multiple iterations of \bar{G}_{ss} in (18) while the second nondiagonal term G_{qrnd} represents indirect (hidden) links between the N_r nodes appearing via the global network. We note that in principle certain matrix elements of G_{qr} can be negative, which is possible due to negative terms in $\mathcal{Q}_c = \mathbf{1} - \mathcal{P}_c$ appearing in (18). However, for all subsets considered in this work the total weight of negative elements was negligibly small (about 10^{-10} for the data of UK politicians, 0 for data of politicians of other countries, and 10^{-5} for the G20 state leader data, of the total weight 1 for G_R).

It is convenient to characterize the strength of 3 components in (16) by their respective weights W_{rr} , W_{pr} , W_{qr} given respectively by the sum of all matrix elements of G_{rr} , G_{pr} , G_{qr} divided by N_r . By definition we have $W_{rr} + W_{pr} + W_{qr} = 1$.

In the following sections we will see that all three components of (16) play important roles. We present here only results for G_R obtained from G . The results for the net-

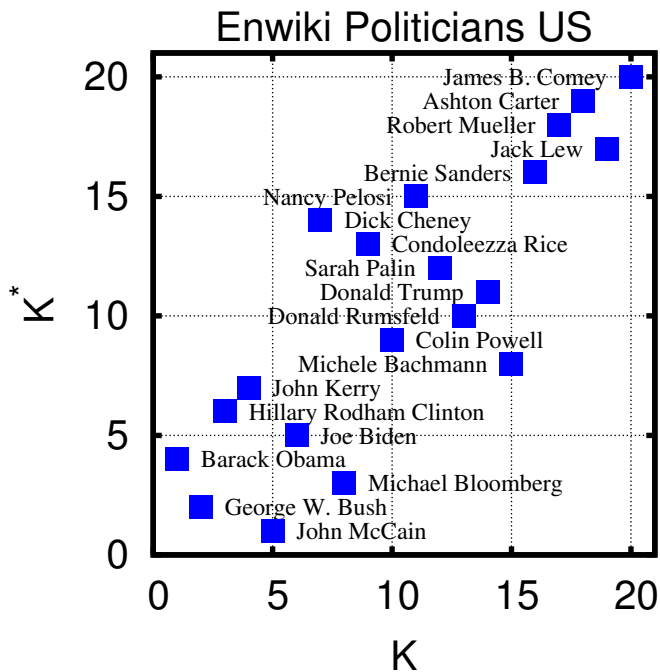


Fig. 2. Position of nodes in the local (K, K^*) plane of the reduced network for 20 US politicians in the Enwiki network. The names are shown on the same lines of the corresponding data points.

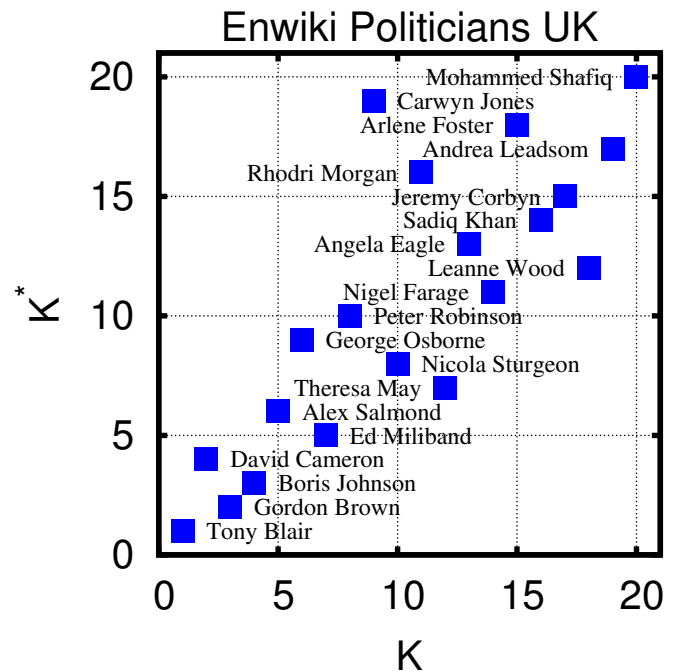


Fig. 3. Position of nodes in the local (K, K^*) plane of the reduced network for 20 UK politicians in the Enwiki network. The names are shown on the same lines of the corresponding data points.

Names (US)	K	K^*	K_G
Barack Obama	1	4	2
George W. Bush	2	2	1
Hillary Rodham Clinton	3	6	6
John Kerry	4	7	4
John McCain	5	1	3
Joe Biden	6	5	7
Dick Cheney	7	14	5
Michael Bloomberg	8	3	14
Condoleezza Rice	9	13	10
Colin Powell	10	9	12
Nancy Pelosi	11	15	8
Sarah Palin	12	12	9
Donald Rumsfeld	13	10	11
Donald Trump	14	11	20
Michele Bachmann	15	8	16
Bernie Sanders	16	16	15
Robert Mueller	17	18	13
Ashton Carter	18	19	19
Jack Lew	19	17	18
James B. Comey	20	20	17

Table 2. List of names of 20 selected US politicians and the corresponding PageRank index values K , K^* and K_G of the reduced network matrices G_R , G_R^* and G_{rr+qr} given as the sum of G_{rr} and G_{qr} (without diagonal elements). All matrices were computed for the English Wikipedia edition of 2013.

work with inverted direction of links, corresponding to CheiRank of G_R^* of G^* are given at [27].

3 PageRank-CheiRank plane of Wikipedia editions

For our studies we choose 6 independent groups of articles of 20 US and 20 UK politicians from Enwiki, 40 German and 40 French politicians from Dewiki and Frwiki respectively, 20 Russian politicians from Ruwiki and the 20 G20 state leaders from Enwiki. The information about number of nodes and links for each Wikipedia edition is available at [14]. In the selection of names of political leaders of each country we used the names appearing at the top of Google search on e. g. “politicians of Russia”, in addition we take only those politician who were active in the period not more than 10 - 20 years before the collection date of our Wikipedia editions of 2013. A few names are used to have a group of 20 or 40. We do not pretend that we selected all important politicians of a given country but we suppose that the main part of them is present in our selection.

For each group (or subset of N_r nodes) we order politicians by their PageRank probability in the corresponding global Wikipedia network. After such ordering we obtain local rank PageRank index K changing from 1 to 20 (or 40). The best known politicians are found to be at the top values $K = 1, 2, \dots$. In addition we determine the local CheiRank index K^* of the selected names using the

Names (UK)	K	K^*	K_G
Tony Blair	1	1	2
David Cameron	2	4	3
Gordon Brown	3	2	1
Boris Johnson	4	3	7
Alex Salmond	5	6	5
George Osborne	6	9	6
Ed Miliband	7	5	4
Peter Robinson	8	10	12
Carwyn Jones	9	19	13
Nicola Sturgeon	10	8	9
Rhodri Morgan	11	16	16
Theresa May	12	7	8
Angela Eagle	13	13	15
Nigel Farage	14	11	10
Arlene Foster	15	18	14
Sadiq Khan	16	14	11
Jeremy Corbyn	17	15	17
Leanne Wood	18	12	18
Andrea Leadsom	19	17	19
Mohammed Shafiq	20	20	20

Table 3. Same as Table 2 for 20 selected UK politicians and the English Wikipedia edition of 2013.

CheiRank vector of the global network. At the top of K^* we have most communicative articles of politicians. Then we present the distribution of politicians or G20 state leaders on the PageRank-CheiRank plane of local indexes (K, K^*) in Figs. 2, 3, 4, 5, 6 and 7 for US, UK, DE, FR, RU and G20 respectively. The full names of political leaders are given respectively in Tables 2, 3, 4, 5, 6 and 7 with corresponding values of local K, K^* indexes (we discuss the meaning of the additional index K_G later).

For the US case shown in Fig. 2 and Table 2 we find that Obama, Bush and Clinton take the top three K positions which appears to be rather natural. However, the most communicative politicians being at the top of CheiRank with $K^* = 1, 2, 3$ are McCain, Bush and Bloomberg.

In Fig. 3 and Table 3 the names and distribution of 20 UK politicians are shown. The top 3 positions of PageRank are taken by UK prime ministers Blair, Cameron, Brown with $K = 1, 2, 3$. The distribution in the (K, K^*) plane is more centered in a diagonal vicinity as compared to the US case and other countries discussed below. For $K^* = 1, 2, 3$ we have Blair, Brown and Johnson. The present prime minister May is rather far in rank indexes.

The PageRank-CheiRank distribution of German politicians is shown in Fig. 4 with the full names and ranks given in Table 4. The top PageRank values $K = 1, 2, 3$ are taken by the chancellors Kohl, Schmidt and Merkel with Schröder at $K = 4$. However, the most communicative politicians are Gauck (DE president, $K^* = 1$), Merkel ($K^* = 2$) and Lafontaine ($K^* = 3$).

The French politicians are presented in Fig. 5 and Table 5. Here we choose names of those who are really active

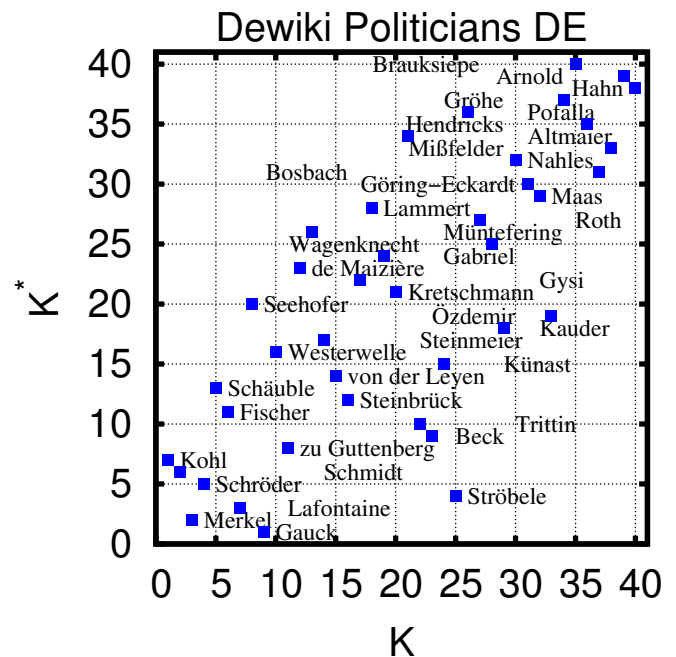


Fig. 4. Position of nodes in the local (K, K^*) plane of the reduced network for 40 DE politicians in the Dewiki network. The names are shown on the same lines of the corresponding data points. Some of the names are horizontally shifted for a better visibility.

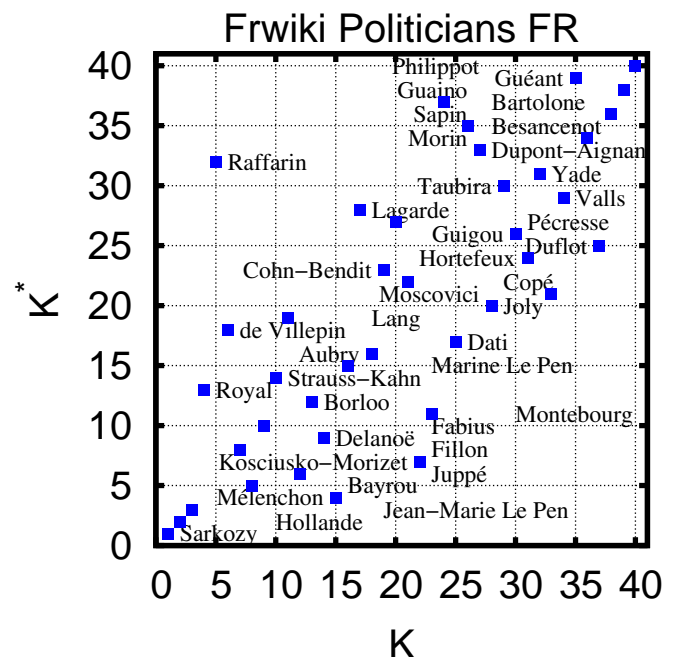


Fig. 5. Position of nodes in the local (K, K^*) plane of the reduced network for 40 FR politicians in the Frwiki network. The names are shown on the same lines of the corresponding data points. Some of the names are horizontally shifted for a better visibility.

Names (DE)	K	K^*	K_G
Helmut Kohl	1	7	3
Helmut Schmidt	2	6	6
Angela Merkel	3	2	1
Gerhard Schröder	4	5	2
Wolfgang Schäuble	5	13	9
Joschka Fischer	6	11	7
Oskar Lafontaine	7	3	4
Horst Seehofer	8	20	20
Joachim Gauck	9	1	24
Guido Westerwelle	10	16	14
Karl-Theodor zu Guttenberg	11	8	13
Thomas de Maizière	12	23	16
Franz Müntefering	13	26	5
Frank-Walter Steinmeier	14	17	10
Ursula von der Leyen	15	14	25
Peer Steinbrück	16	12	8
Gregor Gysi	17	22	17
Norbert Lammert	18	28	22
Sigmar Gabriel	19	24	12
Winfried Kretschmann	20	21	35
Peter Altmaier	21	34	29
Jürgen Trittin	22	10	15
Volker Beck	23	9	32
Renate Künast	24	15	23
Hans-Christian Ströbele	25	4	11
Ronald Pofalla	26	36	19
Claudia Roth	27	27	21
Sahra Wagenknecht	28	25	28
Volker Kauder	29	18	34
Andrea Nahles	30	32	18
Katrin Göring-Eckardt	31	30	26
Heiko Maas	32	29	30
Cem Özdemir	33	19	33
Hermann Gröhe	34	37	31
Ralf Brauksiepe	35	40	38
Barbara Hendricks	36	35	36
Wolfgang Bosbach	37	31	27
Philipp Mißfelder	38	33	37
Rainer Arnold	39	39	39
Florian Hahn	40	38	40

Table 4. Same as Table 2 for 40 selected DE politicians and the German Wikipedia edition of 2013.

Names (FR)	K	K^*	K_G
Nicolas Sarkozy CB	1	1	1
François Hollande CM	2	2	2
Jean-Marie Le Pen CV	3	3	11
Ségolène Royal CM	4	13	3
Jean-Pierre Raffarin CB	5	32	10
Dominique de Villepin CB	6	18	12
François Fillon CB	7	8	4
François Bayrou CB	8	5	6
Laurent Fabius CM	9	10	8
Dominique Strauss-Kahn CM	10	14	7
Jack Lang CM	11	19	13
Alain Juppé CB	12	6	5
Jean-Louis Borloo CB	13	12	19
Bertrand Delanoë CM	14	9	18
Jean-Luc Mélenchon CR	15	4	14
Marine Le Pen CV	16	15	17
Christine Lagarde CB	17	28	25
Martine Aubry CM	18	16	9
Daniel Cohn-Bendit CG	19	23	32
Valérie Pécresse CB	20	27	29
Jean-François Copé CB	21	22	15
Nathalie Kosciusko-Morizet CB	22	7	26
Arnaud Montebourg CM	23	11	16
Claude Bartolone CM	24	37	28
Rachida Dati CB	25	17	35
Olivier Besancenot CR	26	35	37
Nicolas Dupont-Aignan CV	27	33	30
Eva Joly CG	28	20	21
Christiane Taubira CM	29	30	38
Élisabeth Guigou CM	30	26	23
Brice Hortefeux CB	31	24	27
Rama Yade CB	32	31	39
Pierre Moscovici CM	33	21	20
Manuel Valls CM	34	29	22
Claude Guéant CB	35	39	34
Hervé Morin CB	36	34	24
Cécile Duflot CG	37	25	36
Michel Sapin CM	38	36	31
Henri Guaino CB	39	38	33
Florian Philippot CV	40	40	40

Table 5. Same as Table 2 for 40 selected FR politicians and the French Wikipedia edition of 2013. Color marks membership in political parties: red CR (far-left parties), magenta CM (socialist party PS), green CG (green parties), blue CB (right parties UMP, UDI), CV violet (far-right and FN).

in the period 2007 - 2013, thus e. g. Jacques Chirac is not included in the list. The top 3 positions in K and K^* are taken by two presidents Sarkozy, Hollande with (Jean-Marie) Le Pen at the third position. We note a large dispersion of positions on the (K, K^*) plane for the main part of politicians. Thus Royal ($K = 4$) and Raffarin ($K = 5$) have rather high values of $K^* = 13$ or 32 respectively. For French politicians in Table 5 we mark by color their membership in political parties. The effect of interactions and links between different parties is discussed in the last Section.

The names of 20 Russian politicians and their distribution on the (K, K^*) plane are presented in Fig. 6 and Table 6. Similar to the case of France the two top PageRank positions are taken by presidents Putin, Medvedev with Zhirinovskiy at the third position with $K = 3$. For $K^* = 1, 2, 3$ we have Putin, Matvienko, Yavlinsky while Medvedev is at the far value $K^* = 12$ showing low communicative properties of his Wikipedia article.

All the groups of politicians have been considered above in the frame of Wikipedia editions corresponding to their

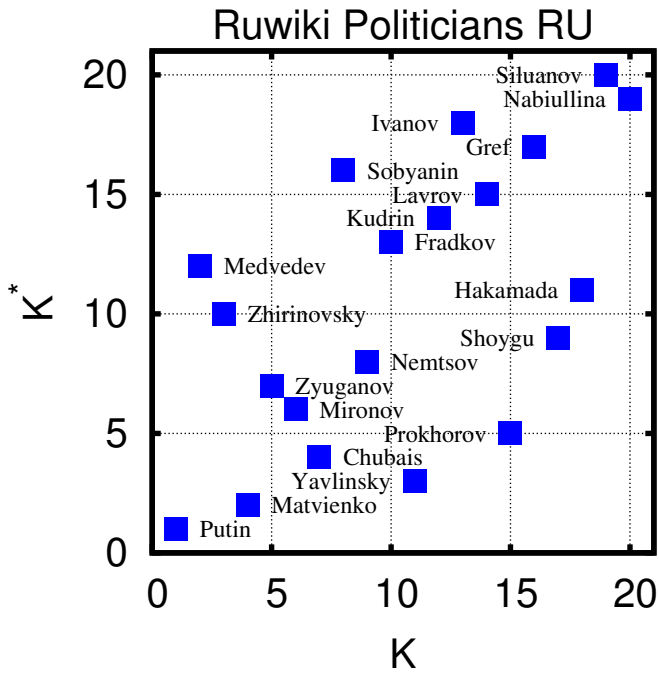


Fig. 6. Position of nodes in the local (K, K^*) plane of the reduced network for 20 RU politicians in the Ruwiki network. The names are shown on the same lines of the corresponding data points.

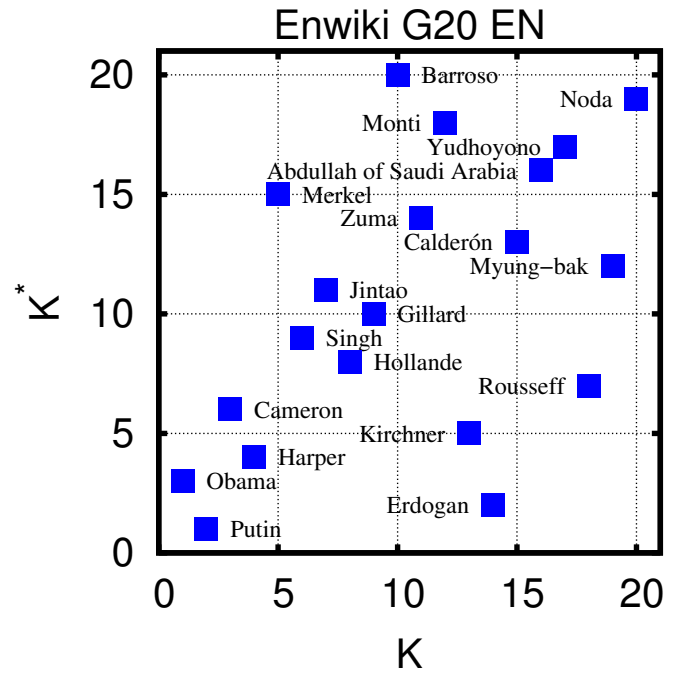


Fig. 7. Position of nodes in the local (K, K^*) plane of the reduced network for 20 state leaders of G20 states in the Enwiki network. The names are shown on the same lines of the corresponding data points.

Names (RU)	K	K^*	K_G
Putin, Vladimir Vladimirovich	1	1	1
Medvedev, Dmitry Anatolyevich	2	12	2
Zhirinovskiy, Vladimir Wolfovich	3	10	15
Matvienko, Valentina Ivanovna	4	2	11
Zyuganov, Gennady Andreyevich	5	7	13
Mironov, Sergey Mikhailovich	6	6	8
Chubais, Anatoly Borisovich	7	4	6
Sobyanin, Sergey Semenovich	8	16	9
Nemtsov, Boris Yefimovich	9	8	10
Fradkov, Mikhail Yefimovich	10	13	3
Yavlinsky, Grigory	11	3	14
Kudrin, Alexei Leonidovich	12	14	7
Ivanov, Sergey Borisovich	13	18	4
Lavrov, Sergey Viktorovich	14	15	20
Prokhorov, Mikhail Dmitrievich	15	5	18
Gref, Herman Oskarovich	16	17	5
Shoygu, Sergei	17	9	19
Hakamada, Irina Mutsuovna	18	11	17
Siluanov, Anton Germanovich	19	20	16
Nabiullina, Elvira	20	19	12

Table 6. Same as Table 2 for 20 selected RU politicians and the Russian Wikipedia edition of 2013.

G20 / Wikipedia edition	EN	DE	FR	RU
Names	K	K	K	K
Barack Obama	1	1	1	2
Vladimir Putin	2	3	2	1
David Cameron	3	4	6	4
Stephen Harper	4	10	7	16
Angela Merkel	5	2	4	3
Manmohan Singh	6	13	13	10
Hu Jintao	7	9	5	5
François Hollande	8	5	3	6
Julia Gillard	9	14	16	11
José Manuel Barroso	10	6	8	14
Jacob Zuma	11	18	18	18
Mario Monti	12	8	9	9
Cristina Fernández de Kirchner	13	12	10	8
Recep Tayyip Erdoğan	14	7	14	12
Felipe Calderón	15	11	12	7
Abdullah of Saudi Arabia	16	15	15	13
Susilo Bambang Yudhoyono	17	19	20	19
Dilma Rousseff	18	17	11	15
Lee Myung-bak	19	16	17	17
Yoshihiko Noda	20	20	19	20

Table 7. List of the 20 state leaders of the G20 states and the corresponding PageRank index values K of the reduced network matrices G_R for the English (EN), German (DE), French (FR) and Russian (RU) Wikipedia editions of 2013. For each state leader the country name is given in ISO 3166-1 alpha-2.

native language. Indeed, we find that in other language editions the articles about many politicians are rather short or sometimes they are even absent (e. g. for 40 French politicians in Enwiki or Ruwiki). However, the political leaders of countries are usually well present in the editions discussed here. Therefore, we take for our analysis 20 world political leaders that have participated in the G20 meeting at Los Cabos summit in 2012 [20]. Their names and local PageRank indexes according to Enwiki, Dewiki, Frwiki, Ruwiki are given in Table 7. The distribution of politicians on PageRank-CheiRank plane is shown in Fig. 7 for Enwiki. We take the name of country leader Abdullah of Saudi Arabia (Saudi Arabia) since the name of the minister of finance, who was representing Saudi Arabia, is not listed in Dewiki, Frwiki, Ruwiki; José Manuel Barroso is taken as EU representative.

Among G20 leaders the top 2 PageRank positions are taken by Obama and Putin (see Table 7) in Enwiki, Frwiki; Putin and Obama in Ruwiki and Obama and Merkel in Dewiki. So there is a definite trend for leaders being promoted in their native editions. The language preference is probably the reason to have Singh (India) ahead of Jintao (China) in Enwiki while in other editions Jintao is well ahead of Singh. At the top CheiRank positions K^* of Enwiki we have Putin, Erdoğan, Obama (see Fig. 7) showing very different communicative strengths of political leaders.

In the next sections we consider interactions between selected political leaders using the reduced Google matrix approach.

4 Direct and hidden links of US politicians

The reduced Google matrix G_R of 20 US politicians, listed in Fig. 2 and Table 2, is shown in Fig. 8 with its three matrix components G_{pr} , G_{rr} , G_{qr} from (16). The amplitudes of matrix elements are shown by color with maximum for red and minimum for blue. We mention that for the data of US Politicians in this section there are no negative matrix elements of G_{qr} . The same holds for the cases of DE and FR politicians (see sections below). However for UK politicians (G20 data, also see sections below for both cases) there are few very (rather) small negative elements of G_{qr} of order $\sim 10^{-8}$ (or $\sim 10^{-5}$) with no (or very small) effects on colors. The data of US politicians in Fig. 8 clearly show that the main contribution to G_R is given by the projector component G_{pr} with a weight of approximately 96%. The remaining weight is distributed between the component of direct links G_{rr} (1.9%) and the one of indirect links G_{qr} (1.6%) (see Fig. 8). Of course, the total weight of three components is equal to unity by construction.

Thus the main component G_{pr} imposes to G_R a large contribution proportional to the PageRank probability $P(K)$ which is mainly produced by the environment of the huge remaining part G_{ss} of the global network with $N - N_r \gg N_r$ nodes. Due to this the structure of G_R is close to a solution (discussed above) when each column of G_R is roughly given by the same PageRank vector of N_r nodes which up

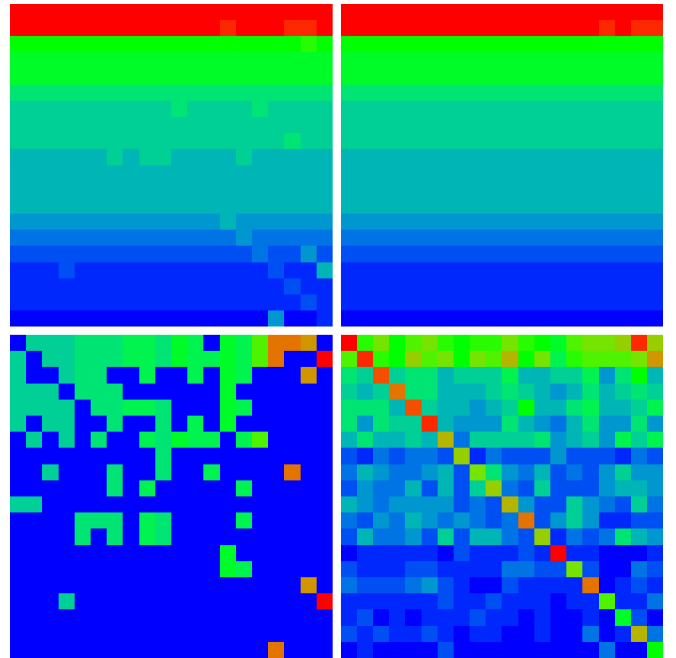


Fig. 8. Density plots of the matrices G_R (top left), G_{pr} (top right), G_{rr} (bottom left) and G_{qr} (bottom right) for the reduced network of 20 US politicians in the Enwiki network. The nodes N_r are ordered in lines by PageRank index changing from $K = 1$ to N_r (left to right) and in columns from $K' = 1$ to N_r from top to bottom. The weights of the three matrix components of G_R are $W_{pr} = 0.9640$ for G_{pr} , $W_{rr} = 0.01970$ for G_{rr} , $W_{qr} = 0.01630$ for G_{qr} . The colors represent maximum (red), intermediate (green) and minimum (blue) values.

to a constant factor coincides with the PageRank probabilities of the selected N_r nodes in the global network of N nodes. Due to the simple rank-one structure of G_{pr} the smaller contributions of the other two components G_{rr} of direct and G_{qr} of indirect links (friends) play an important role even if their weight is significantly smaller as compared to the weight of G_{pr} .

The global structure of G_{rr} of 20 US politicians is shown in Fig. 8 (bottom left panel) and in more detail in Fig. 9 where the lines and columns are marked by short names of politicians (up to 7 letters). We can say that large matrix elements in a column of a given politician can be considered as direct friends to whom he/she points in his/her Wikipedia article. However, we should note that by construction of G all elements inside a given column have the same amplitudes given by a fraction of total number of outgoing links of a given article which point inside N_r nodes of selected subset. Thus on the basis of the G_{rr} component it is not possible to say that some direct friend (link) of a given politician is more preferable than another one: all of them have the same amplitude. Of course, when we are speaking about a friend we simply mean that one politician points to another but at present we cannot say if this link has a positive or negative content. Such a classification would require further extension of our G_R analysis. However, since the PageRank probability is on average

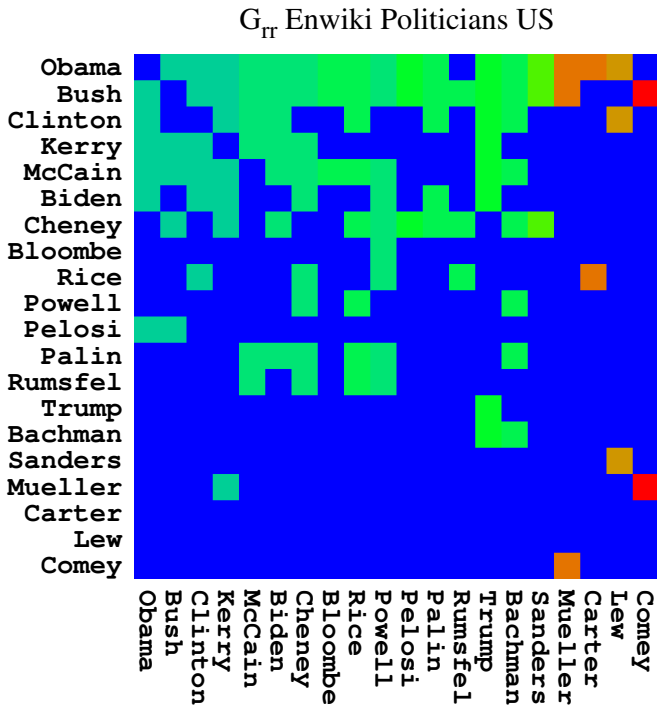


Fig. 9. Density plot of the matrix G_{tr} for the reduced network of 20 US politicians in the Enwiki network with short names at both axes (up to 7 letters, full names are in Table 2).

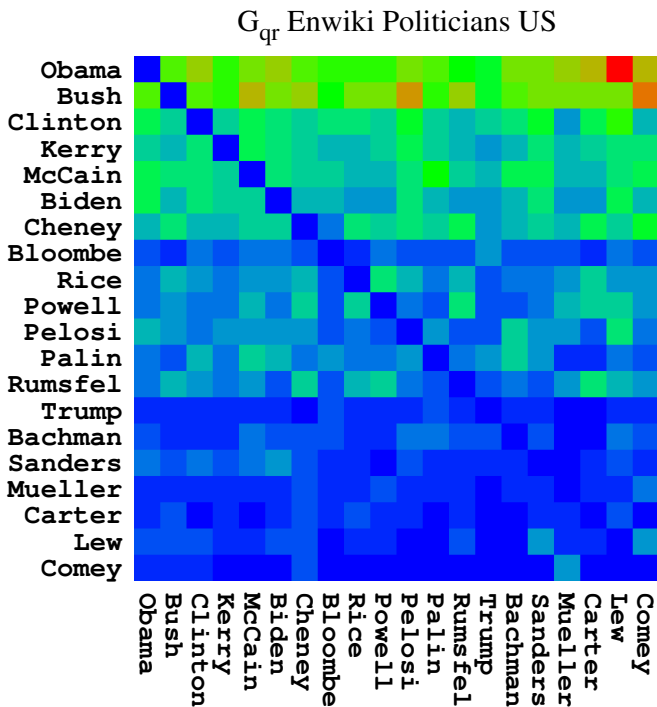


Fig. 10. Density plot of the matrix G_{qr} without diagonal elements for the reduced network of 20 US politicians in the Enwiki network with short names at both axes. The weight of this matrix component without diagonal is $W_{qrnd} = 0.01143$ (see Fig. 8 for W_{qr} weight with diagonal).

Politicians	US	Enwiki
Name	Friends	Followers
Obama	Bush Clinton Biden	Lew Comey Carter
Bush	Obama Cheney McCain	Comey Pelosi McCain
Clinton	Obama Bush McCain	Lew Sanders Pelosi
Kerry	Obama Bush Clinton	Pelosi McCain Biden
McCain	Bush Obama Clinton	Palin Comey Sanders

Table 8. List of leading politicians (in PageRank order) of the group of US politicians together with the leading “Friends” (“Followers”) in this group defined by maximal values of the elements of G_{qr} corresponding to links “to” (“from”) the Friends (Followers) for the English Wikipedia edition. Links to/from the same politician, corresponding to diagonal elements of G_{qr} , are omitted.

proportional to the number of ingoing links we can assume that the content is mainly positive.

The large amplitudes inside a given line, attributed to a given politician, mark the followers of this politician (similar to a situation in Twitter). Thus from Fig. 9 we can say that the first (strongest) direct follower of Bush is Comey who is the present FBI director and is closely linked with Bush. At the same time he has no direct link with Obama. Of course the direct links are important but only on their own basis it is not possible to obtain a correct relationship between selected persons. Indeed, an attempt [36] to make ranking of historical figures of Wikipedia considering only links between their biographical articles gave a rather strange unrealistic result clearly showing that this approach is not working (see also discussion in [14]).

Due to the above reasons the most interesting matrix component G_{qp} is the one of indirect links shown in Fig. 8 (bottom right panel) and in Fig. 10 where diagonal elements have been removed. The weight $W_{qrd} = 0.00487$ of diagonal components of G_{qr} is approximately twice as small than the weight of the nondiagonal part with $W_{qrnd} = 0.01143$: nondiagonal elements play a more significant role. One can understand that indirect links produce considerable diagonal contributions. However, for the analysis of indirect links between *different* nodes there are not of interest.

As discussed above the largest matrix elements of G_{qr} in a column of a given politician give his/her strongest indirect or hidden friends while those in a line give his/her strongest hidden followers. The names of top friends and top followers of top 5 PageRank US politicians are given in Table 8. Surprisingly we find that the strongest hidden

followers of Obama are not his colleagues from the democratic party but Lew (actual secretary of treasury), Comey (actual FBI director), Carter (actual secretary of defense). Interestingly, Clinton has as main followers Lew, Sanders, Pelosi so that already in 2013 there existed the strong hidden links between Sanders and Clinton. At the same time the top 3 hidden friends (followers) of Trump are Obama, Bush, Clinton (Palin, Bloomberg, Clinton) highlighting hidden links between these two political leaders who are now fighting for the US presidency (see data at [27]).

It is possible to try not to take into account the projector component and construct a modified reduced Google matrix G_R obtained from $G_{rr} + G_{qrd}$ by renormalization of each column to unity. Then the PageRank vector of \tilde{G}_R gives the new ranking of the selected group with index K_C given in Table 2. We then see that the top 3 positions are taken by Bush, Obama, McCain. This gives a rearrangement of the ranking which stresses in a stronger way previous presidential teams. We will see that for other countries the effects can work in a different direction (e. g. for DE).

5 Direct and hidden links of UK politicians

The reduced Google matrix analysis of 20 UK politicians from Fig. 3 and Table 3 is presented in Figs. 11, 12, 13. As for the US case we find that G_R has the dominant contribution from the projector component G_{pr} which has a similar weight of 96%. The weights of other two components of direct G_{rr} (1.7%) and hidden G_{qr} (1.5%) links is also similar to those of US. However, on average the direct links are distributed in a more homogeneous manner for UK then for US (see Figs. 12, 9). The strongest direct link is from Leadsom to Osborne.

The distribution of hidden links in Fig. 13 is much more broad as compared to the direct ones in Fig. 12. The top 3 hidden friends and followers of top 5 PageRank politicians are given in Table 9. There are strong links between top 3 leaders Blair, Brown and Cameron. More surprisingly we find that already in 2013, May was the strongest follower of Cameron (there is only a moderate direct link between them) and Johnson (there is no direct link). The strongest amplitudes of links are from Sturgeon (actual first minister of Scotland) to Salmond (previous first ministers of Northern Ireland) and from Foster to Robinson (first ministers of Northern Ireland). Even if the direct links are present in G_{rr} they are definitely not so pronounced as in the indirect part G_{qr} . Also we find that Khan (actual mayor of London) is a second by strength indirect follower of Cameron even if there are no direct links between them.

The ranking index K_C of UK politicians from $G_{rr} + G_{qrd}$ is given in the last column of Table 3. It places on top positions Brown, Blair, Cameron followed by Miliband and Salmond. Such a ranking looks to be less natural as compared to the global rank index K . It stresses that that the projector component G_{pr} still plays an important role.

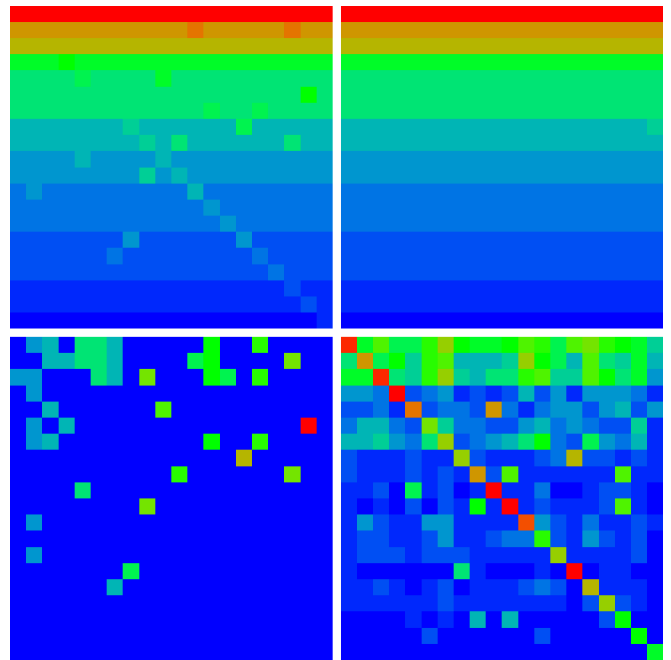


Fig. 11. Same as Fig. 8 for 20 UK politicians in the Enwiki network. The weights of the three matrix components of G_R are $W_{pr} = 0.9668$ for G_{pr} , $W_{rr} = 0.01729$ for G_{rr} , $W_{qr} = 0.01588$ for G_{qr} .

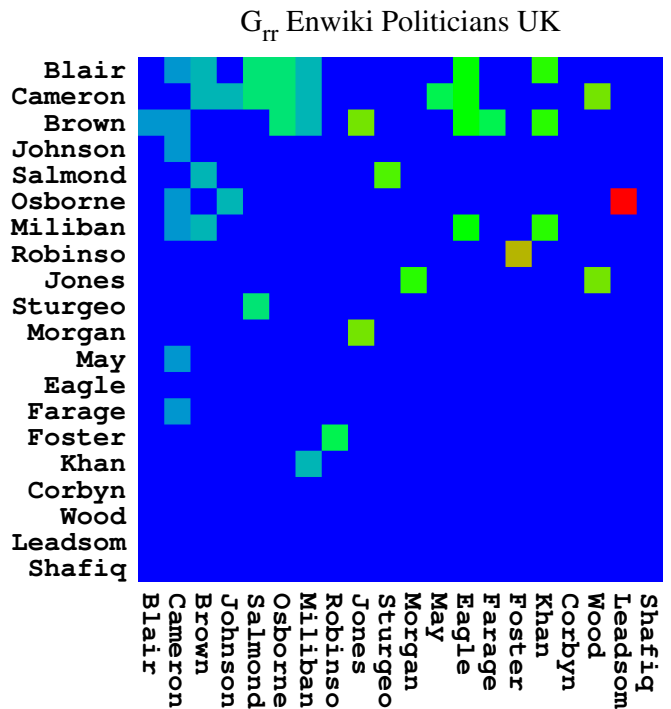


Fig. 12. Density plot of the matrix G_{rr} for the reduced network of 20 UK politicians in the Enwiki network with short names at both axes.

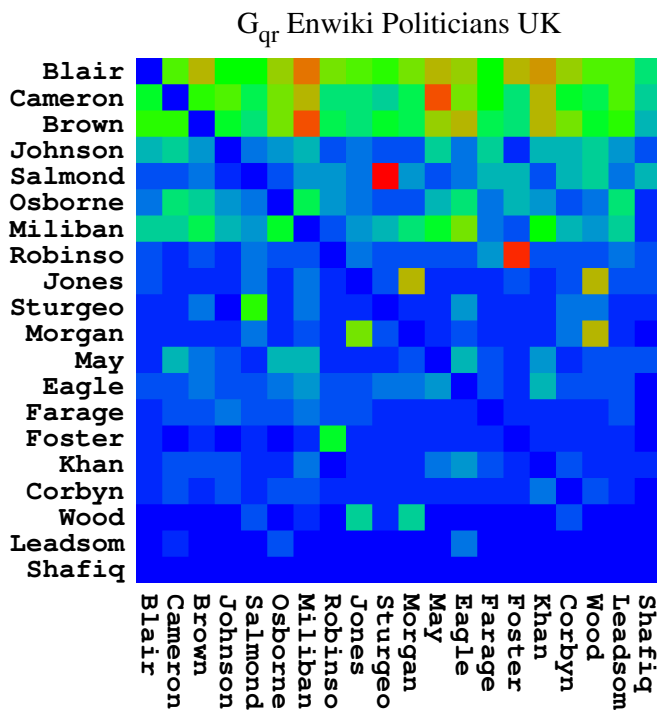


Fig. 13. Density plot of the matrix G_{qr} without diagonal elements for the reduced network of 20 UK politicians in the Enwiki network with short names at both axes. The weight of this matrix component without diagonal is $W_{qrd} = 0.009875$ (see Fig. 11 for W_{qr} weight with diagonal).

Politicians	UK	Enwiki
Name	Friends	Followers
Blair	Brown Cameron Miliband	Miliband Khan Foster
Cameron	Blair Brown Osborne	May Khan Miliband
Brown	Blair Cameron Miliband	Miliband Khan Eagle
Johnson	Cameron Blair Brown	May Wood Cameron
Salmond	Sturgeon Blair Cameron	Sturgeon Wood Foster

Table 9. Same as Table 8 for leading UK politicians and the English Wikipedia edition of 2013.

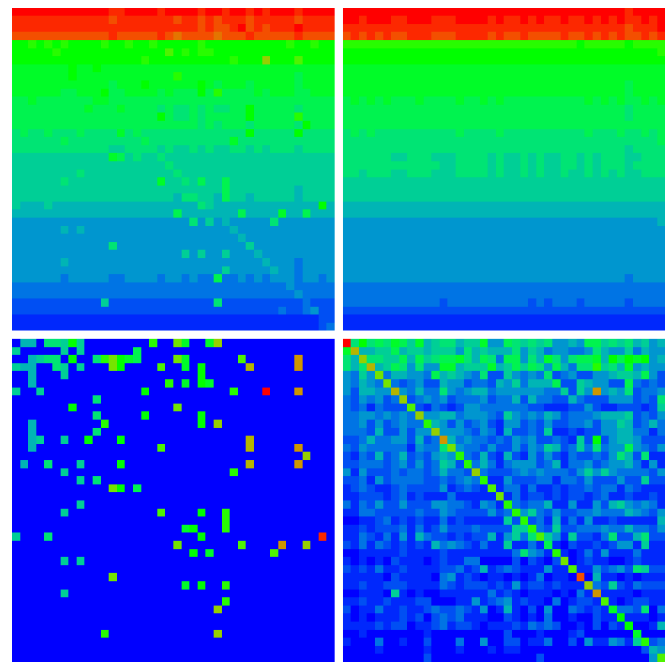


Fig. 14. Same as Fig. 8 for 40 DE politicians in the Dewiki network. The weights of the three matrix components of G_R are $W_{pr} = 0.9590$ for G_{pr} , $W_{rr} = 0.02520$ for G_{rr} , $W_{qr} = 0.01580$ for G_{qr} .

6 Direct and hidden links of DE politicians

For 40 German politicians the matrix G_R and its three components are shown in Figs. 14, 15 and 16. The component weights are similar to the cases of US, UK but the percentage of G_{rr} is now slightly higher.

The strongest direct links are from Maas to Lafontaine (Maas was supported by Lafontaine, chairman of the Social Democratic Party - SPD) and Arnold to Ströbele.

For the indirect links component G_{qr} the strongest links remains the one of Maas to Lafontaine influenced by their direct link. However, in global the number of indirect links is significantly larger compared to direct links. The top 5 PageRank politicians have strongest direct friend links mainly between their own group as it is seen in Table 10. However, the list of followers is rather different, Thus Merkel is the first follower of Kohl who strongly supported her. Kohl is the strongest follower of Schmidt, probably because many Wikipedia articles refer to the change of power between them in 1982 but also because Kohl was the opposition leader during the Schmidt government of 1974-1982. Furthermore, Hendricks is the first follower of Merkel. She is a recent member of Merkel's government (since end 2013 and at least up to 2016) despite being member of the socialist party. The strongest follower of Schöder is Maas. Despite the fact that both belong to the socialist party it is difficult to establish a direct political link due to a considerable time difference of more than 10 years between their time periods in government. However, their names appear both (and together with other names) in certain news articles discussing general marital prob-

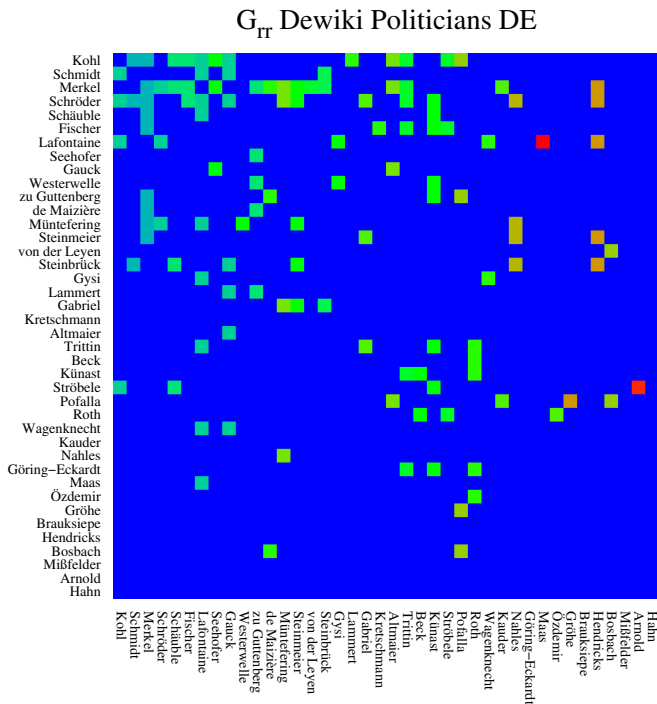


Fig. 15. Density plot of the matrix G_{rr} for the reduced network of 40 DE politicians in the Dewiki network with family names at both axes.

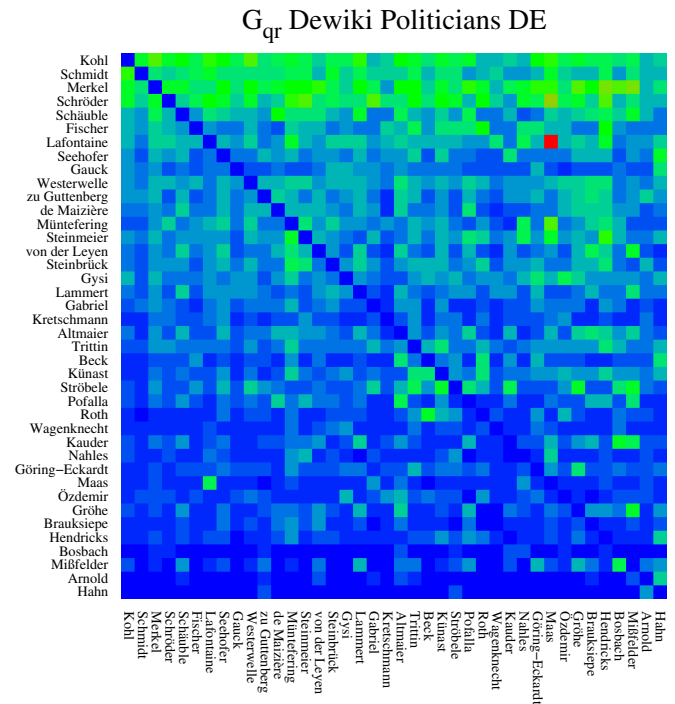


Fig. 16. Density plot of the matrix G_{qr} without diagonal elements for the reduced network of 40 DE politicians in the Dewiki network with family names at both axes. The weight of this matrix component without diagonal is $W_{q\text{rnd}} = 0.01202$ (see Fig. 14 for W_{qr} weight with diagonal).

Politicians	DE	Dewiki
Name	Friends	Followers
Kohl	Schmidt Merkel Schröder	Merkel Westerwelle Lafontaine
Schmidt	Kohl Merkel Schröder	Kohl Lafontaine Steinbrück
Merkel	Kohl Schröder Schäuble	Hendricks Mißfelder Gröhe
Schröder	Kohl Merkel Schmidt	Maas Steinmeier Hendricks
Schäuble	Kohl Merkel Schmidt	Maizière Mißfelder Lammert

Table 10. Same as Table 8 for leading DE politicians and the German Wikipedia edition of 2013.

lems of certain German politicians which may indeed produce indirect links on a different than professional level.

The ranking K_G from the matrix $G_{rr} + G_{q\text{rnd}}$ is given for 40 German politicians in the last column of Table 4 placing on the top positions Merkel, Schröder, Kohl.

7 Direct and hidden links of FR politicians

The reduced Google matrix G_R and its three components for 40 French politicians (see Fig. 5 and Table 5) are shown in Figs. 17, 18, 19. Here the weight of W_{rr} is the largest among all groups of politicians considered in this work. Also the distribution of direct links (see Fig. 18) is very broad compared to the case of 40 DE politicians in Fig. 15. The two strongest direct links go from Philippot to Jean-Marie Le Pen and Marine Le Pen, respectively. They all belong to the far-right FN party.

The hidden links between 40 FR politicians without diagonal terms are shown in Fig. 19. The 3 strongest friends and followers are given in Table 11. Friends of current or former presidents Hollande and Sarkozy are main election opponents or close collaborators (Fillon, Fabius). Raffarin, a former prime minister and leader of UMP party, has only same-party hidden links. This could be explained by the fact that he never participated in the last round of any presidential election. Main relationships of Table 11 are reasonable and can be explained. The most surprising connection is the hidden link from Taubira to J.M. Le Pen (there is no direct link). Taubira issued an important law allowing same-sex marriage in 2013, creating virulent opposition from far-right parties that are documented in Wikipedia.

As pinpointed earlier and in previous works [14], using G_{rr} for extracting knowledge on followers is not meaningful due to column normalization. A good example can

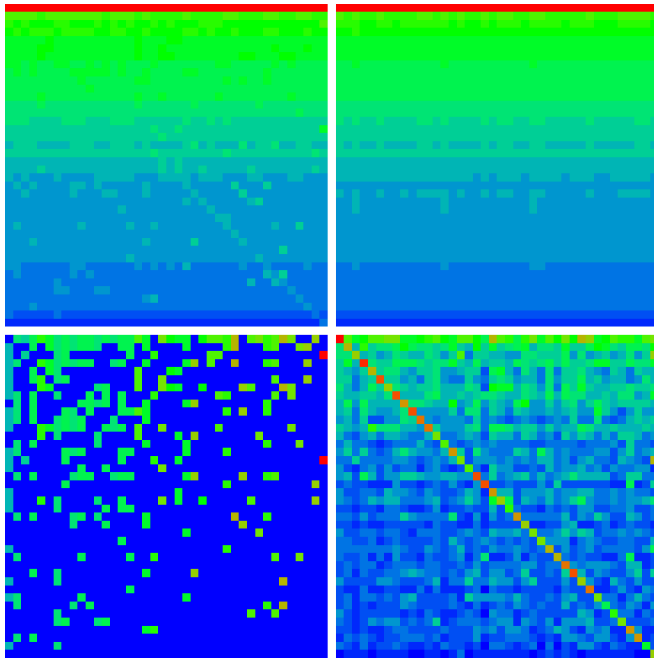


Fig. 17. Same as Fig. 8 for 40 FR politicians in the Frwiki network. The weights of the three matrix components of G_R are $W_{pr} = 0.9481$ for G_{pr} , $W_{rr} = 0.03248$ for G_{rr} , $W_{qr} = 0.01941$ for G_{qr} .

G_{rr} Frwiki Politicians FR

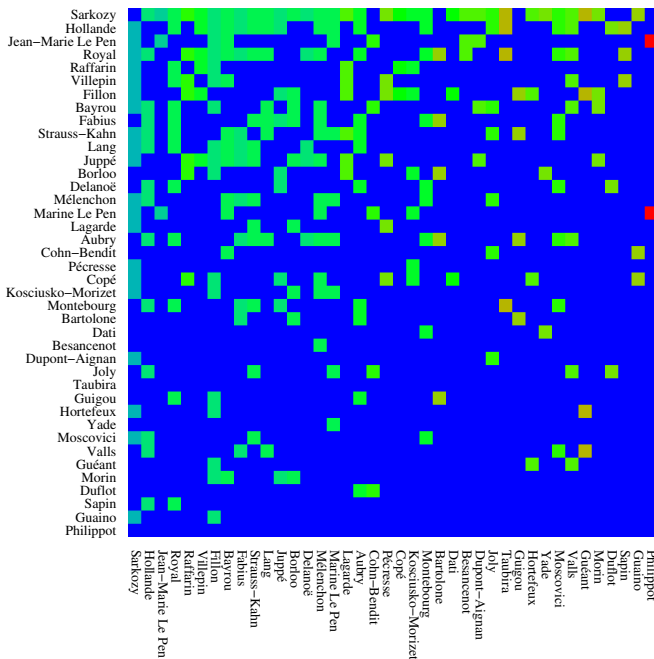


Fig. 18. Density plot of the matrix G_{rr} for the reduced network of 40 FR politicians in the Frwiki network with family names at both axes.

G_{qr} Frwiki Politicians FR

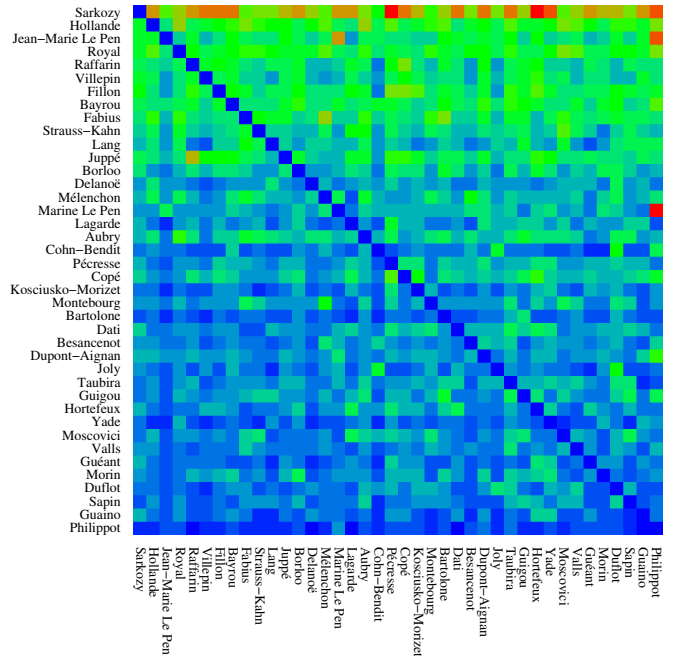


Fig. 19. Density plot of the matrix G_{qr} without diagonal elements for the reduced network of 40 FR politicians in the Frwiki network with family names at both axes. The weight of this matrix component without diagonal is $W_{qrd} = 0.01543$ (see Fig. 17 for W_{qr} weight with diagonal).

Politicians	FR	Frwiki
Name	Friends	Followers
Sarkozy CB	Fillon CB J.-M. Le Pen CV Hollande CM	Hortefeux CB Pécresse CB Yade CB
Hollande CM	Sarkozy CB Royal CM Fabius CM	Royal CM Aubry CM Fabius CM
J.-M. Le Pen CV	Sarkozy CB M. Le Pen CV Bayrou CB	Philippot CV M. Le Pen CV Taubira CM
Royal CM	Hollande CM Sarkozy CB Fabius CM	Moscovici CM Philippot CV Hollande CM
Raffarin CB	Sarkozy CB Juppé CB Fillon CB	Copé CB Pécresse CB Hortefeux CB

Table 11. Same as Table 8 for leading FR politicians and the French Wikipedia edition of 2013.

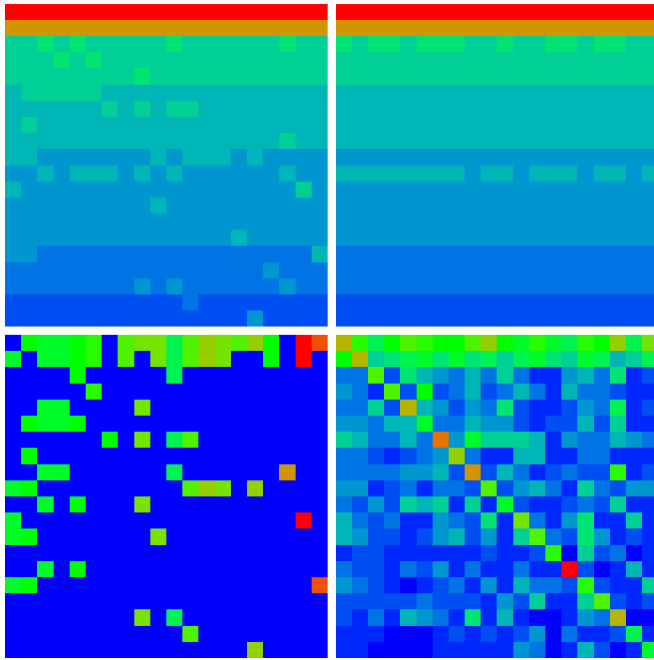


Fig. 20. Same as Fig. 8 for 20 RU politicians in the Ruwiki network. The weights of the three matrix components of G_R are $W_{pr} = 0.9644$ for G_{pr} , $W_{rr} = 0.02408$ for G_{rr} , $W_{qr} = 0.011148$ for G_{qr} .

be highlighted here as in G_{rr} , top three followers of J.-M. Le Pen (FN party founder) are F. Philippot, N. Dupont-Aignan and A. Montebourg (socialist party), while the leader of FN, Marine Le Pen, only ranks as its 27th follower (due to her increased number of outgoing links compared to others). Looking now at the hidden links in Fig. 19 and at the 3 strongest followers of J.M. Le Pen, Marine Le Pen ranks two, just after F. Philippot, the other important figure of that party.

Several hidden links appear in matrix G_{qr} as well in this network: Philippot to Sarkozy is strong in Fig. 19 and non-existing in Fig. 18 ; same with Pécresse to Hollande or Strauss-Kahn to Sarkozy. An in-depth analysis of the network of FR politicians is given in Section 10.

8 Direct and hidden links of RU politicians

The reduced Google matrix G_R for 20 Russian politicians from Fig. 6 and Table 6 are shown in Figs. 20, 21, 22. The weight of the component of direct links is a bit larger than in the case of US and UK but comparable to DE. The strongest direct links are from Siluanov (minister of finance) to Putin and Medvedev and Kudrin (previous minister of finance).

The matrix of hidden links G_{qr} without diagonal is shown in Fig. 22 with the list of 2 top friends and followers for top 5 RU politicians in Table 12. Among hidden friends of Putin we find naturally Medvedev, Chubais, Ivanov who are closely linked with him during his political career. More surprising is that his strongest followers are

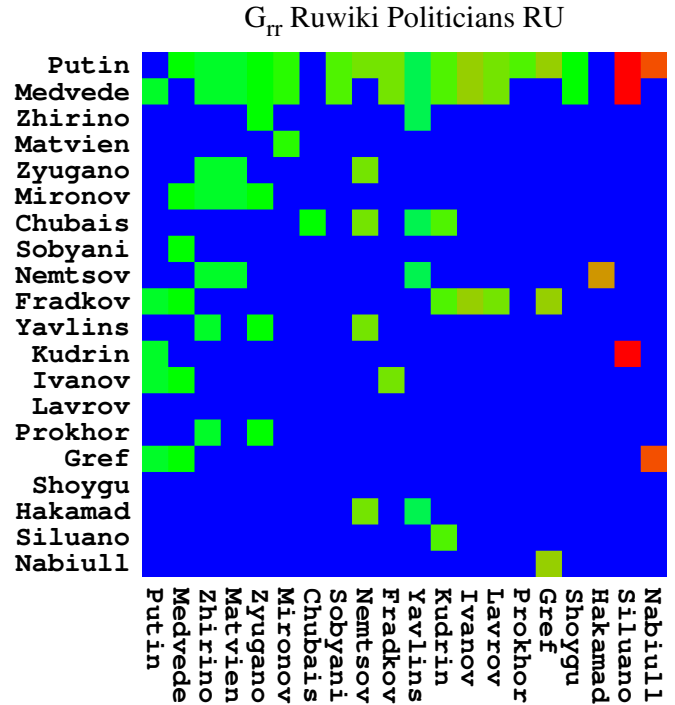


Fig. 21. Density plot of the matrix G_{rr} for the reduced network of 20 RU politicians in the Ruwiki network with short names at both axes.

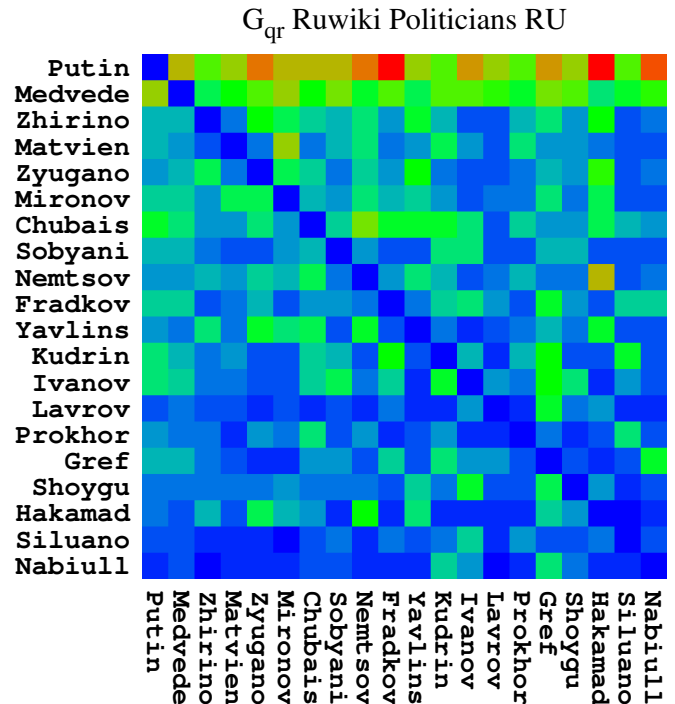


Fig. 22. Density plot of the matrix G_{qr} without diagonal elements for the reduced network of 20 RU politicians in the Ruwiki network with short names at both axes. The weight of this matrix component without diagonal is $W_{qrd} = 0.007912$ (see Fig. 20 for W_{qr} weight with diagonal).

Politicians	RU	Ruwiki
Name	Friends	Followers
Putin	Medvedev Chubais Ivanov	Hakamada Fradkov Nabiullina
Medvedev	Putin Chubais Ivanov	Putin Mironov Gref
Zhirinovskiy	Putin Medvedev Zyuganov	Zyuganov Hakamada Yavlinsky
Matvienko	Putin Medvedev Mironov	Mironov Kudrin Nemtsov
Zyuganov	Putin Medvedev Zhirinovskiy	Hakamada Yavlinsky Zhirinovskiy

Table 12. Same as Table 8 for leading RU politicians and the Russian Wikipedia edition of 2013.

Hakamada, Fradkov, Nabiullina. Among friend relations it is somewhat unexpected that Zhirinovskiy and Zyuganov are friends of each other in a third position. It is also interesting that Russian democratically oriented politicians (Hakamada, Yavlinsky) are among top followers of Zhirinovskiy and Zyuganov. More naturally, top 2 to 5 PageRank politicians (Medvedev, Zhirikovskiy, Matvienko, Zyuganov) have Putin as the first friend (see data at [27]). In fact for all 19 politicians the first friend from the matrix of indirect links G_{qr} is Putin confirming well the vertical power of politics in Russia. On a level of direct friends from G_{rr} only Chubais and Hakamada do not point directly to Putin but on a level of indirect links of G_{qr} all 19 politicians point first to Putin. For RU democrats like we have among top 3 friends (followers) for Yavlinky: Putin, Zyuganov, Chubais (Hakamada, Nemtsov, Zyuganov); for Nemtsov: Putin, Chubais, Hakamada (Hakamada, Chubais, Yavlinsky); for Hakamada: Putin, Nemtsov, Zyuganov (Nemtsov, Zyuganov, Yavlinsky). This shows relatively strong links between democrats even contacts with Zyuganov (leader of communist party) are surprisingly well present.

The rank index K_G from the matrix $G_{rr} + G_{qnd}$ is given in last column of Table 6 with the top leaders being Putin, Medvedev, Fradkov that seems to overestimate the importance of Fradkov, even if he is followed by Ivanov and Gref that looks to be more reasonable. So the above analysis of politicians of US, UK, DE, FR, RU suggests that the projector component should be taken into account even if we want to analyze the relations inside the selected group of politicians since the environment links of the global matrix G still play an important role.

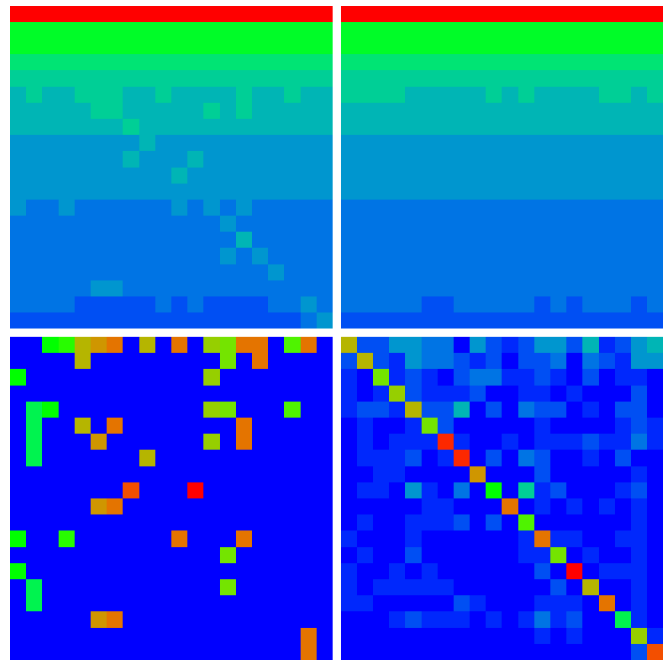


Fig. 23. Same as Fig. 8 for 20 state leaders of G20 states in the Enwiki network. The weights of the three matrix components of G_R are $W_{pr} = 0.9793$ for G_{pr} , $W_{rr} = 0.009976$ for G_{rr} , $W_{qr} = 0.01070$ for G_{qr} .

9 Direct and hidden links of G20 state leaders

Above we considered interactions between political leaders of the same country from the view point of the Wikipedia edition of their main language. It is interesting to see the results of the reduced Google matrix analysis for interactions of state leaders of the G20 summit of 2012 [20]. We analyze these interactions from the view point of 4 Wikipedia editions EN, DE, FR, RU.

The list of names of politicians of G20 and their distribution over PageRank-Cheirank plane are given in Fig. 7 and Table 7. In our presentation below we keep the names in the PageRank order of Enwiki of Table 7. For EN Wikipedia the reduced Google matrix G_R and its three components are shown in Figs. 23, 24, 25. We see that, compared to previous cases inside one country, the weights W_{rr} and W_{qr} are reduced approximately by a factor 2. Indeed, there are significantly less direct links between leaders of different states (see Fig. 24). There are even less direct links for DE, FR, RU editions (see data at [27]). For example, the only direct links for RU Wikipedia are between Merkel and Putin, Erdoğan, Putin and Jintao.

Thus the importance of indirect links from G_{qr} becomes more significant even if the weight of nondiagonal matrix elements is reduced by a factor 4-5 compared to the case of politicians in the same country. In Fig. 25 for Enwiki we find the strongest indirect links between Monti and Barroso (EU), Abdullah (SA) and Obama (even if direct links also exist), Hollande and Merkel, Noda (JP) and Obama (without direct links). The list of top 3 friends

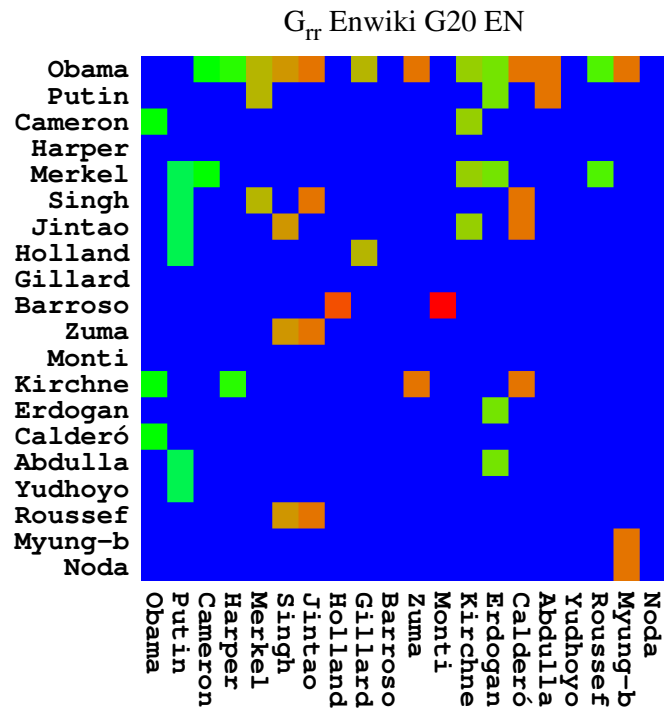


Fig. 24. Density plot of the matrix G_{rr} for the reduced network of 20 state leaders of G20 states in the Enwiki network with short names at both axes.

(followers) of top 5 PageRank leaders of Enwiki is given in Table 13. Among friends the first positions are taken by Putin, Merkel, Obama. For Merkel the top friends are Barroso, Putin, Obama; for Cameron we have Putin, Obama, Merkel. Among top followers of Obama we do not find EU leaders but Noda (JP), Abdullah (SA), Myung-bak (KR). The followers of Putin are Noda, Myung-bak, Merkel. The first follower of Merkel is Hollande while Merkel is the top follower of Harper (CA).

The reduced Google matrices G_R from 4 Wikipedia editions are shown in Fig. 26. It is clear that each culture (which, in first approximation, can be associated with the language) has its own view on relations between state leaders of G20. Indeed, even top 3 PageRank leaders are different for these cultures which creates different structures of matrix elements.

The indirect links of G_{qr} are shown in Fig. 27 for Dewiki, Fig. 28 for Frwiki and Fig. 29 for Ruwiki. For Dewiki the strongest indirect links are from Gillard (AU) and Barroso to Merkel, Abdullah to Erdoğan. For Frwiki the strongest link is from Yudhoyono (ID) to Putin, Monti and Merkel to Barroso. In contrast for Ruwiki the strongest links are from Barroso to Obama, Myung-bak (KR) to Putin. Being at the top of PageRank in Ruwiki, Putin accumulates the largest number of followers. This demonstrates a large variety of cultural views on interactions between state leaders.

The English version provides the richest information thanks to its volume and the variety of its contributions. However, main trends in other countries are not necessar-

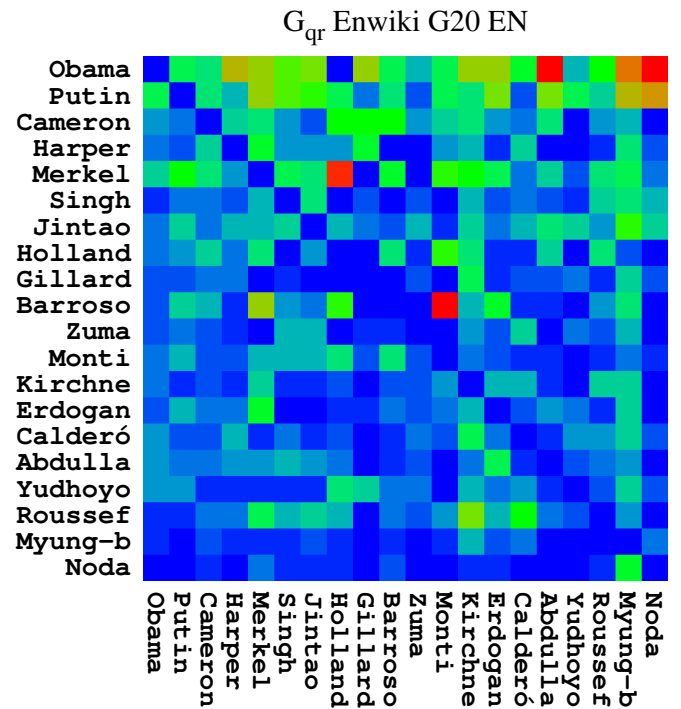


Fig. 25. Density plot of the matrix G_{qr} without diagonal elements for the reduced network of 20 state leaders of G20 states in the Enwiki network with short names at both axes. The weight of this matrix component without diagonal is $W_{qrd} = 0.001878$ (see Fig. 23 for W_{qr} weight with diagonal).

G20	EN	Enwiki
Name	Friends	Followers
Obama	Putin Merkel Calderón	Noda Abdullah Myung-bak
Putin	Merkel Obama Barroso	Noda Myung-bak Merkel
Cameron	Putin Obama Merkel	Gillard Barroso Hollande
Harper	Obama Cameron Putin	Merkel Gillard Myung-bak
Merkel	Barroso Putin Obama	Hollande Monti Kirchner

Table 13. Same as Table 8 for leading G20 state leader and the English Wikipedia edition of 2013.

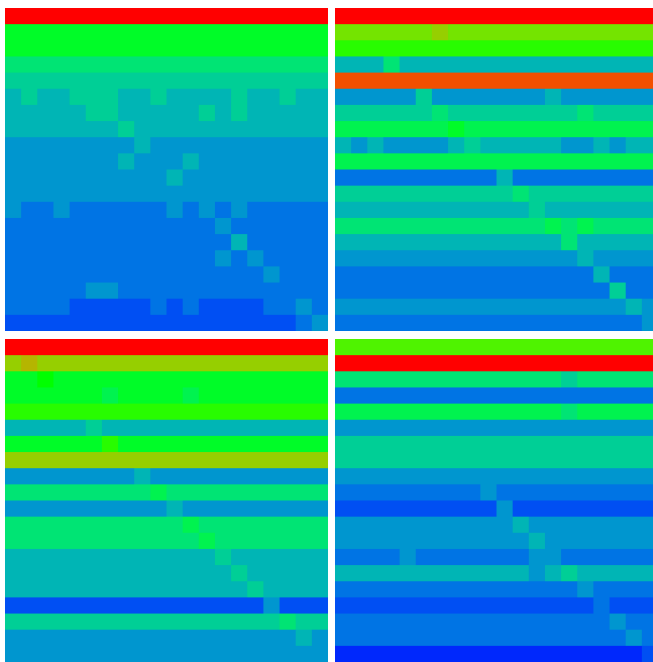


Fig. 26. Density plots of the matrix G_R for the reduced networks of 20 state leaders of G20 states in the Enwiki (top left), Dewiki (top right), Frwiki (bottom left) and Ruwiki (bottom right) networks. The order of the 20 state leaders is in all cases given by the PageRank order for the Enwiki network and corresponds to the same order of Figs. 23-25.

ily pictured in the English version due to cultural bias, and vice versa. For instance, the very strong link from Hollande to Merkel in Enwiki is really thin in the French edition, while it is clearly visible in the German one.

10 Network of political leaders of France

To have a more direct pictorial representation of interactions in the framework of the reduced matrix G_R we choose the case of 40 politicians of France discussed above (see Fig. 5 and Table 5). Here all politicians are attributed to the main political parties marked by corresponding 5 colors in Table 5. For each color we take the top PageRank politicians: Sarkozy (blue, right parties, UMP-UDI), Hollande (magenta, left, PS), Jean-Marie Le Pen (violet, ultra-right, FN), Mélanchon (red, ultra-left), Cohn-Bendit (green, green party). These 5 “leaders”¹ are positioned on a circle of fixed radius forming the first level. Then for the directed network of G_R of these 5 leaders we show for each of them 4 strongest links from them to other politicians (considered as their direct friends). These links are shown by directed bold black lines with arrows in the top panel of Fig. 30. The new (secondary) politicians appeared from

¹ This definition of leader does not necessarily correspond to the official “leader” of a political party or group since in the Wikipedia PageRank index also older, more historical aspects, e. g. “former leadership”, are taken into account.

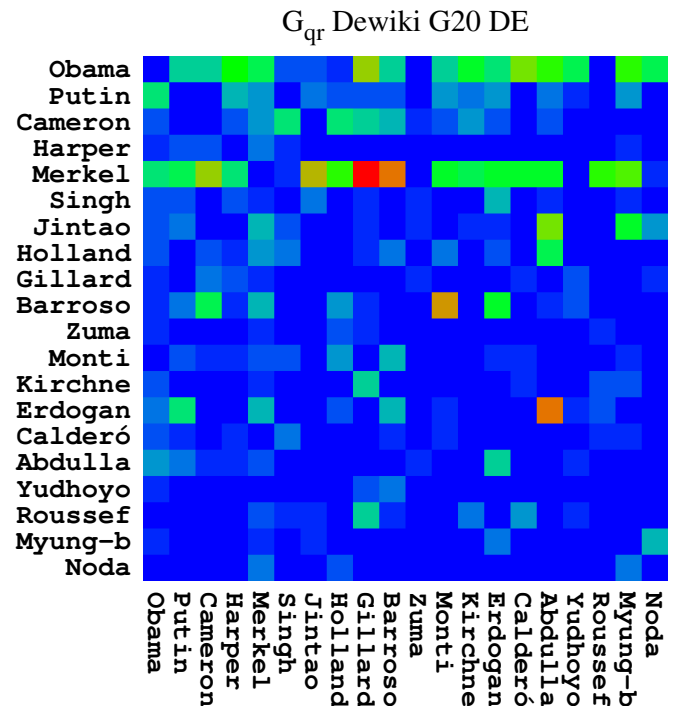


Fig. 27. Density plot of the matrix G_{qr} without diagonal elements for the reduced network of 20 state leaders of G20 states in the Dewiki network with short names (in English) at both axes using the same order of Fig. 25. The weight of this matrix component without diagonal is $W_{qrd} = 0.001125$.

these links are placed on second level circles around the (primary) politicians of the first level to which they correspond (the preference is given to the primary politicians with the same color if there are several corresponding primary politicians). For the politicians of the second level new red links with arrows are drawn for each of them with top 4 strongest links forming the third level circles if any. After these two iterations of 4 strongest friends we obtain the network of friends of G_R with only 7 politicians. The politicians are marked by their PageRank numbers K from Table 5. At blue color we find Sarkozy with Raffarin, at magenta we have Hollande with Royal, and for other colors we have only one politician of the first circle. Thus we find that the circle of close friends is very narrow. We also mention that for this case the subnetwork actually saturates completely at level 3 (i.e. including the group of tertiary politicians visible in Fig. 30) with only these 7 politicians and does not increase even if we try to include higher level circles/politicians. The reason of the saturation on a small sub network is that G_R is dominated by the rank one contribution G_{pr} which selects essentially top PageRank nodes. Therefore we simply find the top 5 PageRank nodes plus the two late PageRank position nodes of Mélanchon and Cohn-Bendit that have been selected to belong to the set of primary nodes.

We now perform the same procedure for the followers in G_R , using the strongest 4 incoming links on each level, instead of friends, as it is shown in the bottom panel of

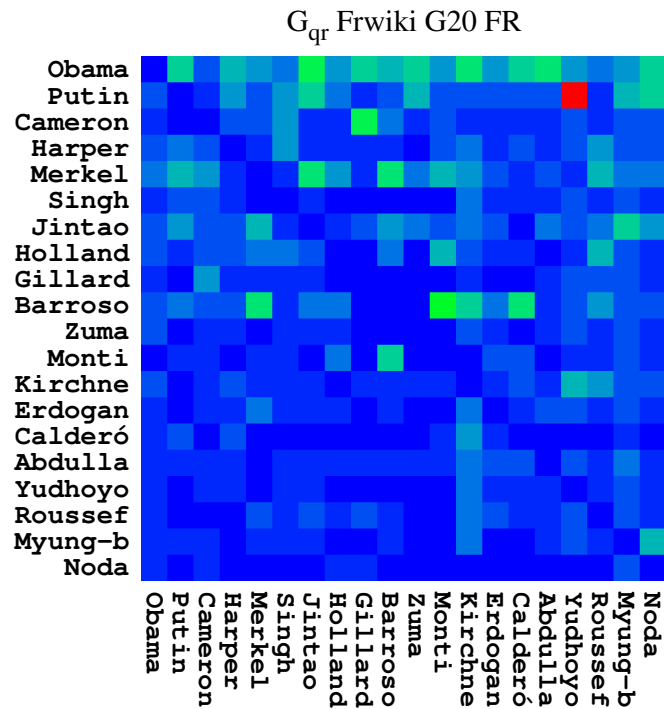


Fig. 28. Density plot of the matrix G_{qr} without diagonal elements for the reduced network of 20 state leaders of G20 states in the Frwiki network with short names (in English) at both axes using the same order of Fig. 25. The weight of this matrix component without diagonal is $W_{qrd} = 0.001554$.

Fig. 30. Again after two iterations (black and red links) we obtain 32 politicians showing that the number of followers is significantly larger than the number of friends. The network of followers has an interesting structure: violet, blue and green parties form mainly compact groups while the magenta party is clearly divided on two groups one centered around Hollande and another centered around Mélanchon. This clearly indicates fundamental structural issues in French PS. We note that for this case of followers (for G_R) the subnetwork would saturate at all 40 politicians after 5 iterations (the corresponding higher level links are not shown in Fig. 30 in order to keep the presentation simpler).

We also used the same approach to construct the network of hidden friends or hidden followers using the matrix G_{qr} (instead of G_R) with the result shown in both panels of Fig. 31. Now, the network of hidden friends contains a significantly larger number of politicians after two iterations (15 instead of 7 for G_R but also with saturation at level 3). At the same time the network of followers has a similar number of nodes (34 instead of 32 for G_R with saturation at 35 after level 4). For the hidden follower network the separation of the magenta PS party in two groups is less pronounced with a smaller number of followers for Mélanchon. The group of the violet party becomes more compact having all its 4 members being grouped together. There is also Taubira ($K = 29$) who is closely linked to

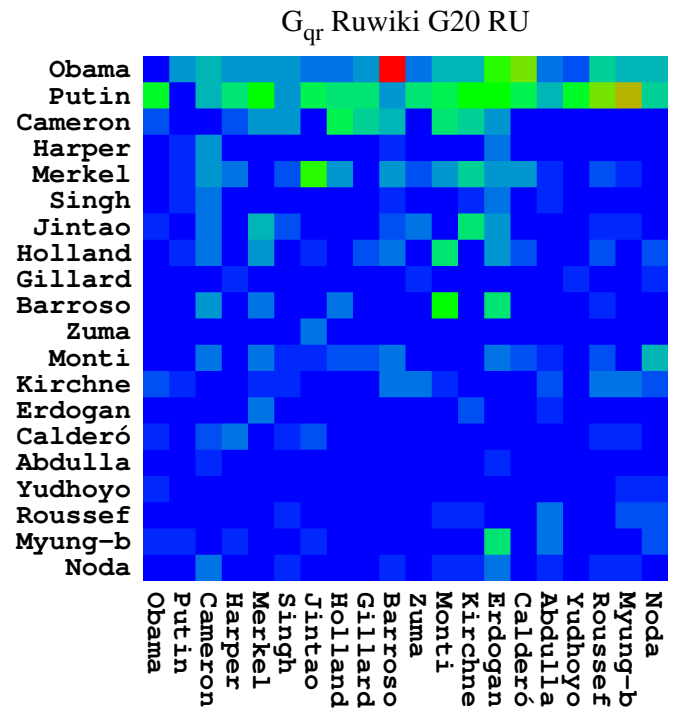


Fig. 29. Density plot of the matrix G_{qr} without diagonal elements for the reduced network of 20 state leaders of G20 states in the Ruwiki network with short names (in English) at both axes using the same order of Fig. 25. The weight of this matrix component without diagonal is $W_{qrd} = 0.001089$.

this violet group for similar reasons as the ones presented earlier.

Several follower interactions seem more reasonable with G_{qr} . For instance, Dufflot ($K = 37$, Green party) isn't linked anymore to PS but mainly to its own party members. Similarly, ultra-left members are connected together at the first level, and not only at the secondary level as it is the case for G_R . Interesting to notice is that in G_{qr} Philippot ($K = 40$, ultra-right FN party) clearly follows four of the main party leaders (J.-M. Le Pen, Sarkozy, Hollande and Cohn-Bendit), while this is not the case in G_R . This is clearly related to his political acquaintances of before 2009 where he backed up Chevènement (socialist party) and Mélenchon, among others.

11 Discussion

In conclusion, we have presented a new mathematical method which establishes an effective directed network for a selected subset of nodes belonging to a significantly larger network. This approach was tested on examples of several groups of political leaders of 5 countries and world state leaders of G20 analyzed in the frame of several Wikipedia networks. Our results show that the proposed method allows in a reliable way to determine direct and hidden links between political leaders. We think that this approach can provide firm mathematical grounds for the LMX studies

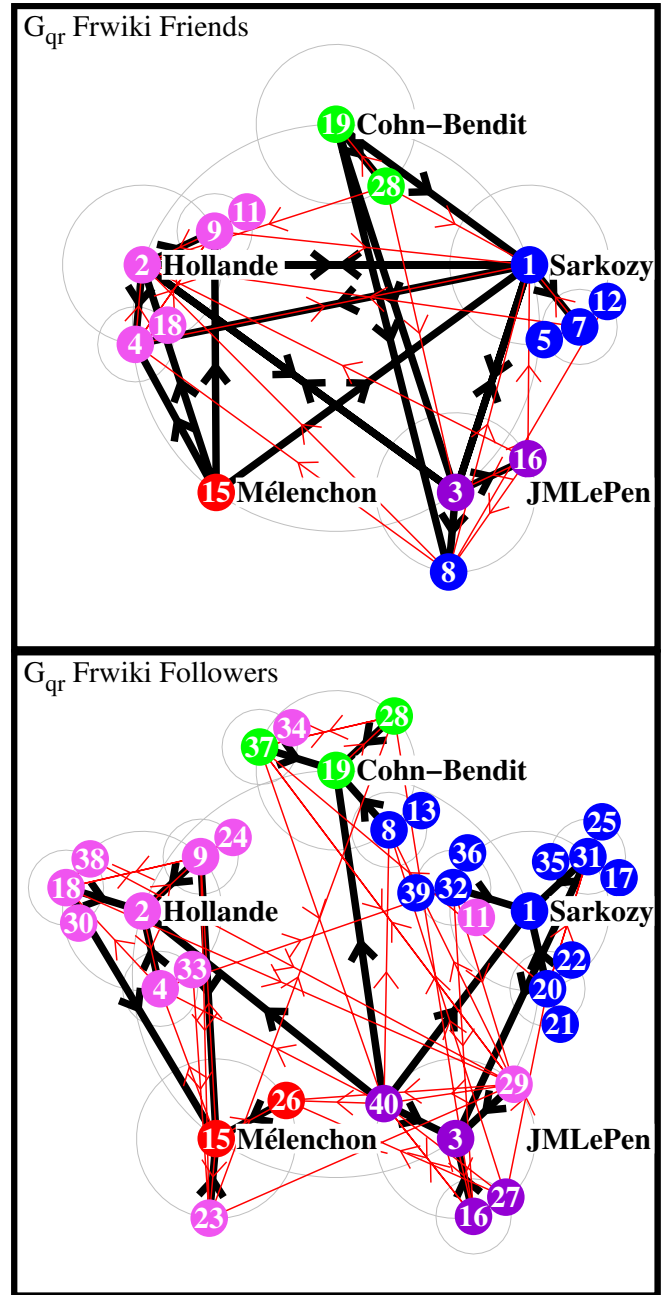
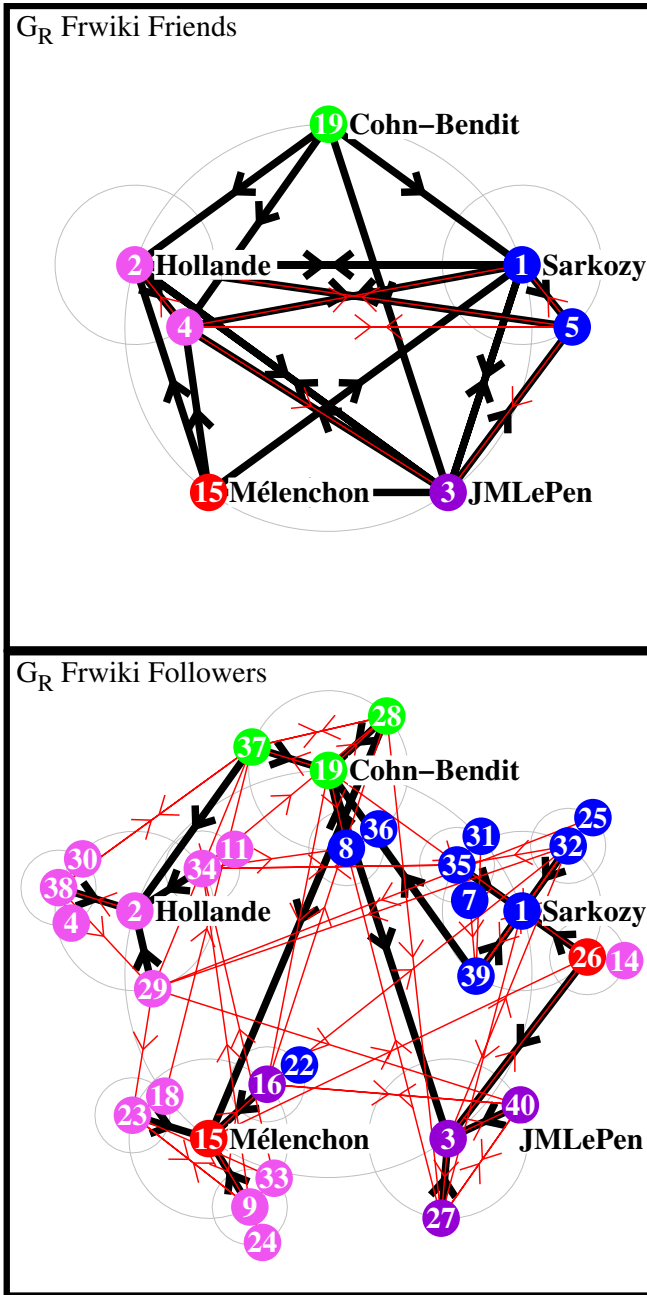


Fig. 30. Network structure of friends (followers) in top (bottom) panel induced by the largest four nondiagonal elements per column (row) of the matrix G_R obtained for the group of 40 French politicians in the French Wikipedia edition of 2013. The five colors of the nodes represent the five main political movements and the numbers of the nodes represent the local PageRank index in Table 5. For each movement the leading node in PageRank ordering is drawn on the largest primary circle with name. The thick black arrows represent links from (to) each primary node to (from) its four most relevant nodes and the thin red arrows represent links from (to) each secondary node to (from) its four most relevant nodes. New secondary (tertiary) nodes are drawn on the secondary (tertiary) circles. Most of these circles contain less than four nodes due to possible multiple links with center nodes on other circles of the same level. When possible, a secondary (tertiary) node was associated to a primary (secondary) node of the same color.

Fig. 31. Same as Fig. 30 for the matrix G_{gr} , which recovers hidden links between politicians.

[22, 23, 24] in social and political sciences. Our results show that the Wikipedia network can be used in an efficient way to determine direct and hidden relations between different subjects appearing in Wikipedia. We also point that the reduced Google matrix approach can be applied to a variety of directed networks where the relations between selected subgroup of nodes are not straightforward to identify.

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