

Kolmogorov turbulence defeated by Anderson localization for a Bose-Einstein condensate in a Sinai-oscillator trap

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We study the dynamics of a Bose-Einstein condensate in a Sinai-oscillator trap under a monochromatic driving force. Such a trap is formed by a harmonic potential and a repulsive disk located in the center vicinity corresponding to the first experiments of condensate formation by Ketterle group in 1995. We argue that the external driving allows to model the regime of weak wave turbulence with the Kolmogorov energy flow from low to high energies. We show that in a certain regime of weak driving and weak nonlinearity such a turbulent energy flow is defeated by the Anderson localization that leads to localization of energy on low energy modes. A critical threshold is determined above which the turbulent flow to high energies becomes possible. We argue that this phenomenon can be studied with ultra cold atoms in magneto-optical traps.

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The Kolmogorov turbulence [1, 2] is based on a concept of energy flow from large spacial scales, where an energy is pumped by an external force, to small scales where it is absorbed by dissipation. As a result a polynomial energy distribution over wave modes is established which has been obtained first from scaling arguments for hydrodynamics turbulence [1, 2]. Later the theory of weak turbulence, based on diagrammatic techniques and the kinetic equation, demonstrated the emergence of polynomial distributions for various types of weakly interacting nonlinear waves [3–5]. However, this theory is based on a fundamental hypothesis directly stated in the seminal work of Zhakharov and Finonenko: *“In the theory of weak turbulence nonlinearity of waves is assumed to be small; this enables us, using the hypothesis of the random nature of the phase of individual waves, to obtain the kinetic equation for the mean square of the wave amplitudes”*. But in finite systems the dynamical equations for waves do not involve Random Phase Approximation (RPA) and the question of RPA validity, and hence of the whole concept of energy flow from large to small scales, remains open.

Indeed, it is known that in a random media with a fixed potential landscape the phenomenon of Anderson localization [6] breaks a diffusive spreading of probability in space due to quantum interference effects, even if the underline classical dynamics of particles produces an unlimited spreading. At present the Anderson localization has been observed for a large variety of linear waves in various physical systems [7]. A similar phenomenon appears also for quantum systems in a periodically driven *ac*-field with a quantum dynamical localization in energy and number of absorbed photons [8–12]. This dynamical localization in energy has been observed in experiments with Rydberg atoms in a microwave field [12, 13] and

cold atoms in driven optical lattices [14, 15]. Thus in the localized phase the periodic driving is not able to pump energy to a system even if the classical dynamics is chaotic with a diffusive spreading in energy.

Of course, the Anderson localization takes place for linear waves. The question about its robustness in respect to a weak nonlinearity attracted recently a significant interest of nonlinear science community [16–20] with the first experiments performed in nonlinear media and optical lattices [21, 22]. These studies show that below a certain threshold the Anderson localization remains robust in respect to a weak nonlinearity while above the threshold a subdiffusive spearing over the whole system size takes place. However, the studies are done for conservative systems without external energy pumping. The numerical simulations for a simple model of kicked nonlinear Schrödinger equation on a ring gave indications that an energy flow to high energies is stopped by the Anderson localization for a weak nonlinearity [23] but such a model is rather far from real experimental possibilities with nonlinear media or cold atoms.

In this Letter we consider a realistic system of Bose-Einstein condensate (BEC) of cold atoms captured in a Sinai-oscillator trap under a monochromatic force. In fact this system in three dimensions (3D) had been used for a pioneering realization of BEC reported in [24] (see also [25, 26]). It represents a harmonic trap with a repulsive potential in a vicinity of the trap center created by a laser beam. The repulsive potential can be well approximated by a rigid disk which creates scattering of atoms and instability of their classical dynamics. In two dimensions (2D) with a harmonic potential replaced by rigid rectangular walls the systems represents the well-known Sinai billiard where the mathematical results guaranty that the whole system phase space is chaotic with a pos-

itive Kolmogorov entropy [27]. Recently it was shown that the classical phase space remains practically fully chaotic if the rigid walls are replaced by a harmonic potential which is much more suitable for BEC experiments [28]. The corresponding quantum system is characterized by the level spacing statistics of random matrix theory [29] satisfying the Bohigas-Giannoni-Schmit conjecture [30] and thus belonging to the systems of quantum chaos [31].

The effects of nonlinearity for BEC evolution in a Sinai-oscillator trap has been studied in [28] in the frame of the Gross-Pitaevskii equation (GPE) [32]. It was shown [28] that at weak nonlinearity the dynamics of linear modes remains quasi-integrable while above a certain threshold there is onset of dynamical thermalization leading to the usual Bose-Einstein distribution [33] over energies of linear eigenmodes. Even if being chaotic this system has energy conservation and there is no energy flow to high energy modes. However, a monochromatic driving force breaks the energy conservation leading for a classical dynamics to a diffusive energy growth and probability transfer to high energy modes typical for the Kolmogorov turbulence. Here we show that there is a regime where such an energy transfer to waves with high wave vectors is suppressed by the dynamical localization being similar to the Anderson localization in disordered solids.

We note that the Kolmogorov turbulence for BEC in 2D rectangular and 3D cubic billiards has been studied numerically in [34, 35]. However, the integrable shape of these billiards does not allow to realize a generic case of random matrix spectrum of linear modes typical for our billiard belonging to the class of quantum chaos systems [31].

For our model the classical dynamics and quantum evolution in absence of interactions are described by the Hamiltonian

$$\hat{H}_0 = (\hat{p}_x^2 + \hat{p}_y^2)/2m + m(\omega_x^2 \hat{x}^2 + \omega_y^2 \hat{y}^2)/2 + V_d(\hat{x}, \hat{y}) + f \hat{x} \sin \omega t \quad (1)$$

Here the first two terms describe 2D oscillator with frequencies ω_x, ω_y , the third term represents the potential of rigid disk of radius r_d and the last term gives a driven monochromatic field of amplitude f . Here we fixed the mass $m = 1$, frequencies $\omega_x = 1$, $\omega_y = \sqrt{2}$, $\omega = (1 + \sqrt{5})$ and disk radius $r_d = 1$. The disk center is placed at $(x_d, y_d) = (-1/2, -1/2)$ so that the disk bangs a hole in a center vicinity as it was the case in the experiments [24]. In the quantum case we have the usual commutator relations $[\hat{p}_x, \hat{x}] = [\hat{p}_y, \hat{y}] = -i\hbar$ with $\hbar = 1$ for dimensional units.

The BEC evolution in the Sinai oscillator trap is described by the GPE, which reads:

$$i\hbar \frac{\partial \psi(x, y, t)}{\partial t} = \hat{H}_0 \psi(x, y, t) + \beta |\psi(x, y, t)|^2 \psi(x, y, t), \quad (2)$$

where β describes nonlinear interactions for BEC. Here we use the same Sinai oscillator parameters as in [28] with

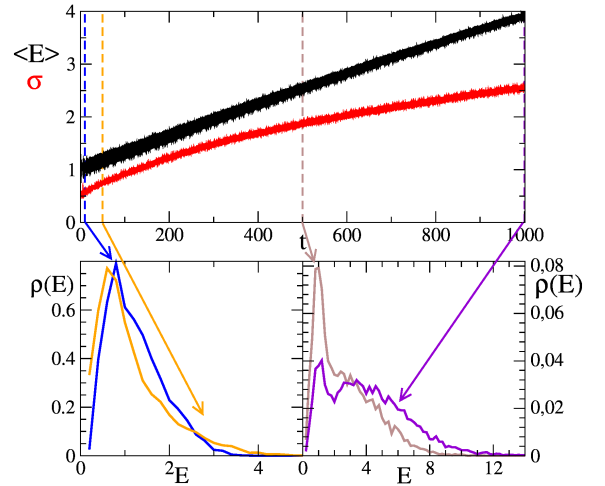


FIG. 1: (Color online) Classical time evolution of average energy $\langle E \rangle$ and its standard deviation σ for $f = 0.4$. The data are obtained from 10^4 trajectories with random initial conditions at $\langle E \rangle = 1$ and $\sigma = 0.5$. Top panel: $\langle E(t) \rangle$ and $\sigma(t)$ are shown by black and red/gray curves respectively. Bottom panels show probability distribution of trajectories $\rho(E, t)$ for (a) $t = 10, 50$ (blue/black, orange/gray curves) and (b) $t = 500, 1000$ (yellow/gray, violet/black curves). Vertical dashed lines in main panels mark snapshot times corresponding to bottom panels.

normalization $\int |\psi|^2 dx dy = 1$. The numerical integration of (2) is done in the same way as in [11, 28] with a Trotter time step ($\Delta t = 0.005$) evolution for noninteracting part of \hat{H}_0 followed by the nonlinear term contribution.

The results for energy E growth with time for classical dynamics (1) are shown in Fig. 1. The energy E and its dispersion σ are steadily growing with time. We expect that at large times the energy increases diffusively with a rate $(\Delta E)^2/t = D \approx C f^2 \omega_x^2 r_d \sqrt{E}/\omega^2$ assuming that $\omega_x \sim \omega_y$ and $\omega > \omega_x$. The data of Fig. 1 give us $C \approx 0.5$ at $t = 10^3$.

We note that the estimate for D comes from the fact that an oscillating velocity component $v_{osc} = f \cos(\omega t)/\omega$ gives a velocity change at disk collision (like with oscillating wall) $\Delta v_x = 2v_{osc}$ and an energy change $\Delta E \approx v_x \Delta v_x$ so that the diffusion is $D \sim (\Delta E)^2/t_c$ where an average time between collisions t_c is defined from the ergodicity relation $\Delta t_c/t_c \sim r_d^2 \omega_x^2/E$ of ratio of disk area and area of chaotic motion at energy E , where $\Delta t_c \sim r_d/E^{1/2}$; thus at large times $E \propto t^{2/3}$. The fit for $E \sim \sigma \propto t^\alpha$ in Fig. 1 gives $\alpha = 0.98 \pm 0.06$ (for E) and 0.58 ± 0.08 (for σ) being comparable to the theoretical value $\alpha = 2/3$. We attributed a deviation from theory to not sufficiently large amplitude of motion $\sqrt{2E}/\omega_x$ required for t_c expression at reached energies.

We also introduce cells of finite energy size δE and determine the probability distribution ρ_k over k energy cells counting a relative number of trajectories inside each cell. The results of Fig. 1 show that the width of proba-

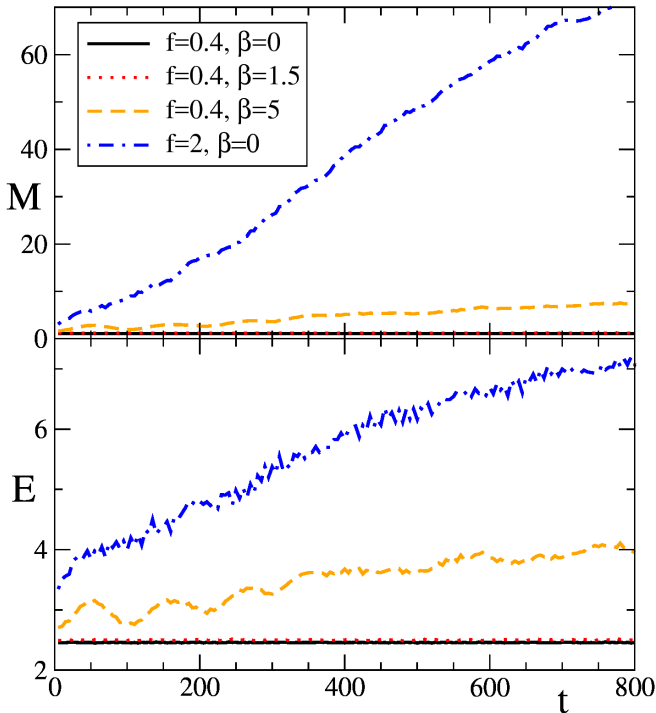


FIG. 2: (Color online) Time evolution of M (top panel) and energy E (bottom panel) for GPE (2) averaged over time intervals $\Delta t = 1$. The initial state is the ground state of (2) at $\beta = 0, f = 0$ (see Fig.5a in [28]). Both panels show the cases of $f = 0.4, \beta = 0$ (black solid lines), $f = 0.4, \beta = 1.5$ (red/gray dotted lines), $f = 0.4, \beta = 5$ (orange/gray dashed lines), $f = 2, \beta = 0$ (blue/gray dot-dashed lines),

bility distribution $\rho(E)$ in energy is growing in time corresponding to increase of E .

The situation is drastically different in the quantum case at $\beta = 0$. Here, at small f , the dynamical localization leads to a complete suppression of energy E and average mode number $M = \sum_k k \rho_k$ growth with their restricted oscillations in time (see Fig. 2). The probability distribution ρ_k over eigenstates ψ_k with eigenenergies E_k of (1) (for stationary case $f = 0$) is shown in Fig. 3. For small $f < f_c$, on average there is a clear exponential decay of probability $\rho_k \propto \exp(-2E_k/\omega \ell_\phi)$ with a number of absorbed photons $N_\phi = E_k/\omega$ and a photonic localization length ℓ_ϕ . Such a localization decay is similar to those discussed for atoms [12, 13] and quantum dots [11] in a microwave field. However, above a certain f_c , e.g. at $f = 2; 3$, we obtain delocalized probabilities ρ_k with a flat plateau distribution at high energies.

According to the theory of dynamical localization described in [11, 12, 36] we have $\ell_\phi \approx 2\pi(D/\omega^2)\rho_c$ where $\rho_c = dk/dE_k$ is the density of E_k states. According to [28] we have $k \approx E^2/2\sqrt{2}$ and $\rho_c \approx E/\sqrt{2}$. With the above expression for the classical diffusion in energy D we obtain $\ell_\phi \approx 2f^2\omega_x^2 E^{3/2}/\omega^4$. Similar to the quantum chaos model [36] we have ℓ_ϕ significantly growing with

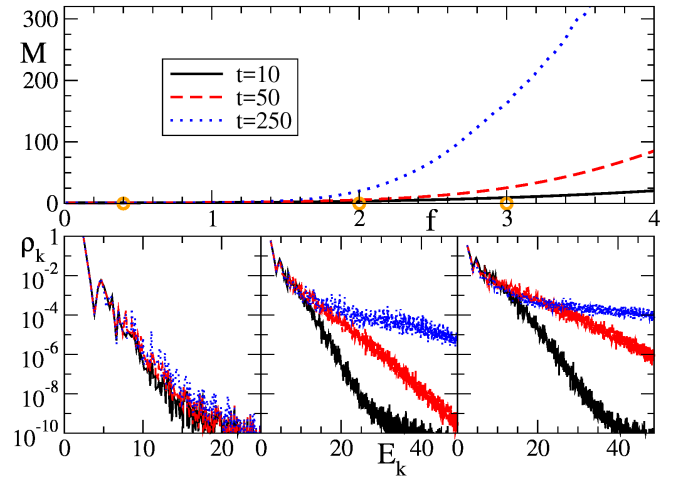


FIG. 3: (Color online) Top panel shows M as a function of driven force f for linear case ($\beta = 0$). Bottom panels show probability distribution ρ_k , averaged over time interval $\Delta t = 5$, as a function of eigenenergies E_k with $t = 10$ in black solid lines, $t = 50$ in red/gray dashed lines, and $t = 250$ in blue/gray dotted lines; Left, center and right bottom panels show the cases of $f = 0.4, 2, 3$ respectively (highlighted with orange/gray circles in top panel).

the number of absorbed photons N_ϕ so that the delocalization of quantum chaos takes place at $\ell_\phi > N_\phi$. As in [36] this leads to a delocalization above a certain border $f > f_c$ with a flat probability distribution on high energies as it is seen in Fig. 3. This gives the delocalization border for quantum states: $f_c r_d / \hbar \omega_x \approx 0.7(\omega/\omega_x)^{3/2} \approx 4$ for the initial ground state at $E \approx \hbar \omega_x = 1$ and $\omega \approx 3.2$. The data for M in Fig. 3 give the critical value $f_c \approx 1.5$ being somewhat smaller than the value given by the above estimate. We attribute this difference to the fact that the above estimate for D , and hence for ℓ_ϕ , is valid in the limit of large spacial oscillations being larger than r_d . The delocalization transition at $f > f_c$ is similar to the Anderson transition, or metal-insulator transition, in disordered systems [6, 7].

The results for $\beta > 0$ are presented in Figs. 2, 4. For $f = 0.4$, when the steady-state probability is well localized at $\beta = 0$, they clearly show that at $\beta = 1.5$ there is no growth of energy E and mode number M . Thus there is no energy flow to high energies and the Anderson localization remains robust for weak nonlinear perturbation. This is also well confirmed by a stable in time probability distribution over energies E_k shown in Fig. 4 (left panel). For larger nonlinearity $\beta = 5$ and $f = 0.4$ there appears a growth of M, E with time (Fig. 2). At larger $f = 1$ and $\beta = 5$ there is emergence of energy flow to high energies and increasing probability ρ_k at high energies E_k (Fig. 4 right panel).

The global dependence of average mode number M on driving amplitude f and nonlinearity β is shown in Fig. 5. We see that there is a stability region of small f, β

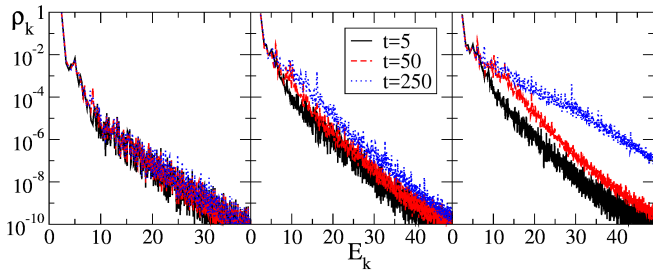


FIG. 4: (Color online) Same as in bottom panels of Fig. 3 for $f = 0.3, \beta = 1.5$ (left panel), $f = 0.5, \beta = 5$ (center panel) $f = 1, \beta = 5$ (right panel).

values where the values M remain small even at large times. This region corresponds to the localized insulator phase (I), from the view point of Anderson localization, of quasi-integrable (or laminar) phase from the view point of nonlinear dynamics (or turbulence). Outside of this region we have large values of number of populated states M so that this regime corresponds to the delocalized metallic or turbulence phase (M-TB). According to the obtained results we conclude that this quasi-stable (or insulator) regime ($f < f_c, \beta < \beta_c$) (see Fig. 5) is approximately described by the relation

$$f_c r_d / \hbar \omega_x \approx 1.5(1 - \beta_c / (6\hbar \omega_x r_d^2)) \quad (3)$$

assuming that $\omega_x \sim \omega_y \sim \omega$. Inside the I-region the turbulent Kolmogorov flow of energy to high modes is suppressed by the Anderson localization. At small nonlinearity β we expect a validity of the Kolmogorov-Arnold-Moser theory (KAM) [37, 38] leading to a quasi-integrable dynamics and trapping of energy on large length modes. At the same time we should note that the mathematical prove of KAM for nonlinear perturbation of pure-point spectrum of Anderson localization and the GPE (2) still remains an open challenge [18, 39, 40].

Outside of the stability region (3) a microwave driving transfers the energy flow from low to high energy modes generating the Kolmogorov energy flow. We expect that the energy dissipation and high modes leads to the the Kolmogorov spectrum of energy distribution [4, 5] over modes. Our results show that the RPA is definitely not valid and that, at small amplitudes of a monochromatic driving and small nonlinearity, the Kolmogorov turbulent flow to high modes is defeated by the Anderson localization and the KAM integrability. The transition from KAM phase to turbulence phase corresponds to the insulator-metal transition in disordered systems with the energy axis corresponding to the spatial distance respectively. The KAM or insulator phase corresponds to a usual observation that a small wind (small f amplitude) is not able to generate turbulent waves.

The experimental realization of our system with BEC in a magneto-optical trap corresponds to the experimental conditions described in [24]. A monochromatic per-

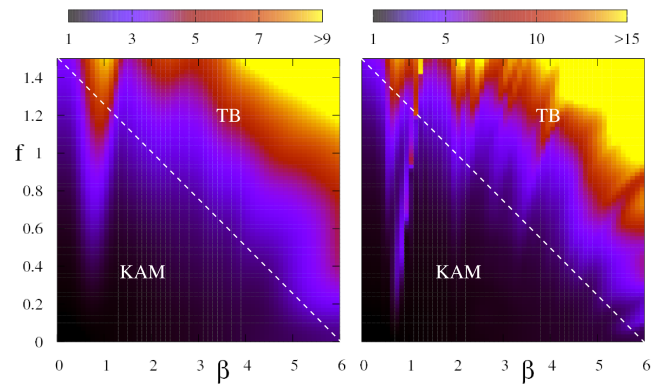


FIG. 5: (Color online) Number of modes M is shown by color/grayness in the plane of parameters f and β (average is done in the time intervals $100 \leq t \leq 150$ and $250 \leq t \leq 300$ in left and right panel respectively). The approximate separation of KAM or insulator phase (KAM) and delocalized turbulent or metallic phase (TB) is shown by the white line (3).

turbation can be created by oscillations of the center of harmonic potential or effectively by oscillations of the disk position created by the laser beam. We note that the experimental investigations of turbulent cascades in quantum gases become now possible [41] as well as a thermometry of energy distribution in ultra cold atom ensembles [42]. Thus we hope that the interesting fundamental aspects of nonlinear dynamics and weak turbulence will be tested with cold atom experiments.

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