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Interactions and Influence of World Painters From the Reduced Google Matrix of Wikipedia Networks

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ABSTRACT This paper concentrates on extracting painting art history knowledge from the network structure of Wikipedia. Therefore, we construct theoretical networks of webpages representing the hyper-linked structure of articles of seven Wikipedia language editions. These seven networks are analyzed to extract the most influential painters in each edition using Google matrix theory. Importance of webpages of over 3000 painters is measured using the PageRank algorithm. The most influential painters are enlisted and their ties are studied with the reduced Google matrix analysis. The reduced Google matrix is a powerful method that captures both direct and hidden interactions between a subset of selected nodes taking into account the indirect links between these nodes via the remaining part of large global network. This method originates from the scattering theory of nuclear and mesoscopic physics and field of quantum chaos. In this paper, we show that it is possible to extract from the components of the reduced Google matrix meaningful information on the ties between these painters. For instance, our analysis groups together painters that belong to the same painting movement and shows meaningful ties between painters of different movements. We also determine the influence of painters on world countries using link sensitivity between Wikipedia articles of painters and countries. The reduced Google matrix approach allows to obtain a balanced view of various cultural opinions of Wikipedia language editions. The world countries with the largest number of top painters of selected seven Wikipedia editions are found to be Italy, France, and Russia. We argue that this approach gives meaningful information about art and that it could be a part of extensive network analysis on human knowledge and cultures.

INDEX TERMS Big data, Google matrix, Markov chains, Wikipedia networks.

I. INTRODUCTION

"The art is the expression or application of human creative skill and imagination, typically in a visual form such as painting or sculpture, producing works to be appreciated primarily for their beauty or emotional power" [1]. Artists use different approaches and techniques to create emotions. Since the beginning of mankind, painters have offered masterpieces in the form of paintings and drawings to the world. Depending on historical periods, cultural context and available techniques, painters have followed different art movements. Art historians group painters into art movements to capture the fact that they have worked in the same school of thought. But a painter could be placed in several movements as his works evolve with time and its individual intellectual path development [2]–[8].

The major finding of this paper is to show that it is possible to automatically extract this common knowledge on art history by analyzing the hyper-linked network structure of the global and free online encyclopedia Wikipedia [9]. The analysis conducted in this work is solely based on a graph representation of the Wikipedia articles where vertices (nodes) represent the articles and the edges (links) provide the hyperlinks linking these articles together. The actual content of articles is never processed in our developments.

Wikipedia has become the largest open source of knowledge being close to Encyclopædia Britannica [10] by the accuracy of its scientific entries [11] and overcoming the later by the enormous quantity of available information. A detailed analysis of strong and weak features of Wikipedia is given in [12] and [13]. Unique to Wikipedia is that articles make citations to each other, providing a direct relationship between webpages and topics. As such, Wikipedia generates a large directed network of article titles with a rather clear meaning. For these reasons, it is interesting to apply algorithms developed for search engines of World Wide Web (WWW), those like the PageRank algorithm [14] (see also [15], [16]), to analyze the ranking properties and relations between Wikipedia articles. For various language editions of Wikipedia it was shown that the PageRank vector produces a reliable ranking of historical figures over 35 centuries of human history [17]–[20] and a solid Wikipedia ranking of world universities (WRWU) [17], [21], [22]. It has been shown that the Wikipedia ranking of historical figures is in a good agreement with the well-known Hart ranking [23], while the WRWU is in a good agreement with the Shanghai Academic ranking of world universities [24].

At present, directed networks of real systems can be very large (about 4.2 million articles for the English Wikipedia edition in 2013 [16] or 3.5 billion web pages for a publicly accessible web crawl that was gathered by the Common Crawl Foundation in 2012 [25]). For some studies, one might be interested only in the particular interactions between a very small subset of nodes compared to the full network size. For instance, in this paper, we are interested in capturing the interactions of nodes using the networks extracted from 7 Wikipedia language editions (FrWiki, EnWiki, DeWiki, ItWiki, EsWiki, NlWiki and RuWiki). We use the network data sets of Wikipedia 2013 described in [20]. The selected nodes (their Wikipedia articles) are embedded in a huge complex directed network with millions of nodes. Thus, the interactions between these selected sets of nodes should be correctly determined taking into account that there are many indirect links between the webpages via all other nodes of the network. In previous works, a solution to this general problem has been proposed in [26]–[28] by defining the reduced Google matrix theory. Main elements of reduced Google matrix $G_{\rm R}$ will be presented in Section II. This approach develops the ideas of scattering theory of nuclear and mesoscopic physics and quantum chaos adapted to Markov chains and Google matrix [26], [27].

In a few words, $G_{\rm R}$ captures in a N_r-by-N_r[1] Perron-Frobenius matrix the full contribution of both direct and indirect interactions existing in the regular Google matrix model of the network, but only for the reduced set of N_r nodes. The number N_r is in the order of a few tens of nodes, which is considerably smaller that the size of the full Wikipedia network which contains millions of nodes. Elements of reduced matrix $G_{\rm R}(i, j)$ can be interpreted as the probability for a random surfer starting at webpage j to arrive in webpage i using direct and indirect interactions. Indirect interactions refer to paths composed in part of webpages different from the N_r ones of interest. Even more interesting and unique to reduced Google matrix theory, we show here that intermediate computation steps of $G_{\rm R}$ offer a decomposition of $G_{\rm R}$ into matrices that clearly distinguish direct from indirect interactions. As such, it is possible to extract a meaningful probability for an indirect interaction between two nodes to happen as shown in the results of [27] and [28]. Thus the reduced Google matrix theory is a perfect candidate for analyzing the direct and indirect interactions between the selected painters.

In this paper, we extract from $G_{\rm R}$ and its decomposition into direct and indirect matrices a high-level *reduced network of N_r painters*. This high-level network is computed with both direct and hidden (i. e. indirect) interactions. More specifically, we deduce from $G_{\rm R}$ a fine-grained classification of painters that captures what we call the *hidden friends* of a given node. The structure of these graphs provides relevant information that offers new information compared to the direct network of relationships.

The aforementioned networks of direct and hidden interactions can be calculated for different Wikipedia language editions. In this paper, reduced Google matrix analysis is applied to the set of 30 painters and the set of 40 painters with 40 countries, from seven different Wikipedia language editions (English, French, German, Spanish, Russian, Italian and Dutch). We will refer to these editions using EnWiki, FrWiki, DeWiki, EsWiki, RuWiki, ItWiki and NlWiki in the remainder of this paper.[2] In total we analyzed the list of 3249 painters taken from [30], restricted to the ones that are present in all 7 language editions. Moreover, we provide hereafter, an analysis of the influence of top PageRank painters on world countries after constructing the reduced Google matrix composed of top 40 PageRank painters and the top 40 PageRank countries investigated in [28]. We present the full lists of painters, rank lists and additional figures at [31].

This paper introduces first the main elements of reduced Google matrix theory in Section II. Next, Section III presents the ranking and selection of painters based on the PageRank algorithm. In Section IV the reduced Google matrices are calculated and described for selected sets for seven different language editions. Specific emphasis is given to the very different English, French and German editions. Then, networks of friendship from direct and hidden interaction matrices are created and discussed in Section V. We show that the networks of friends completely capture the well-established history of painting by *i*) interconnecting densely painters of the same movement and *ii*) showing reasonable links between painters of different movements. We also obtain the global

[1] N_r represents the number of our selected nodes of interest.

[2] The networks of EnWiki, FrWiki, RuWiki, DeWiki, ItWiki, EsWiki and NlWiki contain 4.212, 1.353, 0.966, 1.533, 1.017, 0.974 and 1.14 millions of articles respectively.

TABLE 1. List of 50 top painters from FrWiki, EnWiki, DeWiki, ItWiki, EsWiki, NlWiki and RuWiki by increasing PageRank index .

FrWiki	EnWiki	DeWiki	ItWiki	EsWiki	NlWiki	RuWiki
Pablo Picasso	Leonardo da Vinci	Leonardo da Vinci	Leonardo da Vinci	Leonardo da Vinci	Rembrandt Van Rijn	Leonardo da Vinci
Leonardo da Vinci	Pablo Picasso	Pablo Picasso	Michelangelo	Francisco Goya	Leonardo da Vinci	Pablo Picasso
Michelangelo	Michelangelo	Albrecht Durer	Raphael	Pablo Picasso	Peter Paul Rubens	Michelangelo
Claude Monet	Raphael	Michelangelo	Pablo Picasso	Michelangelo	Vincent Van Gogh	Rembrandt Van Rijn
Vincent Van Gogh	Rembrandt Van Rijn	Raphael	Giorgio Vasari	Raphael	Pablo Picasso	Vincent Van Gogh
Jacques-Louis David	Vincent Van Gogh	Rembrandt Van Rijn	Titian	Diego Velázquez	Johannes Vermeer	Raphael
Eugène Delacroix	Francis Bacon	Peter Paul Rubens	Peter Paul Rubens	Salvador Dali	Piet Mondrian	Albrecht Durer
Raphael	Andy Warhol	Vincent Van Gogh	Caravaggio	Peter Paul Rubens	Pieter Bruegel The Elder	Ilya Repin
Henri Matisse	Peter Paul Rubens	Titian	Vincent Van Gogh	Titian	Claude Monet	Peter Paul Rubens
Salvador Dali	Albrecht Durer	Francis Bacon	Giotto Di Bondone	Francis Bacon	Titian	Nicholas Roerich
Paul Cézanne	William Blake	Andy Warhol	Rembrandt Van Rijn	Albrecht Durer	Sandro Botticelli	Titian
Rembrandt Van Rijn	Titian	Paul Klee	Sandro Botticelli	El Greco	Paul Cézanne	Henri Matisse
Peter Paul Rubens	Claude Monet	Paul Cézanne	Albrecht Durer	Rembrandt Van Rijn	Albrecht Durer	Salvador Dali
Andy Warhol	Salvador Dali	Lucas Cranach the Elder	Francisco Goya	Vincent Van Gogh	Frans Hals	Paul Cézanne
Marcel Duchamp	Henri Matisse	Wassily Kandinsky	Giuseppe Arcimboldo	Sandro Botticelli	Giotto Di Bondone	Viktor Vasnetsov
Édouard Manet	Giorgio Vasari	Claude Monet	Piero Della Francesca	Caravaggio	Jan Van Eyck	Ivan Aivazovsky
Giorgio Vasari	Paul Cézanne	Henri Matisse	Edvard Munch	Henri Matisse	Andy Warhol	Diego Velázquez
Paul Gauguin	Francisco Goya	Salvador Dali	Andrea Mantegna	Eugène Delacroix	Anthony van Dyck	Marc Chagall
Albrecht Durer	Joseph Mallord William Turner	Giorgio Vasari	Masaccio	Paul Cézanne	Paolo Veronese	Claude Monet
Pierre Auguste Renoir	Eugène Delacroix	Edvard Munch	Claude Monet	Andy Warhol	Francisco Goya	Valentin Serov
Joan Miró	Caravaggio	Giotto Di Bondone	Jacques-Louis David	Claude Monet	Salvador Dali	Paul Gauguin
Jean-Auguste-Dominique Ingres	Jackson Pollock	Marc Chagall	Samuel Morse	Giorgio Vasari	Édouard Manet	Hieronymus Bosch
Georges Braque	Édouard Manet	Caspar David Friedrich	Wassily Kandinsky	Paul Gauguin	JAMES ENSOR	Henri de Toulouse-Lautrec
Edgar Degas	Anthony van Dyck	Édouard Manet	Diego Velázquez	Diego Rivera	Wassily Kandinsky	Karl Bryullov
Francisco Goya	Pierre Auguste Renoir	Otto Dix	Pieter Bruegel The Elder	Giotto Di Bondone	Paul Gauguin	Eugène Delacroix
Gustave Courbet	Jacques-Louis David	Caravaggio	Fra Angelico	Jacques-Louis David	Henri Matisse	Wassily Kandinsky
Fernand Léger	Diego Velázquez	Francisco Goya	Salvador Dali	Édouard Manet	William Blake	Édouard Manet
Titian	William Hogarth	Pierre Auguste Renoir	Pierre Auguste Renoir	Tintoretto	Rene Magritte	Francisco Goya
Caravaggio	Paul Gauguin	Paul Gauguin	Andy Warhol	Bartolomé Esteban Murillo	Jacob Jordaens	Kazimir Malevich
Jackson Pollock	Hans Holbein The Younger	Max Ernst	Anthony van Dyck	Anthony van Dyck	Gustav Klimt	Andrei Rublev
Wassily Kandinsky	Edgar Degas	Gustav Klimt	Giovanni Battista Tiepolo	Georges Braque	Eugène Delacroix	Giorgio Vasari
Nicolas Poussin	Johannes Vermeer	Eugène Delacroix	Paul Cézanne	Edgar Degas	Karel Appel	Jacques-Louis David
Marc Chagall	Marcel Duchamp	Joan Miró	Giovanni Bellini	Joan Miró	Jacques-Louis David	Igor Grabar
Honoré Daumier	Sandro Botticelli	Jan Van Eyck	Domenico Ghirlandaio	Wassily Kandinsky	Giorgio Vasari	Pierre Auguste Renoir
Max Ernst	Giotto Di Bondone	Pieter Bruegel The Elder	Pietro Perugino	Hieronymus Bosch	Henry van de Velde	Samuel Morse
Diego Velázquez	Williem De Kooning	Max Liebermann	Jan Van Eyck	Piero Della Francesca	Henri de Toulouse-Lautrec	Caravaggio
Gustave Doré	Nicolas Poussin	Diego Velázquez	Paolo Veronese	Andrea Mantegna	Paul Klee	Edgar Degas
Sandro Botticelli	Pieter Bruegel The Elder	Sandro Botticelli	Giorgione	Jackson Pollock	Marc Chagall	Mikhail Vrubel
Giotto Di Bondone	John Constable	Marcel Duchamp	Nicolas Poussin	Henri de Toulouse-Lautrec	Joseph Mallord William Turner	Nicolas Poussin
Jean-Baptiste Camille Corot	Wassily Kandinsky	Gerhard Richter	Tintoretto	Johannes Vermeer	Edvard Munch	Anthony van Dyck
Henri de Toulouse-Lautrec	Marc Chagall	Max Beckmann	Paul Gauguin	Francisco De Zurbaran	Roger Van Der Weyden	Joseph Mallord William Turner
William Bouguereau	El Greco	Hans Holbein The Younger	Antonio da Correggio	William Blake	Georges Seurat	Jean-Auguste-Dominique Ingres
Pieter Bruegel The Elder	Lucas Cranach the Elder	El Greco	Edgar Degas	Marcel Duchamp	Nicolas Poussin	Alexandre Benois
Antoine Watteau	Benjamin West	Jacques-Louis David	Édouard Manet	Pierre Auguste Renoir	Joan Miró	Giotto Di Bondone
Georges Seurat	Gustave Doré	Georges Braque	Lucas Cranach the Elder	Hans Holbein The Younger	Gustave Doré	Konstantin Korovin
Rene Magritte	Henri de Toulouse-Lautrec	Johannes Vermeer	Eugène Delacroix	Pieter Bruegel The Elder	Edgar Degas	Isaac Levitan
André Derain	Georgia O'keefe	Henry van de Velde	Gustave Doré	Nicolas Poussin	Georges Braque	Gustave Courbet
Paul Klee	James Abbot Mac Neil Whistler	Edgar Degas	Marc Chagall	Jan Van Eyck	Hans Holbein The Younger	William Blake
François Boucher	Jan Van Eyck	Lovis Corinth	Guido Reni	William Bouguereau	Marcel Duchamp	Tove Jansson
Camille Pissarro	Thomas Gainsborough	Franz Marc	William Blake	Gustave Courbet		Ivan Kramskoi

ranking of painters averaged over all 7 Wikipedia editions and analyze the interactions between them. The influence of painters on world countries is analyzed in Section VI. Finally, Section VII discusses featured results and concludes this paper.

II. REDUCED GOOGLE MATRIX THEORY

It is convenient to describe the network of N Wikipedia articles by the Google matrix G constructed from the adjacency matrix A_{ij} with elements 1 if article (node) j points to article (node) i and zero otherwise. Elements of the Google matrix take the standard form $G_{ij} = \alpha S_{ij} + (1-\alpha)/N$ [14]–[16], where S is the matrix of Markov transitions with elements $S_{ij} = A_{ij}/k_{out}(j)$, $k_{out}(j) = \sum_{i=1}^{N} A_{ij} \neq 0$ being the node j out-degree (number of outgoing links) and with $S_{ij} = 1/N$ if j has no outgoing links (dangling node). The quantity $0 < \alpha < 1$ is the damping factor which for a random surfer determines the probability $(1-\alpha)$ to jump to any node; below we use the standard value $\alpha = 0.85$. The right eigenvector of G with the unit eigenvalue gives the PageRank probabilities $P(j)$ to find a random surfer on a node j. We order nodes by decreasing probability P getting them ordered by the PageRank index $K = 1, 2, ...N$ with a maximal probability at $K = 1$. From this global ranking we capture the top 50 painters mentioned in Tab. 1 for 7 editions.

From the computational view point the damping factor α induces a gap between the largest eigenvalue $\lambda = 1$ of G and all other eigenvalues with $|\lambda| \leq \alpha$. Due to this the multiplication of an initial random vector by a matrix G converges exponentially to the PageRank vector with an accuracy about 10^{-15} (about 150 multiplications are needed for $\alpha = 0.85$). Each multiplication of a vector by G involves about 20 N multiplications since on average a Wikipedia network has about 20 links per node. In the range $0.5 < \alpha < 0.9$ the PageRank vector in not very sensitive to α. The usually used value is $\alpha = 0.85$ [15]. The details of PageRank computations are given in [14]–[16] and [27].

Reduced Google matrix is constructed for a selected subset of nodes (articles) following the method described in [26]–[28] and based on concepts of scattering theory used in different fields including mesoscopic and nuclear physics, and quantum chaos. It captures in a N_r-by-N_r Perron-Frobenius matrix the full contribution of direct and indirect interactions happening in the full Google matrix between the N_r nodes of interest. In addition the PageRank probabilities of selected N_r nodes are the same as for the global network with N nodes, up to a constant multiplicative factor taking into account that the sum of PageRank probabilities over N_r nodes is unity. Elements of reduced matrix $G_{\mathrm{R}}(i, j)$ can be interpreted as the probability for a random surfer starting at web-page j to arrive in web-page i using direct and indirect

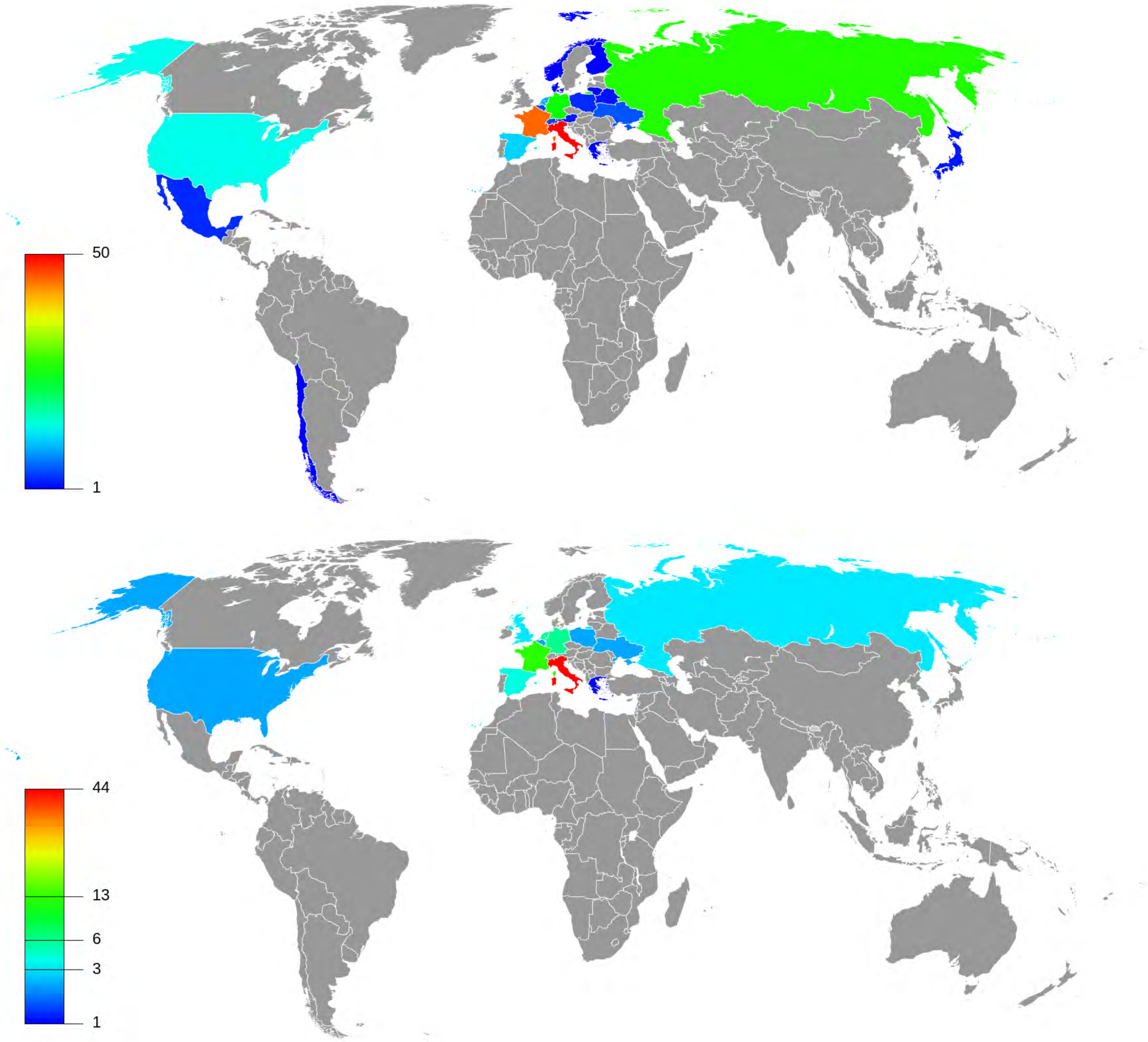

FIGURE 1. Geographic birthplace distribution of the 223 painters that appear at least one time in the PageRank top 100 painters of one of 7 language editions analyzed. Top panel represents 223 painters for all centuries till present, while bottom panel represents 88 painters having middle-age year less than year 1800; countries in gray have zero painters. The birth place is attributed to country borders of 2013.

interactions. Indirect interactions refer to paths composed in part of web-pages different from the N_r ones of interest. Even more interesting and unique to reduced Google matrix theory, we show here that intermediate computation steps of $G_{\rm R}$ offer a decomposition of $G_{\rm R}$ into matrices that clearly distinguish direct from indirect interactions: $G_{\rm R} = G_{rr} + G_{\rm pr} + G_{\rm qr}$ [27]. Here G_{rr} is given by the direct links between selected N_r nodes in the global G matrix with N nodes, $G_{\rm pr}$ is rather close to the matrix in which each column is given by the PageRank vector P_r, ensuring that PageRank probabilities of $G_{\rm R}$ are the same as for G (up to a constant multiplier). Therefore $G_{\rm pr}$ doesn't provide much information about direct and indirect links between selected nodes. The one playing an interesting role is $G_{\rm qr}$, which takes into account all indirect links between selected nodes appearing due to multiple paths via the global network nodes N (see [26]–[28]). The matrix $G_{\rm qr} = G_{\rm qrd} + G_{\rm qrnd}$ has diagonal ($G_{\rm qrd}$) and non-diagonal ($G_{\rm qrnd}$) parts. Thus $G_{\rm qrnd}$ describes indirect interactions between nodes. The matrix elements of $G_{\rm R}$, G_{rr}, $G_{\rm qrnd}$ are represented in a two dimensional density plot in Fig. 2 for a group of 30 painters of EnWiki. The explicit formulas as well as the mathematical and numerical computation methods of all three components of $G_{\rm R}$ are given in [26]–[28]. We discuss the properties of these matrix components below, but before that we introduce our painter selection process for the seven Wikipedia editions.

III. SELECTION OF PAINTERS

A. TOP PAGERANK PAINTERS

We are interested in this part in selecting the most influential painters representative of the seven investigated Wikipedia

TABLE 2. Top 40 painters ranked by decreasing importance following Θ_P-score computed over 7 editions. The average PageRank K_{av} is given as well. It derives from $G_{R_{av}}$, the matrix average of the individual G_R of all 7 editions.

Θ_P rank	K_{av} rank	Painter	Θ_P rank	K_{av} rank	Painter
1	1	Vinci	21	18	Bondone
2	2	Picasso	22	25	Kandinsky
3	6	Van Gogh	23	19	Botticelli
4	4	Rijn	24	21	Caravaggio
5	5	Rubens	25	23	Velázquez
6	8	Durer	26	30	Degas
7	9	Titian	27	26	Bruegel Eld
8	11	Monet	28	29	Dyck
9	12	Dali	29	28	Renoir
10	14	Cézanne	30	31	Chagall
11	3	Michelangelo	31	33	Lautrec
12	7	Raphael	32	27	Vermeer
13	10	Goya	33	36	Poussin
14	13	Vasari	34	37	Turner
15	16	Matisse	35	38	Braque
16	15	Warhol	36	32	Blake
17	17	Delacroix	37	34	Greco
18	22	Manet	38	39	Miró
19	20	David	39	35	Munch
20	24	Gauguin	40	40	Eyck

editions. Importance of nodes is measured in this selection process with the PageRank centrality.

A Matlab script has been written to retrieve all the painters' names from the "List of painters by name" webpage [30] edited by Wikipedia that lists painters from all ages and various parts of the world. We have collected *3334* names. Next, we get their nodes' number in our network representation of each Wikipedia edition. Note that some names are not necessarily known for their painting art production (e.g. Hitler). Thus we have made a second check to remove such cases from our list of painters. This initial sorting leads to a group of 3249 distinct names for our 7 selected editions enlisted in [31].

A Google matrix is constructed for each Wikipedia edition following the standard rules described in Section II. From the Google matrix of a given edition, PageRank index K of all N nodes is determined. From this vector of N values, we extract PageRank nodes of identified painters and we reorder them by decreasing PageRank value getting local PageRank index of painters. Tab. 1 shows the list of the top 50 PageRank painters captured individually by the 7 selected Wikipedia editions. Not surprisingly, the order of top painters changes with respect to the editions due to cultural bias but there are some main trends, e.g.:

- Leonardo Da Vinci ranks first place in 5 out of 7 editions,
- Michelangelo and Picasso belong to the top 4 in all editions,
- Russian painters, like Viktor Vasnetsov and Ivan Aivazovsky, are in the top 20 of RuWiki but don't appear before rank 50 in other editions.

Using the PageRank of all 3249 painters computed for 7 language editions, we have extracted 223 painters by creating the union set of top 100 painters of each language edition. The top panel of Fig. 1 illustrates the statistics of birth countries for these 223 painters (country borders are taken for year 2013). There is a clear predominance of European painters in this selection with a strong part of Russian artists as well. Among these 223 painters, 88 were born before year 1800 and the distribution of these 88 painters over the world map demonstrates a clear dominance of Italy at these times as to be seen in the bottom panel of Fig. 1.

B. GLOBAL RANKING OF PAINTERS

The above results demonstrate different cultural views on the importance of painters in the different language edition of Wikipedia. To get a global, multi-cultural importance of painters we use the approach proposed in [19] and [20]. It defines a global rank Θ_P with:

$$\Theta_P = \sum_E (101 - R_{P,E}). \quad (1)$$

Here $R_{P,E}$ is the rank of the top 100 painters P in Wikipedia edition E retrieved with PageRank algorithm. The painters with the largest Θ_P-score are the most important ones for E Wikipedia editions. Based on Θ_P score, we have selected two different sets of painters for our investigations.

1) TOP 40 PAINTERS

From the Θ_P-score calculated for the $E = 7$ Wikipedia editions of interest (e.g. EnWiki, FrWiki, RuWiki, DeWiki, ItWiki, EsWiki and NlWiki), we have selected the top 40 painters enlisted in Tab. 2 by order of importance.

2) PAINTING CATEGORIES NETWORK

This second set has been chosen to illustrate the existence of painting movements and how the reduced Google matrix analysis captures them automatically. This set is composed of 30 painters that belong to the six following painting

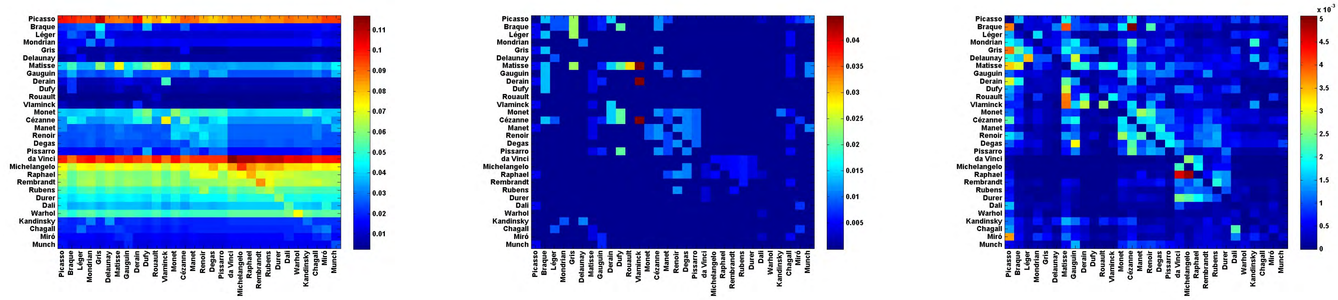

FIGURE 2. Density plots of G_R (left), G_{rr} (middle) and G_{qrnd} (right) for the reduced network of 30 painters grouped by categories from Tab. 3 for EnWiki network. Color scale represents maximum values in red, intermediate in green and minimum close to zero in blue; the absolute values of matrix elements are given by color bars (see also text for weights W_R, W_{rr}, W_{qr} of matrix components).

TABLE 3. List of names of 30 selected painters in the *Painting categories network* set. They are grouped by categories, and in each category they are ranked following the Θ_P-score obtained for FrWiki, EnWiki and DeWiki. Local PageRank order for FrWiki, EnWiki and DeWiki are given as well. A color is assigned to each category.

Name	Category	Colour	FrWiki	EnWiki	DeWiki
Picasso	Cubism	Red	1	2	2
Braque	Cubism	Red	17	20	20
Léger	Cubism	Red	19	24	24
Mondrian	Cubism	Red	25	22	22
Gris	Cubism	Red	29	28	25
Delaunay	Cubism	Red	28	27	26
Matisse	Fauvism	Blue	6	11	12
Gauguin	Fauvism	Blue	13	15	18
Derain	Fauvism	Blue	22	25	27
Dufy	Fauvism	Blue	27	26	29
Rouault	Fauvism	Blue	30	30	28
Vlaminck	Fauvism	Blue	24	29	30
Monet	Impressionists	Green	4	9	11
Cézanne	Impressionists	Green	8	12	9
Manet	Impressionists	Green	12	13	16
Renoir	Impressionists	Green	15	14	17
Degas	Impressionists	Green	18	16	21
Pissarro	Impressionists	Green	23	19	23
da Vinci	Great masters	Orange	2	1	1
Michelangelo	Great masters	Orange	3	3	4
Raphael	Great masters	Orange	5	4	5
Rembrandt	Great masters	Orange	9	5	6
Rubens	Great masters	Orange	10	7	7
Durer	Great masters	Orange	14	8	3
Dali	Modern 20-21	Pink	7	10	13
Warhol	Modern 20-21	Pink	11	6	8
Kandinsky	Modern 20-21	Pink	20	17	10
Chagall	Modern 20-21	Pink	21	18	15
Miró	Modern 20-21	Pink	16	21	19
Munch	Modern 20-21	Pink	26	23	14

categories: Cubism, Impressionism, Fauvism, Great masters and Modern art (20th century).

Following an average ranking Θ_P−score calculated for 3 Wikipedia editions (EnWiki, FrWiki and DeWiki), we have selected the top 5 painters of each category. They are enlisted in Tab. 3 by order of appearance. This Table also lists local PageRank index for painters in the French, English and German Wikipedia editions. Painters that belong to the same movement or having a common piece of history may probably exhibit stronger interactions in Wikipedia. As such, we have created a color code that groups together painters that either belong to the same movement (e.g. Fauvism, Cubism, Impressionists) or share a big part of history (e.g. Great Masters, Modern). Color code is as follows: Red, Blue, Green, Orange and Pink represents Cubism, Fauvism, Impressionists, Great masters and Modern (20-21st century), respectively.

IV. REDUCED GOOGLE MATRICES

To illustrate the matrices derived by the reduced Google matrix analysis, we plot G_R, G_{rr} and G_{qrnd} in Fig. 2 for the set of 30 painters composing the *Painting categories network* in Tab 3. The targeted edition is EnWiki.

Columns and lines are ordered with the order set in Tab 3. Following observations can be made. G_R is per-column normalized and dominated by the projector G_{pr} contribution, which is proportional to the global PageRank probabilities (for more details see in [26] and [27]). As such, we clearly see that the density of each line of G_R is proportional to the importance of the painter in the full network. The matrices are interpreted in the following way: painter of column j is linked with the probability of element (i, j) to the painter of line i.

As it was pointed in Section II the component G_{pr} of the reduced Google matrix G_R is essentially given by almost the same columns given by the PageRank vector of G_R. Due to this reason they mainly represent the PageRank probabilities of N_r nodes while the interactions and direct and indirect links between these nodes are described by G_{rr} and G_{qr}. The contribution of each component is characterized by its weight W_R, W_{pr}, W_{rr}, W_{qr} (W_{qrnd}) respectively for G_R, G_{pr}, G_{rr}, G_{qr} (G_{qrnd}). The weight is the sum of all matrix elements of a matrix component divided by N_r. By definition $W_R = 1$, while for EnWiki we have $W_{pr} = 0.913951$, $W_{rr} = 0.046369$, $W_{qr} = 0.03968$ ($W_{qrnd} = 0.0259$). These weights determine the absolute scale on matrix elements shown in Fig. 2. Similar values are obtained for DeWiki and FrWiki. Even if W_{pr} is significantly higher than W_{rr} and W_{qr} its columns are very close to the PageRank vector being very similar to each other. Due to this reason we concentrate

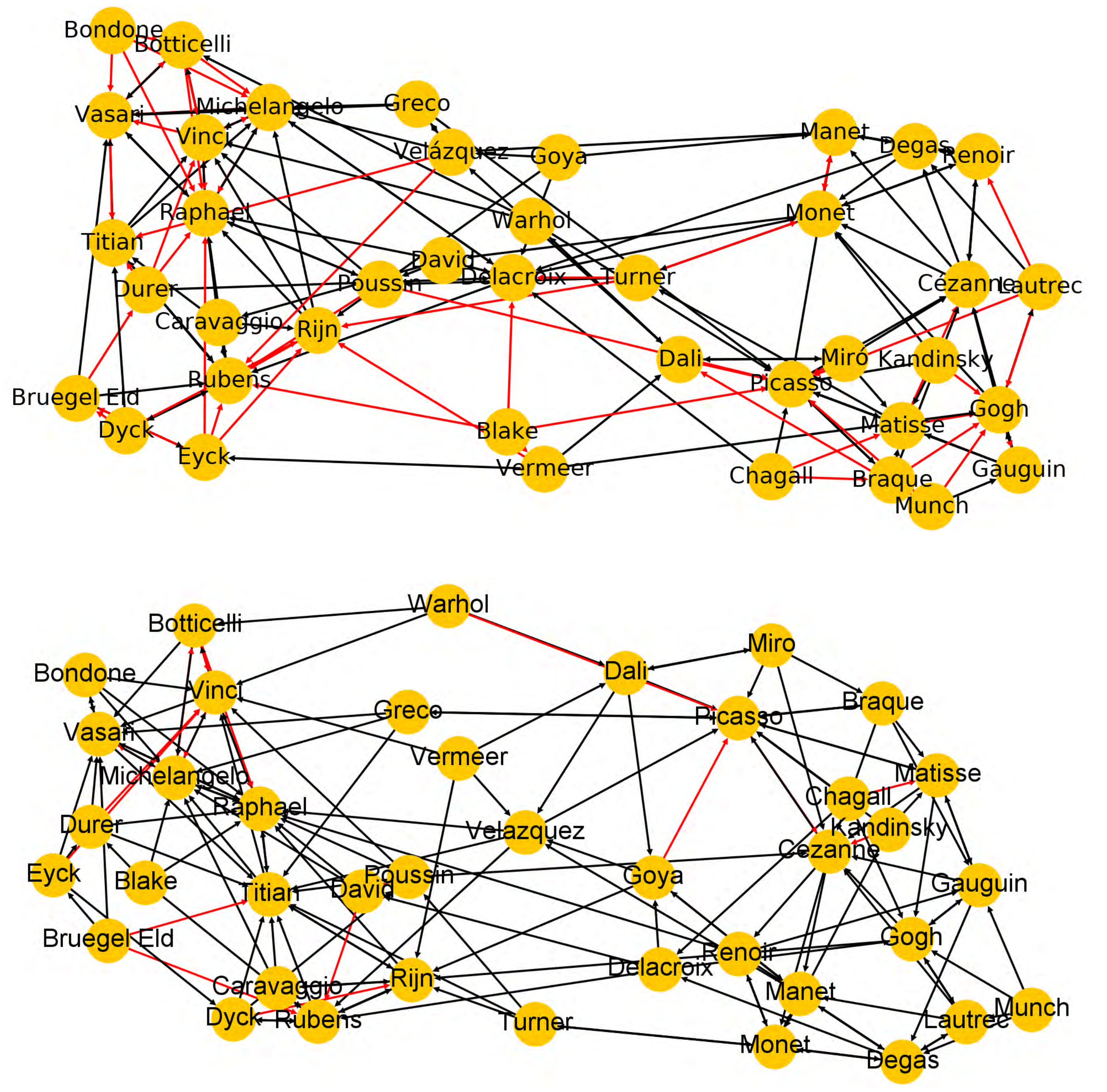

FIGURE 3. Friendship networks for the top 40 painters data set listed in Tab. 2. Top panel: Friendship network extracted from $G_{rr} + G_{\mathrm{qrnd}}$ computed with FrWiki. Bottom panel: Friendship network extracted from the matrix $G_{rr\,av} + G_{\mathrm{qrnd}_{av}}$, where $G_{\mathrm{R}av}$ is the average of the G_{R} matrices obtained for all 7 Wikipedia editions. For each painter in the set, arrows are drawn to its top 4 friends. Arrows are colored in red if component G_{qrnd} is larger than component G_{rr}, and in black otherwise. Both graphs are automatically plotted using Yifan Hu layout with *Gephi* [42].

the analysis on these two matrix components of G_{R}. The properties of G_{R} components had been discussed in detail in [27] and [28] and here we discuss mainly the most interesting contributions induced by G_{rr}, G_{qr}.

The matrix G_{rr} provides information only on direct links between painters. In other words, it represents the probability for a random surfer to reach the painter of line i from the article of the painter of column j using a hyperlink linking article j to article i in Wikipedia. On the contrary, G_{qrnd} offers a much more unified view of painters interactions as it captures more general indirect (or hidden) interactions via the $N - N_r$ other nodes of the full Wikipedia network. In other words, it represents the probability linking the painter of column j to the painter of line i related to all indirect paths linking article j to article i in the full network. An indirect path starts with a hyperlink linking the article of painter j to an article k that doesn't belong to the N_r nodes and ends with a hyperlink ending on the article of node i.

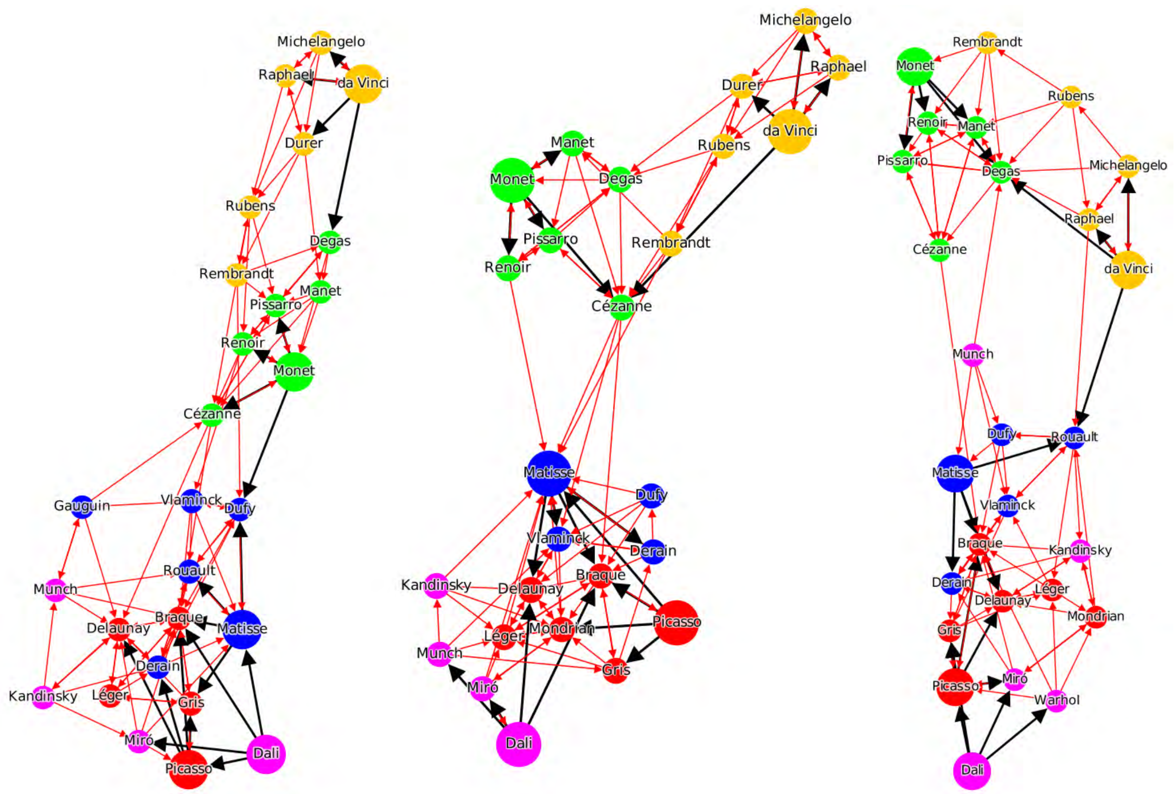

FIGURE 4. Friendship networks for the painting categories data set listed of Tab. 3. Results are extracted from the G_{qrnd} matrix derived from EnWiki (left), FrWiki (middle) and DeWiki (right). Red, Blue, Green, Orange and Pink nodes represents Cubism, Fauvism, Impressionists, Great masters and Modern(20-21) respectively. The top painter node points with a bold black arrow to its top 4 friends. Red arrows represent the friends of friends interactions computed until no new edges are added to the graph. All graphs are automatically plotted using Yifan Hu layout with *Gephi* [42].

Reading Fig. 2, we can extract strong and meaningful interactions between painters. New links appearing in G_{qrnd} and being absent from G_{rr} exist. As an example we list the links between Picasso and Braque, Pissaro and Monet, Rouault and Matisse. These relationships are very well known in art history, but looking at the pure structure of the network (i. e. reading G_{rr} matrix), they are absent. They appear clearly in the higher order mathematical analysis of the network using G_{qrnd}. For instance, it is common knowledge that since his visit to Picasso's studio, Braque became impressed by Picasso's paintings. They even became friends [32], which confirms our result. Pissaro and Monet are both impressionists. Monet succeeded in reaching England after entrusting a number of his works to Pissaro [33]. Rouault and Matisse were both students of Gustave Moreau [34] and were deeply influenced by him throughout their life [35]. Their relationship began in 1906 and lasted all their life. All these interactions can be extracted from the network of Wikipedia webpages using G_{qrnd} matrix.

In order to simplify the reading and interpretation of these matrices, we have introduced in [26]–[28] a set of tools that captures essential features of the reduced network. In Section V, we build the friendship networks for our sets of painters and in Section VI we analyze the influence of painters on countries using the PageRank sensitivity analysis of G_{R}.

V. FRIENDSHIP NETWORKS

A. FRIENDSHIP NETWORK CONSTRUCTION

It is possible to extract from G_{rr} and G_{qrnd} a network of friendship to conveniently illustrate direct and hidden links in the network, or a combination of both. Direct links are extracted from G_{rr} while hidden (i. e. indirect) are extracted from G_{qrnd}. The network of friends is built by considering larger matrix elements in a column j of a given painter as top friends (i. e. there is a high probability to end in node i from node j). It is true that the word *friend* usually represents a symmetrical relationship. But we have chosen this denomination for its ease of use. Clearly, in this paper, a friend represents a node that is an attractor for the node of interest.

From the notion of friendship, we derive the networks of friends shown in Fig. 3 and Fig. 4. In Fig. 3, the set of top 40 painters is analyzed while in Fig. 4 the painting categories set is investigated. Both networks of friends have been derived in the following way:

- For Fig. 3, arrows representing 4 outlinks are drawn from each painter to its top 4 friends in the matrix $G_{rr} + G_{\text{qrnd}}$. In this figure, we mark arrows in red if the G_{qrnd}

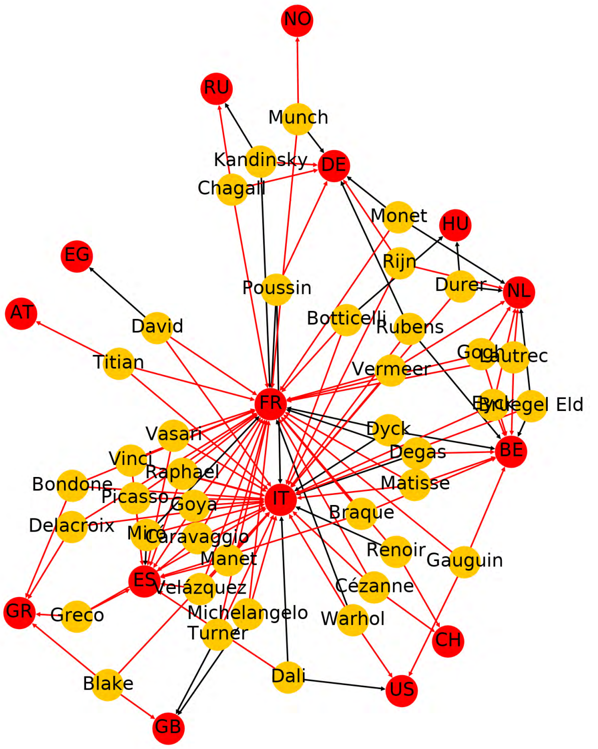

FIGURE 5. **Network structure of top 3 country friends for top 40 painter network for EnWiki Painters are selected from the global rank list of 7 Wikipedia editions from Table 2 for top 40 PageRank countries of EnWiki from Table 4. Arrows are showing links only from a painter to top 3 countries, they are given by links of matrix elements $G_{rr} + G_{\rm qrnd}$, red arrow mark links when an element $G_{\rm qrnd}$ is larger than element G_{rr}, black arrows are drown in opposite case. Countries and shown by red circles and painters are shown by yellow circles.**

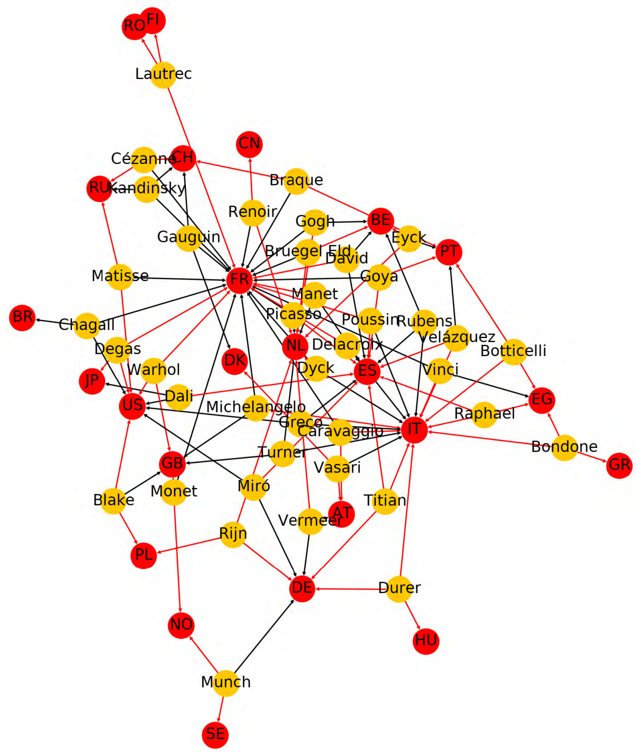

FIGURE 6. **Network structure of top 3 country friends for top 40 painter network for FrWiki Painters are selected from the global rank list of 7 Wikipedia editions from Table 2 for top 40 PageRank countries of EnWiki from Table 4.**

component (i. e. indirect link probability) is larger than the G_{rr} component (i. e. direct link probability). Black arrows thus represent the opposite case. The graphs are automatically plotted with *Gephi* [42] using the Yifan Hu algorithm. In other words, for each painter j of Tab. 2, we extract from sum of both matrices the top 4 *Friends* given by the 4 strongest elements of column j. This figure represents two different views: *i*) a regional view (top panel) as $G_{rr} + G_{\rm qrnd}$ are computed for FrWiki and *ii*) a unified view (bottom panel) as the friendship network is built from $G_{rr\,av} + G_{{\rm qrnd}_{av}}$ which is defined as the average of the corresponding matrices computed over all 7 Wikipedia editions.

- In Fig. 4, we capture for the Painting categories data set the sole indirect interactions provided by $G_{\rm qrnd}$ between the 5 categories of painters. Therefore, we have selected the most influential painter in each category. This category leader is the one with the best (i. e. smallest) average ranking score over all 6 selected Wikipedia editions. The top painters are Pablo Picasso for Cubism, Henri Matisse for Fauvism [36], Claude Monet for Impressionists [37], [38], Leonardo Da Vinci for Great Masters and Dali for Modern. The networks of Fig. 4 are created by marking with black arrows the link between each leading painter and its top 4 friends in $G_{\rm qrnd}$. Red arrows represent the friends of friends interactions computed until no new edges are added to the graph. Three networks are plotted, originating from EnWiki, FrWiKi and EnWiki.

We discuss next the interesting results observed for both networks of painters.

B. TOP 40 PAINTERS NETWORKS

This part discusses the friendship networks of Fig. 3. The point of our analysis is to underline the relative importance of direct and indirect interactions. For black arrows, the direct component G_{rr} is stronger than the indirect one $G_{\rm qrnd}$. For red arrows, the indirect component is the strongest.

The top panel presents the French Wikipedia view of this painter network. A large proportion of arrows are red, meaning that in this case the indirect interaction between nodes is contributing a lot to building the network. The graph has a clear structure of three main clusters: painters who worked in France (right part around Matisse-Picasso-Monet-Van Gogh), the ones that have worked in Italy (left top corner with Da Vinci-Titian-Botticelli) and the ones that have worked in the Netherlands and Belgium (bottom left corner with Rubens-Rembrandt Van Rijn-van Dyck). This shows that our network analysis captures realistic relations between painters.

However, the above presentation takes into account only the opinion of the FrWiki edition with a dominance of French

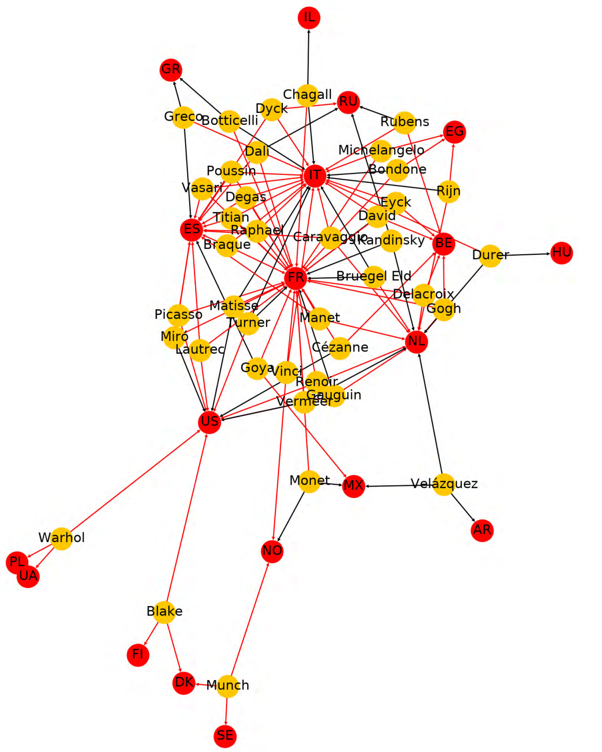

FIGURE 7. Network structure of top 3 country friends for top 40 painter network for DeWiki Painters are selected from the global rank list of 7 Wikipedia editions from Table 2 for top 40 PageRank countries of EnWiki from Table 4.

culture linked to French language. It is interesting to have the network structure which takes into account the opinions of all 7 editions. In fact the approach of the reduced Google matrix is well suited for this. Indeed, to perform the average over different cultures we take G_{R} for 40 painters (size 40) and its components and take the average of these 7 matrices with equal democratic weights getting in this way the average $G_{\mathrm{R}av}$ and its average 3 components. Of course, after averaging $G_{\mathrm{R}av}$ still belongs to the class of Google matrices. The network structure obtained from $G_{rr\,av} + G_{\mathrm{qrnd}_{av}}$ is shown in the bottom panel of Fig. 3. In global the two centers with painters worked in France (on the right) and in Italy (left) is similar to the case of FrWiki in the top panel. However, the number of indirect links (red arrows) is decreased. We attribute this to increased number of direct links present in all 7 editions.

We also note that $G_{\mathrm{R}av}$ has now a new average PageRank vector $P_{av}(K_{av})$, which takes into account opinions of all 7 cultures (it is different from the simple averaged probabilities of 7 individual PageRank vectors). This average rank index K_{av} is shown in Tab. 2. The top two positions are the same as for Θ-rank, however, there is a noticeable change of order in positions 3-12 with more importance given to ancient Italian masters like Michelangelo, Raphael, Titian who moves to the top K_{av} positions while more recent painters such as Van Gogh, Monet, Dali are getting larger K_{av} values. We

TABLE 4. List of PageRank of top 40 countries in EnWiki.

Order	Country	Order	Country
1	US	21	NO
2	FR	22	RO
3	GB/UK	23	TK
4	DE	24	ZA
5	CA	25	BE
6	IN	26	AT
7	AU	27	GR
8	IT	28	AR
9	JP	29	PH
10	CN	30	PT
11	RU	31	PK
12	ES	32	DK
13	PL	33	IL
14	NL	34	FI
15	IR	35	EG
16	BR	36	ID
17	SE	37	HU
18	NZ	38	TW
19	MX	39	KR
20	CH	40	UA

attribute this to the fact that the ancient historical figures are on average better reviewed in various cultures and Wikipedia editions.

C. PAINTINGS CATEGORIES NETWORK

Fig. 4 illustrates the indirect interactions provided by G_{qrnd} in the painting categories network. The 5 leading painters are connected with black arrows to the top 4 friends. And these friends are connected to their top 4 friends with red arrows.

Thus G_{qrnd} seems to emphasize finer-grained regional interactions and by looking at the interactions, we can see the strong relationship between Da Vinci, Michelangelo and Raphael which can be explained by the fact that they are the nucleus of fifteenth-century Florentine art [39]. Another strong relation could be snapped between Mirò and Dali, as both are inspired by Picasso [40], [41].

Impressionists, Fauvism, Cubism and Great masters create, in all editions, a cluster of nodes densely interconnected. The group of Modern painters plays a role by connecting the other categories:

1) Dali seems to be the common interconnection node between Fauvism and Cubism categories in EnWiki.
2) Kandinsky connects Fauvism and Cubism in FrWiki.
3) Munch connects Impressionists and Fauvism in DeWiki. The networks of G_{qrnd} end up almost spanning the full set of 30 painters.

These links show that the interactions between the painters groups are coherent. These graphs picture the essence of painting history by grouping together painters that belong to the same movement and by interconnecting them in a reasonable and close-to reality way.

For instance, our graphs are consistent with the history of modern art which starts with the Impressionists movement

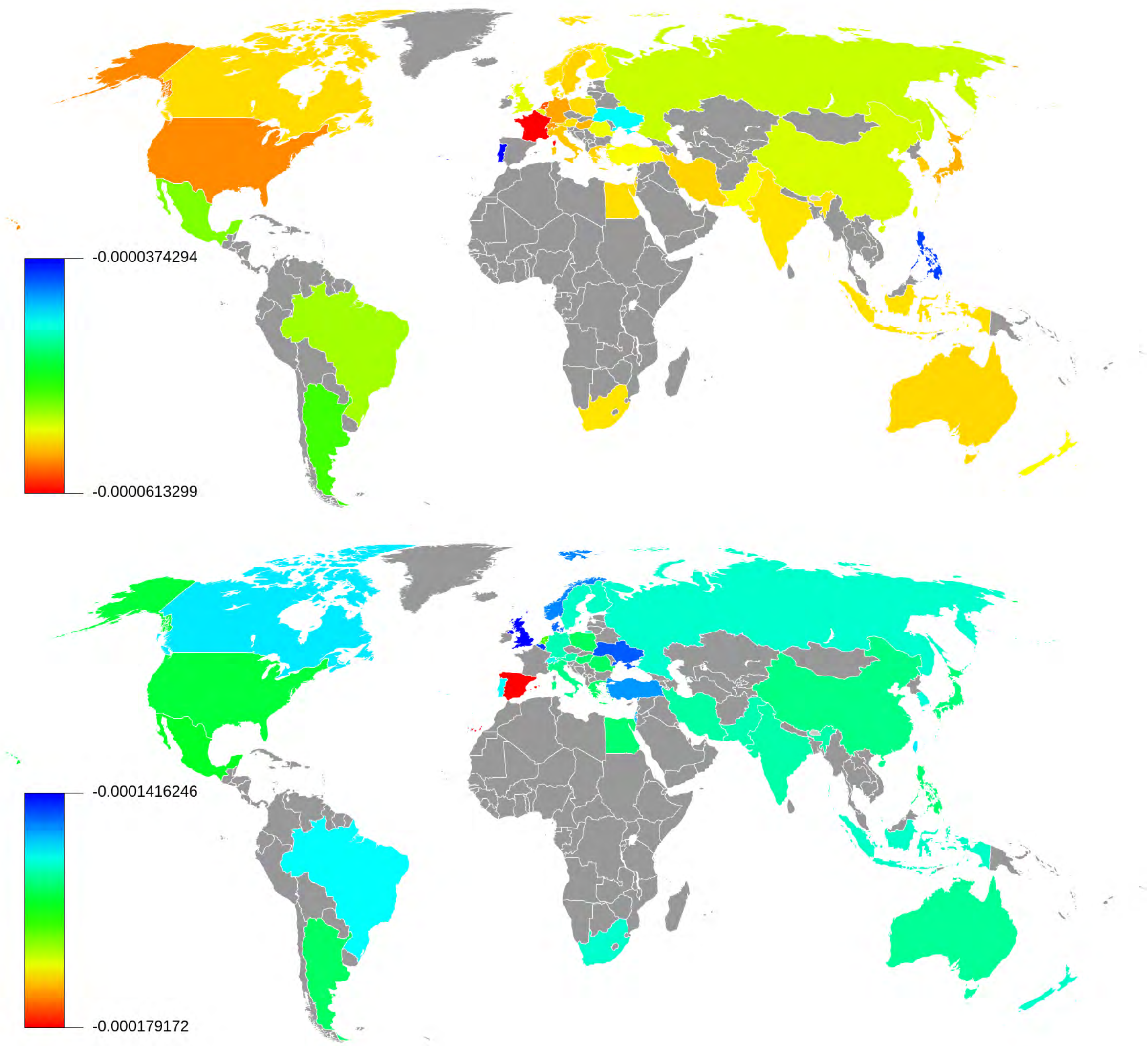

FIGURE 8. **Sensitivity *D* of 40 world countries to the link variation going from Picasso to Spain and Picasso to France. Top panel: Picasso to Spain and bottom panel: Picasso to France. Data is averaged over 7 Wikipedia editions. For a better visibility, sensitivity of Spain (top) and France (bottom) are given in Figure 10. .**

(1870-1890) that searched for the exact analysis of the effects of color and light in nature. The painters we have selected are among the most important ones of the movement and they create a clear cluster of nodes in Fig. 4 (see green nodes) as they exhibit a tight relationship in G_R. The Fauvism movement emerged after impressionist (1899-1908) [43]–[45]. Fauvist painters were concerned with the impression created with colors. This movement was inspired by different artists such as Matisse. The *Fauves* members were a loosely shaped group of artists with shared interests. Henri Matisse became later the leader of the group of artists [36]. He introduced unnatural and intense color into their paintings to describe light and space. The fauvism movement is the precursor of the Cubism movement [46]. Our result shows deep relationships between Fauvism and Cubism, noting that Braque is always the core of this interconnection. Cubism movement (1907- 1922) is pretty distinct from Impressionism, which is underlined as well in our graphs with only a few red links connecting these two clusters of nodes.

VI. INFLUENCE OF PAINTERS ON COUNTRIES

A. DATASETS

Another complementary study is presented here to visualize the influence of painters on countries. To analyze the relation between painters and countries of the world we construct a reduced Google matrix with $N_r = 80$ nodes composed of the top 40 painters shown in Table 2 and the group of 40 countries listed in Table 4. The painters are the ones having top Θ_P-score for $E = 7$: EnWiki, FrWiki, RuWiki, DeWiki, ItWiki, EsWiki and NlWiki. Table 2 only lists short names, however, the full painter names together with their Θ_P-score, birth country and life period are available as well in [31].

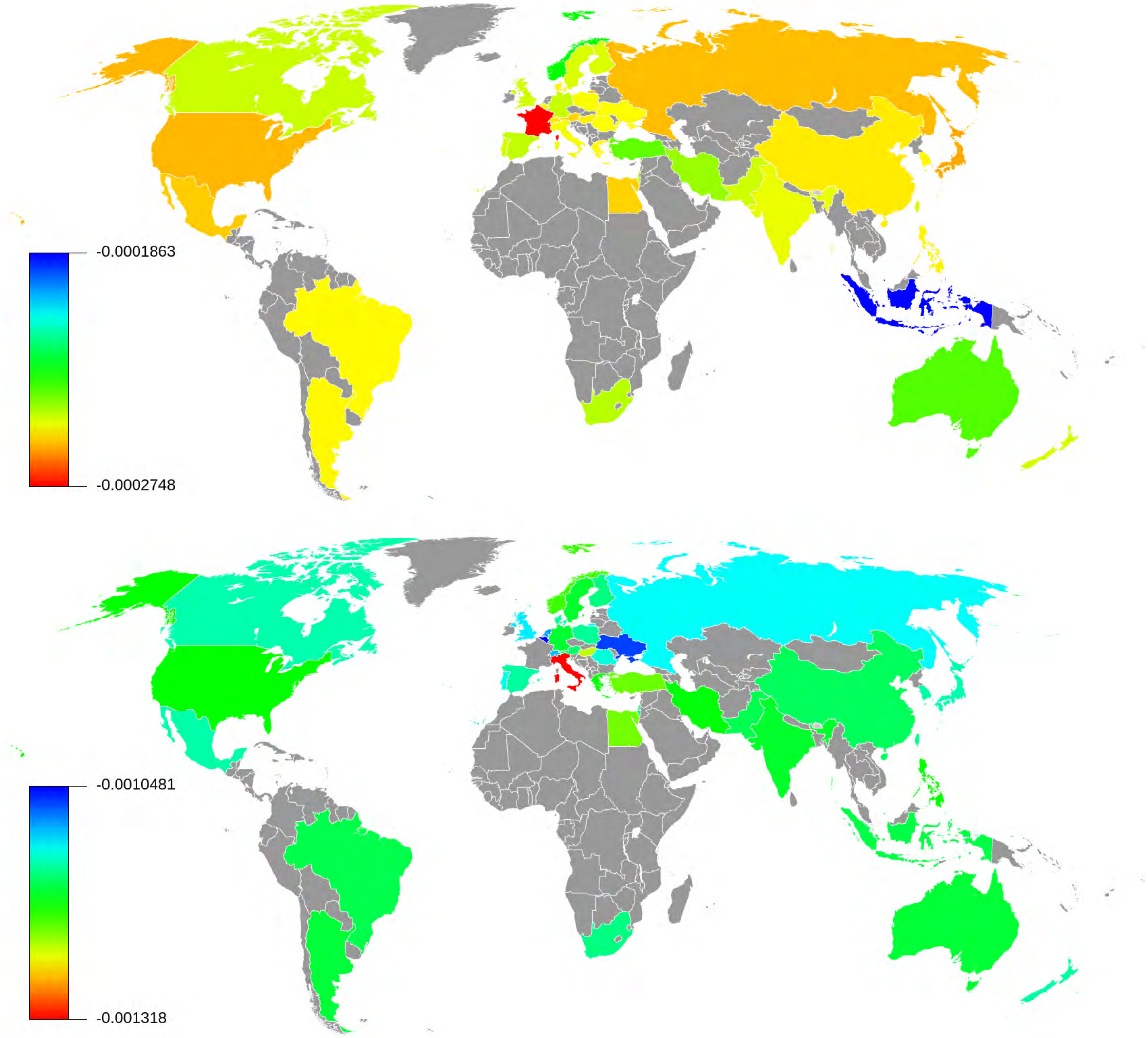

FIGURE 9. Sensitivity *D* of 40 world countries to the link variation going from Van Gogh to the Netherlands and Da Vinci to France. Top panel: Van Gogh-the Netherlands and bottom panel: Da Vinci-France. Data is averaged over 7 Wikipedia editions. For a better visibility, sensitivity of the Netherlands (top) and France (bottom) are given in Figure 10.

The top 40 countries of EnWiki are presented in Table 4. The names of countries are given by ISO 3166-1 alpha-2 code (see [47]).

B. NETWORKS OF PAINTERS AND COUNTRIES

The three painters with the largest Θ_P-score are:

1) Leonardo Da Vinci with $\Theta = 698$, born in Italy,
2) Pablo Picasso with $\Theta = 688$, born in Spain,
3) Vincent Van Gogh with $\Theta = 656$, born in the Netherlands.

The following painters are the most important one for their country of birth:

- Peter Paul Rubens for Germany with $\Theta = 651$ (but worked mainly in Netherlands),
- Claude Monet for France with $\Theta = 605$,
- Wassily Kandinsky for Russia with $\Theta = 515$,
- Joseph Mallord William Turner for United Kingdom (UK or GR) with $\Theta = 386$.

The top 6 countries with the largest number of painters from the global list of 223 painters are Italy (50), France (45), Russia (27), Germany (26), USA (14), Spain (11) (note that the 223 other countries are listed as well in [31]).

The geopolitical relations between painters and countries has been analyzed more precisely for EnWiki, FrWiki and DeWiki data. Therefore, we have plotted a network of friendship between our 40 painters and their top 3 most friendly countries. This network has been calculated using G_{rr} and G_{qrnd} calculated for the union of 40 painters and 40 countries. For each painter column, we select the top 3 countries in the sum matrix $G_{rr}+G_{\text{qrnd}}$ to account for direct and indirect interactions and mitigate the effect of the projector component. Resulting networks are shown in Figure 5, 6 and 7 for EnWiki,

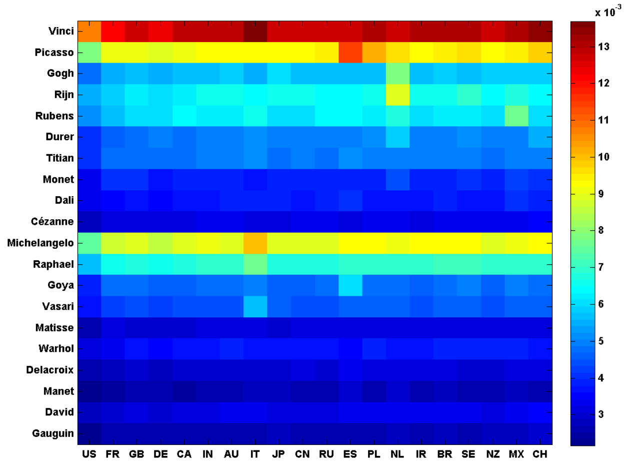

FIGURE 10. Diagonal sensitivity of the top 20 countries to bidirectional link variations between painter/country pairs (i. e. painter to country and country to painter) Color bar shows the sensitivity values. Data is averaged over 7 Wikipedia editions and are shown for top 20 entries of Table 2 and Table 4.

FrWiki and DeWiki, respectively. In these figures, arrows are colored in red if $G_{\rm qrnd}(i,j) > G_{rr}(i,j)$ and in black other wise. The network structure is different for each edition due to different cultural views and preferences. However, the central role of France and Italy is well visible in all 3 editions.

C. INFLUENCE OF PAINTERS ON COUNTRIES

To analyze in a more direct way the world influence of painters we average $G_{\rm R}$ matrix and its three components $G_{\rm pr}$, G_{rr}, $G_{\rm qr}$ over 7 Wikipedia editions that allows us to account for different cultural views on selected 40 painters of Table 2 and 40 countries of Table 4. The reduced Google matrix $G_{{\rm R}av}$ averaged over these 7 different editions allows us to obtain a balanced view of various cultural opinions of Wikipedia language editions for a selected group of nodes representing Wikipedia articles. We determine the PageRank probability of this averaged $G_{\rm R}$ matrix and compute its logarithmic derivative (sensitivity) with respect to a weight variation of a selected link going from a specific painter to a specific country. For instance, we vary the intensity in $G_{\rm R}$ of the link going from Picasso to Spain, and observe the variation of PageRank for other countries. This PageRank probability variation is defined as the sensitivity $D(i)$ of a node i to a link change. We refer the reader to [29] for a precise definition of D (D essentially is given by a logarithmic derivative of PareRank probability in respect to a relative link weight variation).

1) INFLUENCE OF PICASSO LINK TO SPAIN AND FRANCE

Figure 8 shows the sensitivity D of 40 world countries with respect to a link variation from Picasso to Spain (top panel) and from Picasso to France (bottom panel). Pablo Picasso, the son of the Spanish painter Don José Ruiz y Blanco, was born in Spain in 1881. Pablo began painting since he was eight, and in 1896, he has joined the art and design school of Barcelona "Escola de la Llotja". In 1904, Picasso married Fernande Olivier a French artist and model. Since that, Picasso spent most of his life in France and died there at 92 years old. This could explain the results we have obtained from our sensitivity analysis, which underlines that France and Spain are the countries that are mostly affected for a Picasso-Spain and a Picasso-France link variation, respectively.

2) INFLUENCE OF VAN GOGH LINK TO NETHERLANDS AND DA VINCI LINK TO FRANCE

Figure 9 shows the sensitivity D of 40 world countries with respect to a link variation from Van Gogh to Netherlands and from da Vinci to France in top and bottom panels, respectively. Even though Van Gogh has only spent the last four years of his life in different places of France, these years were important to Van Gogh's painting career. Van Gogh has built there strong relationships with leading French painters. He has worked at that time with Emile Bernard, Henri de Toulouse-Lautrec, Georges Seurat, Paul Signac and Gauguin. These relationships and the work achieved by Van Gogh in France explain our results in the top panel of Figure 9, which shows that France is strongly influenced by a link variation from Van Gogh to the Netherlands.

The Italian painter Leonardo Da Vinci learned painting in the workshop of Verrochio in Florence, and crafted there its first painting between 1472 and 1474. Da Vinci was based in Italy until 1516, when Francois I (King of France) invited him to join the Royal court as: "The King's First Painter, Engineer and Architect". Da Vinci died in France four years after his arrival. Da Vinci's works was highly noted by French statesmen. Louis XII and (later) Napoleon though to bring "The Last Supper" to France. "Madonna of the Yarnwinder" is a painting done by Da Vinci to respond the demand of the secretary of state of Louis XII of France. Leonardo brought a version of the "Virgin of the Rocks" to France. One of the most important painting of Da Vinci is "Mona Lisa", currently displayed at Louvre Museum in Paris, was finalized in the Royal court of Francois I. All these elements about the relations between Da Vinci, France and Italy, explain the fact that Italy is strongly influenced by a link variation from Da Vinci to France, as shown in the bottom panel of Figure 9.

3) DIAGONAL SENSITIVITY OF COUNTRIES

Finally in Figure 10 we present the diagonal sensitivity of countries to their links with painters. This measure is computed by calculating the 2-way sensitivity of Eq. (2) for each painter/country couple. It is the sum of the logarithmic PageRank sensitivity for the painter to country link and the one for the country to painter link. In Eq. (2), c is the index of a country and p of a painter.

$$D_{(p\leftrightarrow c)}(c) = D_{(p\rightarrow c)}(c) + D_{(c\rightarrow p)}(c) \qquad (2)$$

Also, in Figure 11, we represent the same diagonal sensitivity of countries to Da Vinci and Picasso influence using a world map. In other words, we color the countries with the intensities found on the Da Vinci line (top panel) and the Picasso

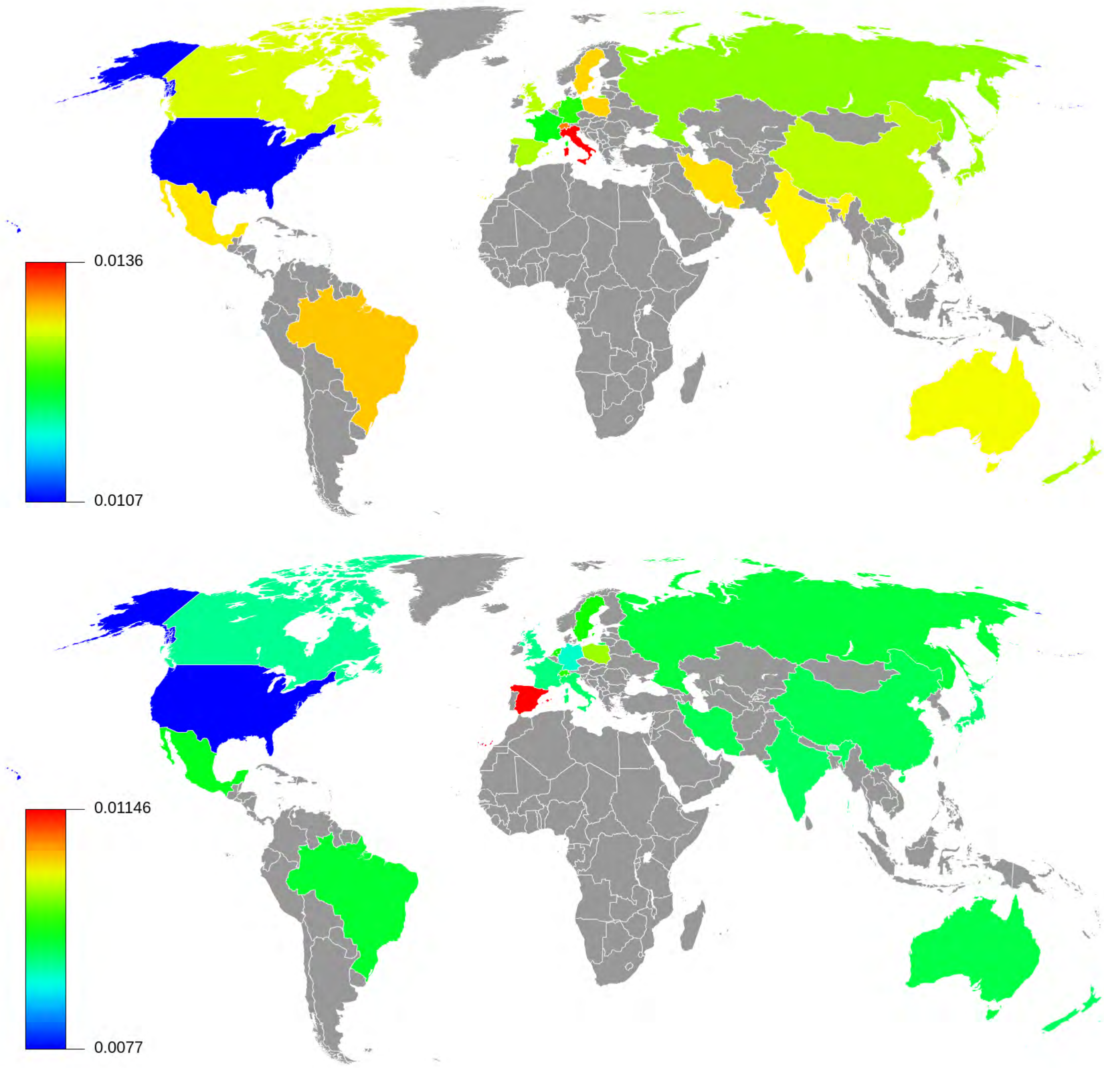

FIGURE 11. Map representation of the diagonal sensitivity of countries to influence of Da Vinci (top panel) and Picasso (bottom panel) .

line (bottom panel) of the matrix of Figure 10. We have previously discussed the relationship between Picasso and Spain and the relation between Da Vinci and Italy which are most sensitive countries in Figure 11 respectively. In this figure, we picture the secondly affected countries for each painter using the 2-way sensitivity metric for a the same bidirectional link variations. It is seen in Figure 11 (bottom) that Poland is greatly impacted by Picasso. The reason is that the Mermaid of Warsaw is a symbol of Warsaw represented on city's coat as well as in a many imagery and statues. Picasso's drawing of Warsaw Mermaid explains the weight of 2-way sensitivity between Picasso and Poland and clarify why Poland is highly linked to Picasso [49]. On the other hand, from Figure 11 (top) we find that the second most influenced country by Da Vinci is Switzerland, possibly due to the fact that a central masterpiece of Da Vinci is to be found in Switzerland: Isabella d'Este [50].

To finally conclude this analysis, we can underline that according to Figure 10 Da Vinci, Picasso and Michelangelo are the most influential painters for selected world countries.

VII. DISCUSSION

This paper shows that our sensitivity analysis captures the importance of relationships on network structure. This analysis relies on the reduced Google matrix and leverages its capability of concentrating all Wikipedia knowledge in a small stochastic matrix. We stress that the friendship networks and the sensitivity analysis of influence of painters on countries helped us extract valuable and realistic knowledge from a pure mathematical analysis without any direct appeal to arts, political, economical and social sciences. In a certain sense the reduced Google matrix approach provides an artificial intelligence analysis (the authors have no specific education in arts) of interactions and influence of top painters on world countries using Wikipedia networks.

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