

Incommensurate standard map

Leonardo Ermann^{1,2} and Dima L. Shepelyansky³

¹*Departamento de Física Teórica, GIyA, Comisión Nacional de Energía Atómica. Av. del Libertador 8250, 1429 Buenos Aires, Argentina*

²*Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Buenos Aires, Argentina*

³*Laboratoire de Physique Théorique, IRSAMC, Université de Toulouse, CNRS, UPS, 31062 Toulouse, France*
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We introduce and study the extension of the Chirikov standard map when the kick potential has two and three incommensurate spatial harmonics. This system is called the incommensurate standard map. At small kick amplitudes the dynamics is bounded by the isolating Kolmogorov-Arnold-Moser surfaces while above a certain kick strength it becomes unbounded and diffusive. The quantum evolution at small quantum kick amplitudes is somewhat similar to the case of Aubry-André model studied in mathematics and experiments with cold atoms in a static incommensurate potential. We show that for the quantum map there is also a metal-insulator transition in space while in momentum we have localization similar to the case of 2D Anderson localization. In the case of three incommensurate frequencies of space potential the quantum evolution is characterized by the Anderson transition similar to 3D case of disordered potential. We discuss possible physical systems with such map description including dynamics of comets and dark matter in planetary systems.

I. INTRODUCTION

The investigation of dynamical symplectic maps allows to understand the fundamental deep properties of Hamiltonian dynamics. The map description originates from the construction of Poincaré sections of continuous dynamics invented by Poincaré [1]. The mathematical foundations for symplectic maps are described in [2, 3]. Their physical properties and applications including numerical studies are given in [4, 5]. The renormalization features of critical Kolmogorov-Arnold-Moser (KAM) invariant curves are analyzed in [6] with the transport properties through destroyed KAM curves investigated in [4, 7].

The seminal example of a symplectic map is the Chirikov standard map [4]

$$\bar{p} = p + K \sin x, \quad \bar{x} = x + \bar{p}, \quad (1)$$

where p and x are canonically conjugated variables of momentum p and coordinate x and bars mark the values of variables after a map iteration. For $K > K_c = 0.9716\dots$ the last KAM curve is destroyed and the dynamics is characterized by a global chaos and diffusion in momentum $\langle p^2 \rangle = Dt$ with a diffusion rate D , where the time t is measured in number of map iterations and the diffusion coefficient is $D \approx K^2/2$ at large K values [4, 6, 7].

The important feature of map (1) is its universality related to an equidistant spacing between resonance frequencies so that a variety of symplectic maps and periodically driven Hamiltonian systems can be locally described by the map (1) in a certain domain of the phase space. The Chirikov standard map finds applications for description of various physical systems including plasma confinement in open mirror traps, microwave ionization of hydrogen atoms, comet and dark matter dynamics in the Solar System and behavior of cold atoms in kicked optical lattices (see e.g. [4, 8–11] and Refs. therein).

The quantum version of map (1) is obtained by considering p and x as the Heisenberg operators with the commutation relation $[\hat{p}, \hat{x}] = -i\hbar$. The corresponding quantum evolution of the wave function $\psi(x)$ is described by the map [12, 13]:

$$\bar{\psi} = \exp(-ik \cos x) \exp(-i\hat{p}^2/2\hbar)\psi, \quad (2)$$

where $\hat{p} = \hbar\hat{n} = -i\hbar\partial/\partial x$, $k = K/\hbar$, $T = \hbar$, $K = kT$ (T can be also considered as a rescaled time period between kicks). Here, the wave function is defined in the domain $0 \leq x \leq 2\pi$ corresponding to the quantum rotator case, or it can be considered on the whole interval $-\infty < x < \infty$ corresponding to motion of cold atoms in an optical lattice. The later case has been realized in experiments with cold atoms in kicked optical lattice [14]. At $K > K_c$ the classical diffusion in p becomes localized by the quantum interference effects with an exponential decay of probability over the momentum states $n = p/\hbar$:

$$\langle |\psi_n|^2 \rangle \propto \exp(-2|n - n_0|/\ell); \quad \ell \approx D/\hbar^2 \quad (3)$$

with the localization length ℓ and n_0 being an initial state [12, 15]. This dynamical quantum localization is analogous to the Anderson localization in disordered solids as pointed in [16]. However, the role of spacial coordinate is played by momentum state level index n and diffusion appears due to dynamical chaos in the classical limit and not due to disorder (see more detail in [13, 15, 17]).

The important feature of the Chirikov standard map is its periodicity in spacial coordinate (of phase) x . The cases with several kick harmonics have been considered for various map extensions (see e.g. [4, 5, 12, 18]) but all of them had periodicity in x . The important example of the map with several harmonics is the generalized Kepler map which provides an approximate description of the Halley map dynamics in the Solar System [19] (see also [11] for Refs. and extensions). In this system Jupiter gives an effective kick in energy of the Halley comet when

it passes through its perihelion. This kick function contains several sine-harmonics of Jupiter rotational phase since the comet perihelion distance is inside the Jupiter orbit. However, other planets, especially Saturn, also give a kick change of comet energy. Since the frequencies of other planets are generally not commensurate with the Jupiter rotation frequency we have a situation where the kick function depends at least on two phases with incommensurate frequencies. Thus it is important to analyze the incommensurate extensions of the Chirikov standard map.

The simplest model is the incommensurate standard (i-standard) map which we investigate in this work:

$$\begin{aligned}\bar{p} &= p + K_1 \sin x + K_2 \sin \nu x \\ \bar{x} &= x + \bar{p}\end{aligned}\quad (4)$$

where ν is a generic irrational number, and K_1, K_2 are amplitudes of two incommensurate kick harmonics. Here the coordinate domain is $-\infty < x < \infty$ corresponding to dynamics of atoms in an incommensurate optical lattice. For $K_1 = 0$ or $K_2 = 0$ the model is reduced to the map (1). We consider here the case of the golden mean value $\nu = (\sqrt{5} - 1)/2 = 0.618\dots$

The incommensurate standard map (4) describes the dynamics of cold atoms in a kicked optical lattice with an incommensurate potential. The static incommensurate optical lattice can be created by laser beams with two incommensurate wave lengths. In fact such incommensurate optical lattice had been already realized in cold atoms experiments [20] where the evolution of atomic wavefunction can be approximately described [21] by the Aubry-André model on a discrete incommensurate integer lattice with the Aubry-André transition from localized to delocalized states [22]. At present the investigation of interactions between atoms on such a lattice attracts a significant interest of cold atoms community (see e.g. [23]).

Due to the above reasons we think that the incommensurate standard map will capture new features of dynamical chaos with possible application to various physical systems. We also note that the quantum evolution of this map may have localization or delocalization properties with a certain similarity with the Anderson transition in disordered solids.

The paper is constructed as follows: Section II describes the properties of classical dynamics; quantum map evolution is analyzed in Section III, effective two-dimensional (2D) and three-dimensional features of quantum evolution are considered in Section IV, the discussion of results is given in Section V.

II. CLASSICAL MAP DYNAMICS

To study the properties of classical dynamics of map (4) we launch a bunch of trajectories in a vicinity of unstable fixed point $x = 0, p = 0$ and compute an effective diffusion coefficient $D = \langle p^2 \rangle / t$ averaged over

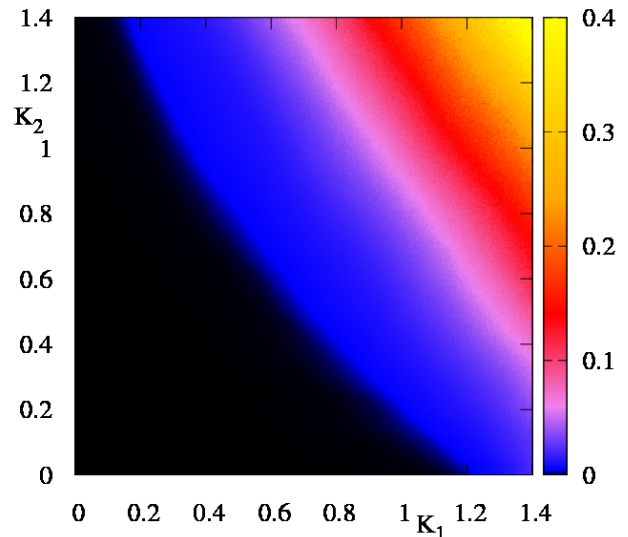


FIG. 1. Diffusion rate in momentum $D = \langle p^2 \rangle / t$ vs K_1 and K_2 . Data are averaged over 1000 trajectories with random initial conditions in the interval $x, p \in [0, 10^{-6}]$ for the number of iterations $t = 10^4$. Color bar shows the values of D .

all initial trajectories. The dependence of D on K_1 and K_2 is shown in Fig. 1. These data show that there is a critical curve $K_{c2} = f(K_{c1})$ below which the momentum oscillations are bounded and above which p grows diffusely with time. At $K_1 = K_2 = K$ we obtain $K_c = K_{c1} = K_{c2} \approx 0.65$ (see below). Of course, the time t used in Fig. 1 is not very large (due to many trajectories and many K_1, K_2 values) so that we obtain only an approximate position of the critical curve. Thus for $K_2 = 0$ we know that $K_c = 0.9716\dots$ [6] that is a bit below than the blue domain with a finite diffusion rate D . This happens due to not very large $t = 10^4$ value and a small diffusion near K_c being $D \approx 0.3(K - K_c)^3$ for the map (1) [4, 15].

To represent trajectories on the Poincaré section it is convenient to use variables $x_1 = x \pmod{2\pi}$, $x_2 = \alpha x \pmod{2\pi}$ and p . We show two sections for $K = K_1 = K_2$ below the critical value at $K = 0.2 < K_c$ and above the critical value at $K = 0.7 > K_c$ (see Fig. 2). For $K = 0.2$ there are smooth invariant KAM surfaces bounding p variations while for $K = 0.7$ there is a chaotic unbounded dynamics in momentum.

When the last KAM surface is destroyed we have a diffusive growth of momentum as it is shown in Fig. 3 for $K = K_1 = K_2 = 0.7 > K_c$. For $K = K_1 = K_2 = 0.2 < K_c$ the values of p remain bounded for all times.

From the time dependence of $\langle p^2 \rangle$ on time t we compute the diffusion rate D . The dependence of D on $K = K_1 = K_2$ is shown in Fig. 4. We find that this dependence is satisfactorily described by the relation $D = D_0(K - K_c)^\alpha$ with $D_0 \approx 0.95$, $K_c \approx 0.65$ and $\alpha \approx 2.5$. The value of the exponent α is close to the one for the Chirikov standard map with $\alpha \approx 3$ [4, 7, 15].

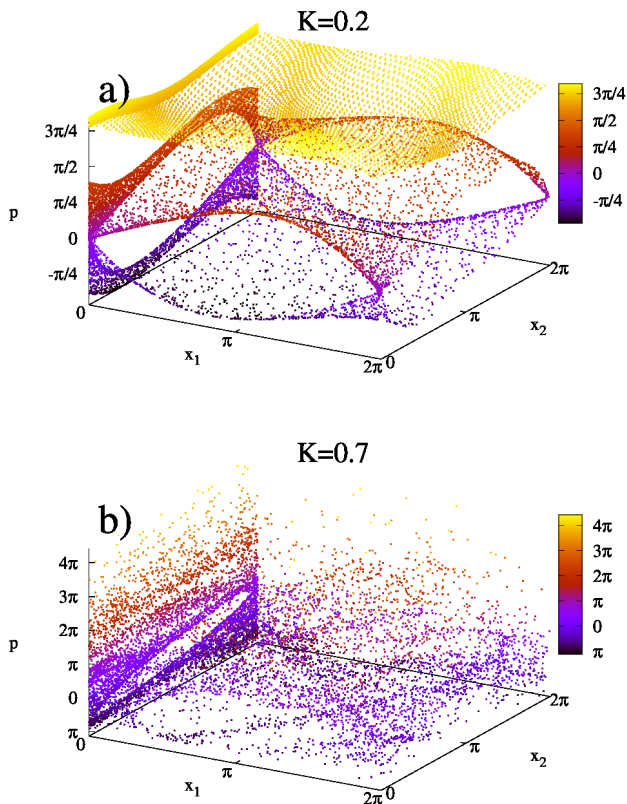


FIG. 2. Poincaré section in the phase space (x_1, x_2, p) for $K = K_1 = K_2 = 0.2$ (a) and $K = K_1 = K_2 = 0.7$ (b). a) The data are obtained from two trajectories with $t = 5000$ map iterations for the initial values $p = x_1 = 0.01, x_2 = 0.01 \times \nu$ with $p = 0.01$ and $p = \pi/\sqrt{2}$; b) the initial phases x_1, x_2 are the same as in (a) and $p = 0.01$. Color bars show p values.

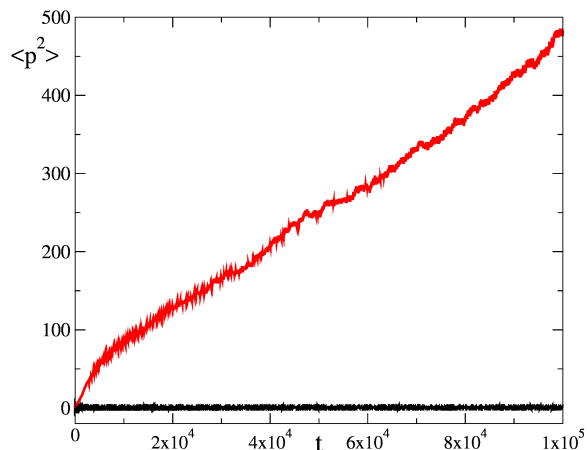


FIG. 3. Dependence of $\langle p^2 \rangle$ on time t for $K_1 = K_2 = 0.2$ in black (bottom) and $K_1 = K_2 = 0.7$ in red (top) solid curves. The average is done over 1000 trajectories with random initial conditions in the range $x, p \in [0, 10^{-6}]$ (similar to initial conditions as in Fig. 1).

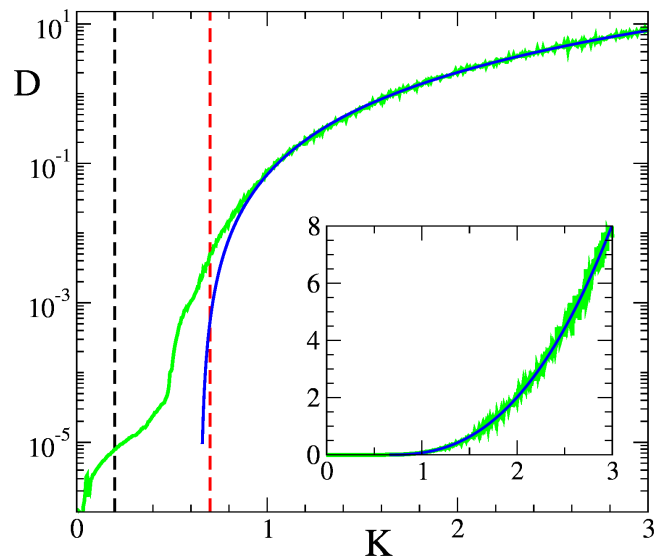


FIG. 4. Dependence of diffusion rate $D = \langle p \rangle / t$ on $K = K_1 = K_2$ shown by green (gray) curve, where vertical black and red (gray) dashed lines mark values $K = K_1 = K_2 = 0.2$ and $K = K_1 = K_2 = 0.7$ respectively. The fit dependence $D = D_0(K - K_c)^\alpha$ is shown by blue (black) curve with the fit values $D_0 = 0.95 \pm 0.04, k_c = 0.65 \pm 0.02$ and $\alpha = 2.5 \pm 0.03$. Inset plot show the same curve for linear scale in D axis. Data are obtained from 1000 trajectories and $t = 10^5$ iterations.

Of course, the number of iterations $t = 10^5$ is not very large and due to that we have only approximate values of the fit parameters D_0, K_c, α . It is possible that the real value K_c is a bit smaller than the its value $K_c = 0.65$ but more exact determination of these values require separate studies.

With the obtained global properties of the classical i -standard map we go to analysis of its quantum evolution in next Sections.

III. QUANTUM MAP EVOLUTION

The quantum evolution of i -standard map is described by the following transformation of wave function on one map period:

$$\bar{\psi} = \exp[-i(k_1 \cos x + (k_2/\nu) \cos \nu x)] \times \exp[-i\hat{p}^2/2\hbar] \psi \quad (5)$$

with $k_1 = K_1/\hbar, k_2 = K_2/\hbar$ and normalization condition $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$. Here, the wave function ψ evolution is considered on infinite domain $-\infty < x < \infty$ corresponding to dynamics of cold atoms in kicked optical lattices; the first multiplier describes kick from optical lattice and the second one gives a free propagation in empty space. The kick potential is $V(x) = k_1 \cos x + (k_2/\nu) \cos \nu x$.

To perform numerical simulations of the quantum map (5) we approximate ν by its Fibonacci series with $\nu_m =$

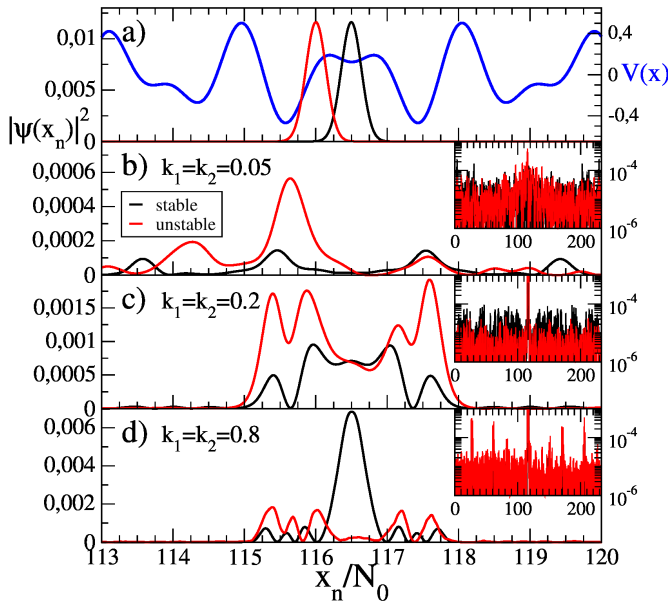


FIG. 5. Probability distribution in coordinate. Panel (a) shows the probability distribution in coordinate for initial state given by a Gaussian centered at stable (black curve) and unstable (red/gray curve) conditions. Blue (dark black) curves show the potential energy $V(x)$ for $k_1 = k_2 = 0.2$ where the scale is shown in right side. Panels (b), (c), (d) (from top to bottom) show respectively the distributions for $k_1 = k_2 = 0.05$, $k_1 = k_2 = 0.2$ and $k_1 = k_2 = 0.8$ after $t = 10^4$ iterations with black and red curves showing the cases of stable and unstable initial conditions. In each panel, top-right insets show the same distribution in logarithmic scale and for the whole Hilbert space. In all cases $T = 76\pi/3^5 \approx 0.983$.

$r_m/q_m = 144/233$ considering time evolution on a ring of size $[0, 2\pi q_m)$ with periodic boundary conditions. Here $r_m = f_{m-1}$ and $q_m = f_m$, are the $(m-1)^{\text{th}}$ and m^{th} Fibonacci numbers given by $f_m = f_{m-1} + f_{m-2}$ with $f_1 = f_2 = 1$, and therefore $\lim_{m \rightarrow \infty} \nu_m = \nu$. Then one iteration of the map is obtained as

$$\bar{\psi} = \hat{F}^{-1} e^{-i \left[k_1 \cos\left(\frac{2\pi x_n}{N_0}\right) + \frac{k_2}{\nu} \cos\left(\frac{2\pi \nu_m x_n}{N_0}\right) \right]} \hat{F} e^{-i \frac{T}{2} \left(\frac{p_n}{q_m}\right)^2} \psi \quad (6)$$

where $T = \hbar$ and the Hilbert space dimension is $N = N_0 \times q_m$, with q_m (r_m) periodic space cells of internal dimension N_0 for k_1 (k_2) harmonic of $V(x)$. Here positions and momentum have integer values $x_n = n$ and $p_n = (n - [N/2])$ with $n = 0, \dots, N-1$; and \hat{F} and \hat{F}^{-1} are the operators of discrete Fourier transform from momentum to coordinate representation and back. For numerical simulations we have chosen $N_0 = 3^5 = 243$, $r_m = 144$ and $q_m = 233$, with $\hbar = \frac{76\pi}{3^5} \simeq 0.983$.

We choose as an initial configuration a Gaussian wave packet centered at stable (or unstable) point of the kick potential $V(x) = k_1 \cos x + (k_2/\nu) \cos \nu x$. On a discrete lattice of $x_n = n$ ($0 \leq n < N$) this distribution is $\psi(x_n) = A \exp[-(x_n - X_0)^2 / (2(N_0/5)^2)]$ with $\langle x \rangle = X_0$ and A a normalization factor. The corresponding distribution, in momentum space p_n , is obtained by the dis-

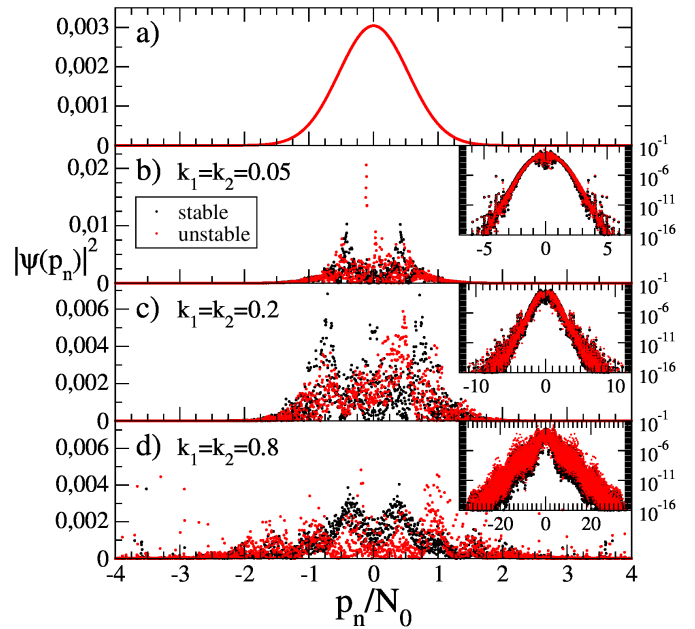


FIG. 6. Probability distribution in momentum for the same cases than 5. Top panel show the probability distribution in momentum for initial state given by a Gaussian centered at zero (the same for stable and unstable conditions). In second, third and fourth panels (from top to bottom) the distribution is shown for $k_1 = k_2 = 0.05$, $k_1 = k_2 = 0.2$ and $k_1 = k_2 = 0.8$ after $t = 10^4$ iterations where black and red dots illustrate the cases of stable and unstable conditions. In each panel, top-right insets show the same distribution in logarithmic scale and for the whole Hilbert space. In all cases $T = 76\pi/3^5 \approx 0.983$.

crete Fourier transform, where in this case $\langle p \rangle = P_0 = 0$.

We define $P_{3,x}(t)$ and $P_{3,p}(t)$ as the probability to stay in 3 cells centered at initial X_0 and P_0 values respectively:

$$P_{3,x}(t) = \sum_{n=[X_0 - \frac{3N_0}{2}] }^{[X_0 + \frac{3N_0}{2}]} |\psi(x_n)|^2 \quad (7)$$

$$P_{3,p}(t) = \sum_{n=[P_0 - \frac{3N_0}{2}] }^{[P_0 + \frac{3N_0}{2}]} |\psi(p_n)|^2 \quad (8)$$

where $[x]$ is the integer part of x .

The initial probability distributions placed in a vicinity of stable and unstable fix points of the kick potential $V(x)$ are shown in Fig. 5. We note that for small $k_1 \sim k_2 \ll 1$ the map approximately describes a continuous time evolution in a static potential. In [21] it is shown that in this case the quantum evolution is approximately reduced to the Aubry-André model on a discrete lattice with the eigenstates of the stationary Schrödinger equation:

$$\lambda \cos(\hbar n + \beta) \phi_n + \phi_{n+1} + \phi_{n-1} = E \phi_n. \quad (9)$$

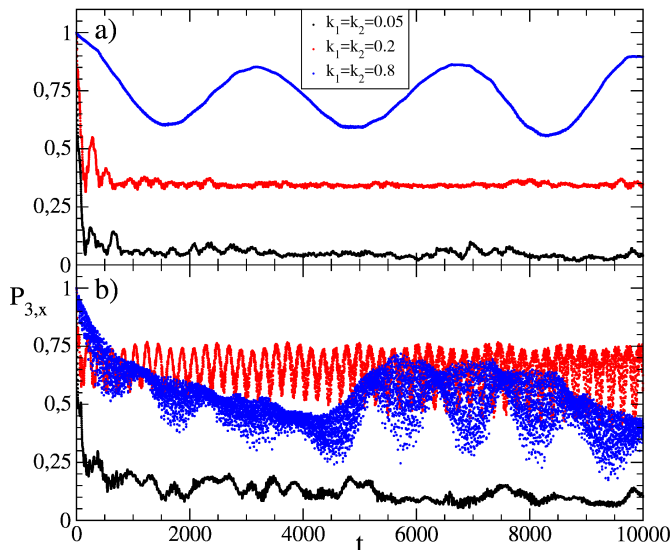


FIG. 7. Time evolution of the probability to stay in the first 3 space cells $P_{3,x}$ for the cases of Figs 5 and 6 with $N_0 = 243$ and $q_m = 233$. Top (a) and bottom (b) panels show the stable and unstable conditions of Fig. 5 respectively for $k_1 = k_2 = 0.05$ (black circles), $k_1 = k_2 = 0.2$ (red circles) and $k_1 = k_2 = 0.8$ (blue circles) ordered from bottom to top in each panel.

Here λ is an effective dimensional energy of the quasiperiodic potential and the hopping amplitude being unity. A metal-insulator transition (MIT) takes place from localized states at $\lambda > 2$ to delocalized eigenstates at $\lambda < 2$ [22]. A review of the properties of the Aubry-André model can be found in [24] and the mathematical prove of MIT is given in [25]. An estimate obtained in [21] shows that $\lambda \propto k_2 \exp(-C_1/k_1^{C_2} + 2C_3\sqrt{k_1})$ for an irrational $\nu \sim 1$ with C_1, C_2, C_3 being numerical constants order of unity. This approximate reduction of the Schrödinger equation in a continuous quasiperiodic potential $V(x)$ to the discrete lattice Aubry-André model have been used in experiments with cold atoms where the MIT was found at $\lambda = 2$ [20, 23].

The signs of MIT are visible in Fig. 5 with a delocalization of probability in space at $k_1 = k_2 = 0.05$ and localization at $k_1 = k_2 = 0.2; 0.8$. At the same time the results of Fig. 6 show that the probability distribution in momentum remains exponentially localized for the above k_1, k_2 values.

The time evolution of probabilities of stay in a vicinity of initial cell $P_{3,x}$ and $P_{3,p}$ are shown in Fig. 7 and Fig. 8. We see that at $k_1 = k_2 = 0.05$ only a small fraction of probability (about 10 percent) remains in a vicinity of initial space cell while in contrast it is rather large for $k_1 = k_2 = 0.2; 0.8$. In contrast for the distribution in momentum about 99 percent remains in a vicinity of initial cell for $k_1 = k_2 = 0.05; 0.2$ and 65 percent for $k_1 = k_2 = 0.8$. Thus this data confirms localization in momentum and MIT transition is space similar to the MIT discussed in [21] in the limit of energy conservative system corresponding to the dynamics of our kicked

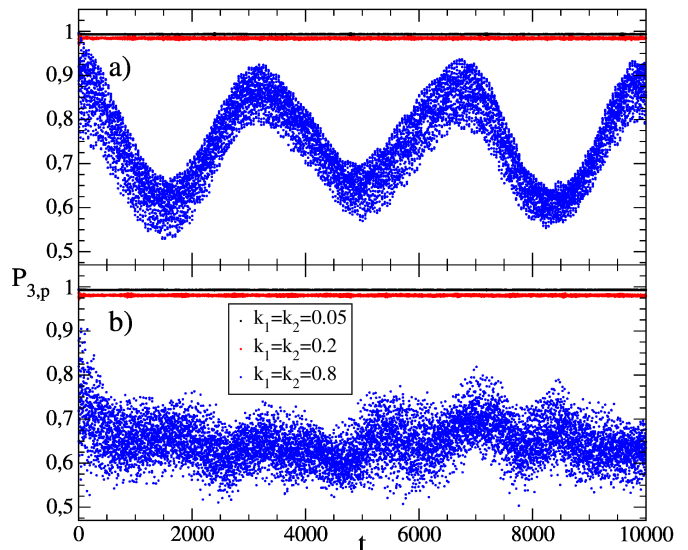


FIG. 8. Time evolution of the probability to stay in the first 3 momentum cells $P_{3,p}$ for the cases of Figs 5 and 6 with $N_0 = 243$ and $q_m = 233$. Top (a) and bottom (b) panels show the stable and unstable conditions of Fig. 6 respectively for $k_1 = k_2 = 0.05$ (black circles), $k_1 = k_2 = 0.2$ (red circles) and $k_1 = k_2 = 0.8$ (blue circles) ordered from top to bottom in each panel.

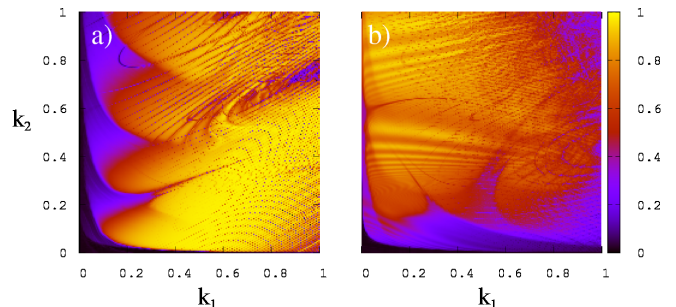


FIG. 9. Probability $P_{3,x}(t)$ averaged in time within interval $t \in (9000, 10000)$ is shown as a function of k_1 and k_2 . Here $T = 0.983\dots$ and the initial condition is centered at stable and unstable points of the potential in (a) and (b) panels respectively.

model at small k_1, k_2 values.

The global dependence of probabilities $P_{3,x}(t)$ and $P_{3,p}(t)$ on kick amplitudes k_1, k_2 are shown in Figs. 9, 10 for initial packet centered in a vicinity of stable or unstable point. For $P_{3,x}(t)$ there is a clear region on k_1, k_2 -plane where the probability $P_{3,x}(t)$ drops significantly corresponding to delocalization in space (see Fig. 9)). In contrast, the probability $P_{3,p}(t)$ remains always rather high showing that the classical chaotic diffusion in momentum is localized by quantum interference effects.

In Fig. 11 we consider a case with a smaller value of $T = 0.2$ so that the system becomes more close to the case of stationary potential analyzed in [21]. This data are similar to the case at larger $T = 0.983\dots$ with a domain of

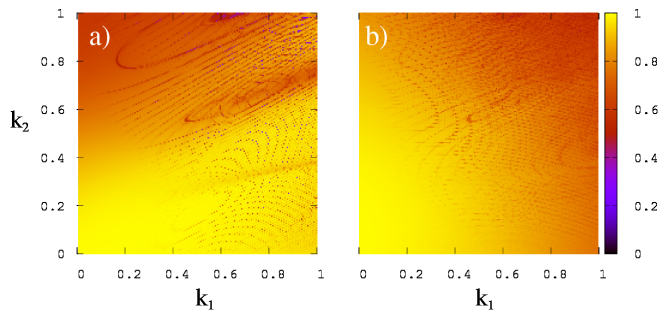


FIG. 10. Probability $P_{3,p}(t)$ in momentum averaged in time within interval $t \in (9000, 10000]$ is shown as a function of k_1 and k_2 . Here $T = 0.983\dots$ and the initial condition is centered at stable and unstable points of the potential in left (a) and right (b) panels respectively.

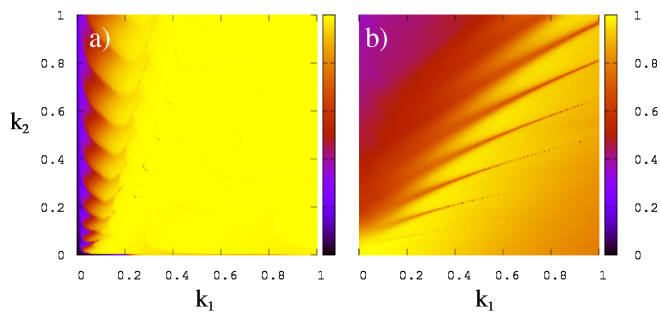


FIG. 11. Probability $P_{3,x}(t)$ in space (a) and $P_{3,p}(t)$ in momentum (b) averaged in time within interval $t \in (9000, 10000]$ are shown as a function of k_1 and k_2 . The initial condition in x is centered at *stable points of the potential* and $T = 0.2$.

small probability $P_{3,x}(t)$ values at small k_1, k_2 indicated the delocalization of probability in space. At the same time the probability $P_{3,p}(t)$ in momentum remains always localized.

The obtained results show the presence of certain space delocalization of the quantum incommensurate map at small kick amplitudes while at large amplitudes the probability remains localized. The probability in momentum remains localized for all k_1, k_2 values at irrational $T/2\pi$ values.

At the same time we note that the rigorous prove of probability delocalization in our model remains a mathematical challenge since the map of our system (or even its stationary version considered in [21]) on the Aubry-André model (9) works only approximately. Indeed, in our model it is possible to have excitation of high energy states (even if the numerical results indicate localization in momentum) that complicates the dynamics. We hope that the skillful mathematical tools developed in [25] will allow to obtain mathematical results for the quantum incommensurate standard map.

Below we consider the localization properties in momentum in more detail.

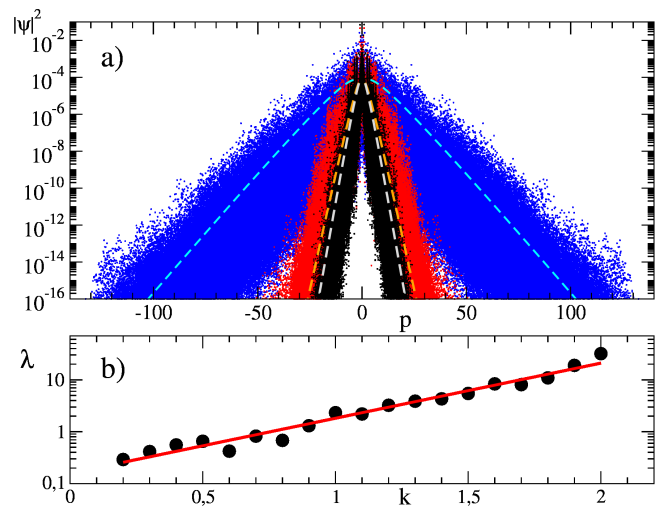


FIG. 12. (a) Probability distribution $|\psi(j_1, j_2)|^2$ in the model (10) as a function of $p = j_1 + \nu j_2$ shown at time $t = 10^4$. Here $T = 2$ and $N_2 = 3^7 = 2187$ ($N = 4782969$). Black (internal domain), red/gray (middle domain) and blue/dark (external domain) circles represent $k = k_1 = k_2 = 0.5, 0.7, 1.2$ respectively. Numerical fits of $|\psi|^2 \sim e^{-\frac{p}{\lambda}}$ with localization length values are shown by dashes lines with $\lambda(k = 0.5) \approx 0.656$, $\lambda(k = 0.7) \approx 0.822$ and $\lambda(k = 1.2) \approx 3.22$. Panel (b) shows the exponential dependence of fitted values of λ as a function of k . Numerical fit is shown by the straight line with the exponential growth $\lambda \approx 0.16 \exp(2.44k)$. Initial state is at $j_1 = j_2 = 0$.

IV. 2D AND 3D MODELS OF I -STANDARD MAP

To analyze the localization properties in momentum we note that the kick with k_1 generates integer harmonics of $\exp(-ij_1x)$ while the kick with k_2 generates only harmonics $\exp(-ij_2\nu x)$ with integer j_1, j_2 values. Due to that the wave function contains only these two types of harmonics and the system evolution is described by

$$\bar{\psi} = \hat{F}^{-1} e^{-i[k_1 \cos x + k_2 \cos \nu x]} \hat{F} e^{-iT(j_1 + \nu j_2)^2/2} \psi, \quad (10)$$

where \hat{F} is the 2D-fast Fourier transformation from momentum to space representation and \hat{F}^{-1} gives a back transformation from space to momentum. The integers j_1 and j_2 number the correspondent harmonic numbers with the energy phase of free propagation between kicks being $\phi_E = Tp^2/2 = T(j_1 + \nu j_2)^2/2$ with $p = (j_1 + \nu j_2)$. If we would have $\phi_E(j_1, j_2)$ taking random values for each j_1, j_2 then we would have 2D kicked rotator with the Anderson type localization in 2D. In such a case we would expect that the localization length λ grows exponentially with the diffusion rate $\ln \lambda \sim D \sim (k_1^2 + k_2^2)/2$ (see e.g. [15, 26, 27]). However, the phases $\phi_E(j_1, j_2)$ are not random but incommensurate and the appearance of 2D Anderson localization is not so obvious.

For investigation of this expected 2D localization we take $N_2 = 3^7$ harmonics j_1 and j_2 so that the total

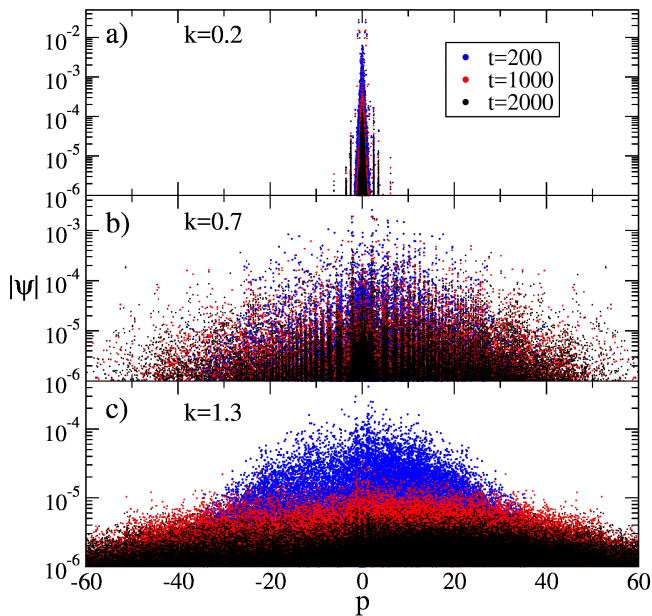


FIG. 13. Probability distribution $|\psi(j_1, j_2, j_3)|^2$ in the model (11) as a function of $p = j_1 + j_2/\theta + j_3/\theta^2$ where θ is the solution of $\theta^3 - \theta - 1 = 0$. Here $T = 2$ and $N_3 = 529$ ($N = N_3^3 = 148035889$) at evolution times $t = 200, 1000, 2000$ with blue (top dark points), red (middle gray points) and black (bottom) points respectively. Top (a), middle (b) and bottom (c) panels show the cases of $k_1 = k_2 = k_3 = 0.2$, $k_1 = k_2 = k_3 = 0.7$ and $k_1 = k_2 = k_3 = 1.3$. Initial state is at $j_1 = j_2 = j_3 = 0$. The Anderson transition takes place at $k = k_1 = k_2 \approx 0.7$.

number of states becomes $N = N_2^2 = 4782969$. The results of numerical show an exponential decay of probability with momentum p as it is shown in Fig. 12(a) for a few $k = k_1 = k_2$ values. We fit this decay by an exponential dependence $|\psi(p)|^2 \sim \exp(-|p|/\lambda)$ thus determining the localization length λ . The results presented in Fig. 12(b). show the expected exponential growth of localization length $\ln \lambda \sim 2.3k$. The fact that $\ln \lambda$ is proportional to k and not to expected k^2 can be attributed to the fact that we are still relatively close to the chaos border (see Fig. 4) and that the diffusion rate is small while the estimate $\ln \lambda \sim D \sim k^2$ assumes well developed chaotic regime with a relatively high D [27].

There is no delocalization in 2D but in 3D there is the Anderson transition to delocalization [28] if a disorder is below a critical value or chaotic diffusion rate is higher a certain border (see e.g. [27]). We argue that 3D case can be realized in our model if we add kick with one more incommensurate potential $V_3(x) = k_3 \cos \nu_3 x$. Then the wave function additional harmonics $\exp(-ij_3 \nu_3 x)$ and in analogy with (10) the time evolution is described by

$$\begin{aligned} \bar{\psi} &= \hat{F}^{-1} e^{-i[k_1 \cos x + k_2 \cos \nu x + k_3 \cos \nu_3 x]} \\ &\times \hat{F} e^{-iT(j_1 + \nu j_2 + \nu_3 j_3)^2/2} \psi, \end{aligned} \quad (11)$$

where momentum integer harmonics are $j_1, j_2, j_3 = 1, \dots, N_3$ and with the total dimension $N = N_3^3 = 529^3 =$

148035889 and $p = j_1 + j_2/\theta + j_3/\theta^2$, with irrational $\theta = 1.32471795724475\dots$ being the solution of equation $\theta^3 - \theta - 1 = 0$. Thus $\nu = 1/\theta$, $\nu_3 = 1/\theta^2$.

The results for time evolution of probability are presented in Fig. 13. They clearly show that at $k = k_1 = k_2 = 0.2$ there is exponential Anderson localization of probability over momentum [28]. For $k = k_1 = k_2 = 1.3$ there is spreading of probability in time over momentum states. The case at $k = k_1 = k_2 = 0.7$ is close to a critical parameter value where the Anderson transition takes place. Thus the MIT point is located in the range $0.7 \leq k_c < 1.3$. Additional studies should be performed to obtain the critical parameter more exactly but the presented results definitely show that the transition takes place for k -parameter in this range.

V. DISCUSSION

In our research we determined the main properties of the incommensurate standard map for its classical dynamics and for its quantum evolution. In the classical case the invariant KAM surfaces are destroyed above certain kick amplitudes given us a critical curve on the plane of kick amplitudes K_1, K_2 (see Fig. 1). We find that above the critical curve at its vicinity the diffusion rate is characterized by a critical exponent $\alpha \approx 2.5$ which is not so far from the case of the Chirikov standard map.

The quantum evolution at small quantum kick amplitudes $k_1 = K_1/\hbar$, $k_2 = K_2/\hbar$ is similar to the Aubry-André type transition [22] as discussed in [21] and observed in cold atom experiments with a static incommensurate potential [20, 23]. However, at larger values of k_1, k_2 the evolution remain localized both in space and momentum. We show that the localization in momentum is similar to the case of Anderson localization in 2D. While a significant progress has been reached with rigorous results for Aubry-André model [25], we point that the mathematical prove of space and momentum localization for the quantum incommensurate standard map represents a high challenge for mathematicians.

The quantum evolution for the quantum i -standard map is always localized in momentum, as in the case of 2D Anderson localization. However, for three kick-harmonics the situation becomes similar to the 3D Anderson localization with MIT from localized to delocalized evolution as the kick amplitude is increased. We note that this behavior has similarities with the frequency modulated kicked rotator introduced in [29] which also demonstrates the Anderson localization in effective 2 and 3 dimensions [26] observed in the cold atoms experiments [30, 31]. Thus the kicked rotator with one additional modulation frequency in time domain is similar to the case of 2D Anderson localization in agreement with predictions done in 1983 [29]. The case of 2 additional modulation frequencies is similar to the case of 3D Anderson transition as discussed in [26]. We hope that the results presented here will allow to investigate the Anderson lo-

calization in 2D and 3D for periodically kicked rotator with kicked incommensurate potential discussed in this work.

As it was pointed in Section I the incommensurate standard map naturally appears for a description of dynamics of dark matter or comets in the Solar System and other planetary systems with 2 or more planets rotating around the central star. Recently it has been shown that in the case of star and one rotating planet the quantum effects can play a significant role for escape of very light dark matter from the planetary system due to the Anderson localization of energy transitions [32]. The obtained results show that a presence of second planet leads to the dynamics described by the incommensurate standard map with significant effects on the quantum localization of dark matter.

Since the Chirikov standard map has many universal features and appears in the description of evolution of many very different physical systems we argue that the incommensurate standard map will also find a broad field of applications.

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