

Quantum computer with cold ions in the Aubry pinned phase

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Abstract. It is proposed to modify the Cirac-Zoller proposal of quantum computer with cold ions in a global oscillator trap potential by adding a periodic potential with an incommensurate average ratio of number of ions to number of periods being order of unity. With the increase of the periodic potential amplitude the system enters in the Aubry pinned phase characterized by quasi-frozen positions of ions and a gap of their first phonon excitations becomes independent of number of ions. This makes this quantum computer really scalable. It is augured that the usual single- and two-qubit gates can be realized between the nearby ions in the Aubry phase. The possibilities of experimental realizations of a periodic potential with microtrap arrays or optical lattices are discussed.

1 Introduction

The creation of a scalable quantum computer for generic computational tasks is an important challenge of modern quantum technology [1]. One of the first physical proposals of such a computer is the Cirac-Zoller quantum computer of 1995 with a chain of cold ions placed in an oscillator trap potential [2]. Indeed, at that time the storage of cold ions already allowed to keep several tens of ions in a storage ring [3]. Thus soon after the proposal a two-qubit gate with a conditioned phase shift had been realized [4] followed later by realization of a few other two-qubit gates [5,6,7]. Simple quantum algorithms [8], a set of universal gates with two ions [9] and a creation of various entangled states [10] had been also reported. The experimental progress with cold ion experiments is reviewed in [11,12,13,14]. At present up to 100 ions can be routinely trapped for hours in a linear trap configuration [15]. This experimental progress makes cold ions to be very attractive for scalable quantum computer realization. Their important physical advantages are related to possibilities of individual addressing of a selected ion by a laser beam and low temperatures reached experimentally.

However, the scalable quantum computation with ion-trap computers is not so easy to reach even if about 100 ions can be now trapped for hours. Thus, the original Cirac-Zoller proposal [2] is not really scalable for a very large number of ions. Indeed, the coupling between ion chain and the internal ion levels decreases with the number N of trapped ions as $1/\sqrt{N}$ (see Eq.(1) in [2]). Also, the ion chain oscillation frequency ω_r is unavoidably decreasing if the number of ions in the trap is growing with a constant average distance between ions. Thus the gap between the ground state and the first excitation of ion chain drops with N . It is proposed to avoid these problems with a modular type architecture with quasi-separated groups of ions with a further adiabatic transfer of quantum information between groups. However, the practical realization of this concept is not an easy task.

Here I propose another concept of quantum computer with cold ions in a linear configuration based on the Aubry pinned phase [16]. In this proposal the linear chain of ions is placed in a periodic potential (or lattice), created by external fields, and a global oscillator trap potential. It is assumed that there is an incommensurate density of ions $\nu = N/L \sim 1.618$ (ratio of number of ions N per number of potential periods L). In the limit of small potential amplitude the system is reduced to the Cirac-Zoller proposal. In this regime the spectrum of ion excitations have an almost acoustic spectrum starting from ω_r which goes to zero in the limit of large number of ions. However, when the lattice amplitude K becomes larger than a certain critical value K_c the chain enters in the Aubry pinned phase with the appearance of optical gap ω_g of excitations being independent of the chain length and the number of ions placed in it. The physics of this transition is related to the dynamical symplectic maps, invariant Kolmogorov-Arnold-Moser (KAM) curves and the fractal cantori replacing these curves above the transition to the Aubry pinned phase corresponding to the chaotic map dynamics. Since the spectral gap ω_g is independent of the system size it is possible to place unlimited number of ions in such a system.

The first analytical and numerical studies of ions in a periodic potential had been done in [17] where its physical properties and the critical point of Aubry had been determined. The cold ion experiments had been started in [18] and the signatures of the predicted Aubry transition have been reported recently by the Vuletic group [19,20] with up to 5 ions. The Aubry phase with chains of larger number of ions is under investigations [21]. Recently, the transport properties of charges in a periodic 1D and 2D lattices have studied analytically and numerically in [22,23]. However, in these studies [17,22,23] ions or charges were considered without internal states while they are essential since they form a qubit for a given ion and the interactions between internal ion states (usually S and D states) are used that give a qubit lifetime of about a second [2,13]). Also the coupling between internal ion states and spacial mo-

tion of ions is essential for the realization of universal quantum gates. These features are discussed in this work with arguments about the advantages of ions placed in a lattice of Aubry phase.

The paper is constructed as follows: the system description and its physical properties are given in Section 2, the quantum gates with ions in the Aubry phase are discussed in Section 3 and the discussion of the results and possible experimental realizations are given in Section 4.

2 System description and properties

The motion of ions in a periodic potential and a global oscillator potential is described by the Hamiltonian [17]:

$$H = \sum_{i=1}^N \left(\frac{P_i^2}{2} + \frac{\omega_{tr}^2}{2} x_i^2 - K \cos x_i \right) + \sum_{i>j} \frac{1}{|x_i - x_j|}. \quad (1)$$

Here P_i, x_i are ion momentum and position, K gives the amplitude of periodic potential and all N ions are placed in a harmonic trap potential with frequency ω_{tr} . The Hamiltonian is written in dimensionless units where the potential period is $\ell = 2\pi$ and ion mass and charge are $m = e = 1$. In these atomic-type units the physical system parameters are expressed in units: $r_a = \ell/2\pi$ for length, $\varepsilon_a = e^2/r_a = 2\pi e^2/\ell$ for energy, $E_{adc} = \varepsilon_a/er_a$ for applied static electric field, $v_a = \sqrt{\varepsilon_a/m}$ for particle velocity, $t_a = er_a\sqrt{m/\varepsilon_a}$ for time t .

The physical properties of this system has been analyzed in detail in [17]. They are not sensible to the boundary conditions so that instead of global oscillator potential one can consider the ion chain with fixed ends or hard wall boundary conditions [23,24].

The equilibrium positions of ions are determined by the condition $P_i = 0$ and $\partial H/\partial x_i = 0$. In the approximation of interactions only between nearest neighbors this give the recursive map for equilibrium ion positions x_i :

$$p_{i+1} = p_i + Kg(x_i), \quad x_{i+1} = x_i + 1/\sqrt{p_{i+1}}. \quad (2)$$

Here $p_i = 1/(x_i - x_{i-1})^2$ is the effective momentum conjugated to x_i and the kick function is $Kg(x) = -\omega^2 x - K \sin x$. The numerical simulations performed in [17,22,23,24] confirm that this approximation provides a good description of real ion positions obtained by numerical simulations. Thus the nearest neighbor interactions between ions are dominant.

The map description (2) provides important links with the generic properties of dynamical symplectic maps (see e.g. [25,26,27]). The equation for x_{i+1} can be locally linearized in p_{i+1} near the resonant values of $p_r \approx 2\pi/v$ defined by the condition $x_{i+1} = x_i + 2\pi m$ where m are integers (see examples in [25,26]). This leads to the local description of dynamics by the Chirikov standard map [17]:

$$y_{i+1} = y_i - K_{eff} \sin x_i, \quad x_{i+1} = x_i - y_{i+1}, \quad (3)$$

where $y_i = \alpha(p_i - p_r)$, $\alpha = 1/(2p_r^{3/2}) = (2\pi/v)^3/2$ and the dimensionless chaos parameter $K_{eff} = \alpha K = K(2\pi/v)^3/2$.

This local description corresponds to the linear-spring forces locally acting between particles that in fact represents the Frenkel-Kontorova model describing commensurate-incommensurate transition in solid states systems [28]. Thus the properties of this

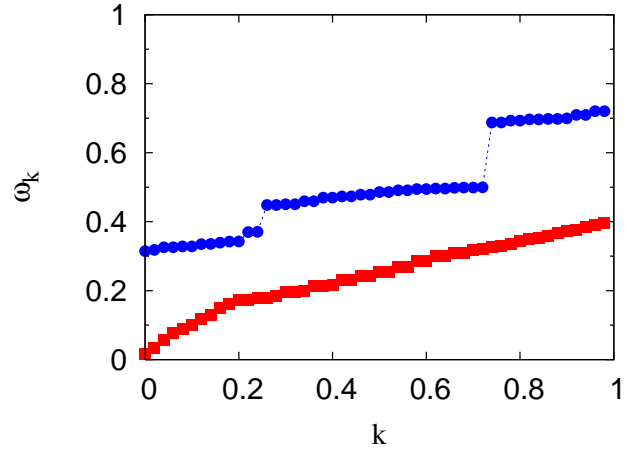


Fig. 1. (Color online) Phonon spectrum $\omega(k)$ as a function of scaled mode number $k = i/N$ ($i = 0, \dots, N-1$) for the KAM sliding phase at $K = 0.03$ (bottom curve, red squares) and the Aubry pinned phase at $K = 0.2$ (top curve, blue points) for $N = 50$ ions in a trap with frequency $\omega_{tr} = 0.014$ which approximately gives the golden mean density in the central 1/3 part of the chain (after [17]).

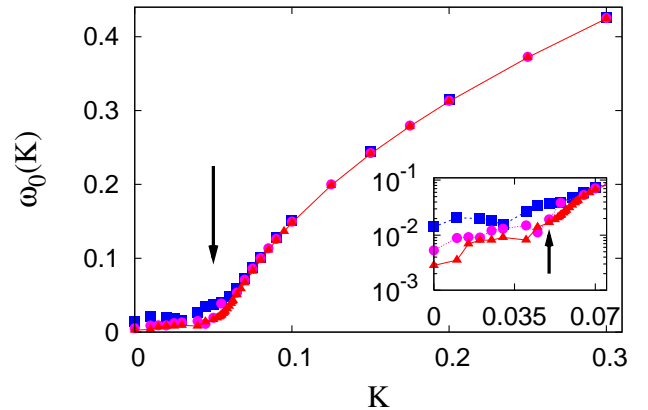


Fig. 2. (Color online) Minimal excitation frequency $\omega_0(K)$ as a function of periodic potential strength K for the golden mean ion density $v_g = 1.618\dots$ and number of ions $N = 50$ (red squares; $\omega_{tr} = 0.014$), $N = 150$ (magenta circles; $\omega_{tr} = 0.00528$), $N = 300$ (blue triangles, $\omega_{tr} = 0.00281$). The critical point $K_c \approx 0.05$ of Aubry transition is marked by arrow; inset shows data near K_c (after [17]).

system of ions in a periodic potential can be understood from the properties of the Chirikov standard map which describes the local dynamics of various physical systems (see e.g. [29]).

At small K or K_{eff} the phase space of maps (2) and (3) is covered by the invariant KAM curves characterized by irrational rotation number $r = \langle x_i - x_0 \rangle / 2\pi i = v$ which gives

an average distance (phase) between ions related to the average ion density ν . The oscillations of ions near the equilibrium positions has the acoustic excitation spectrum $\omega_k \approx C_\nu k + \omega_{tr}$ where $k = i/N$ plays the role of wavevector number and $C_\nu \sim 1$ is the sound velocity.

For the Chirikov standard map the last invariant curve with the golden mean rotation number $r_g = \nu_g = 1.618\dots$ is destroyed at $K_{eff} \approx 1$ leading to a global chaos and diffusion in y [25]. For the case of ions with density ν this gives the critical amplitude of potential [17]:

$$K_c(\nu) \approx 0.034(\nu/\nu_g)^3, \quad \nu_g = 1.618\dots \quad (4)$$

This theoretical dependence is recently confirmed by extensive numerical simulations [23]. For $\nu = \nu_g$ the numerical results give $K_c = 0.0462$ [17,23,24] that is slightly higher than the theoretical value due to the local approximation used in the reduction to the Chirikov standard map.

For $K > K_c(\nu)$ the invariant KAM curve is destroyed and it is replaced by a fractal cantori invariant set as proved by Aubry in [16]. The configuration of particles corresponding to this invariant set has the minimal energy and thus represents the ground state of the system. The spectrum of ion oscillations near these ground state positions is characterized by the optical gap $\omega_g \sim \sqrt{K}$. Thus in difference from the KAM sliding phase at $K < K_c$ for $K > K_c$ we have the Aubry pinned phase where the ion chain is pinned by the lattice.

The example of excitation spectrum for the KAM and Aubry phases is shown in Fig. 1 taken from [17]. The dependence of the minimal excitation frequency $\omega_0(K)$ on potential amplitude K is shown in Fig. 2 taken from [17]. For these data the trap frequency ω_{tr} is chosen in such a way that, at a given number of ions N in the trap, the central 1/3 part of the chain keeps the fixed density $\nu \approx 1.618$ when the number of ions N is growing. Due to this condition at $K = 0$, corresponding to the Cirac-Zoller proposal [2], the trap becomes more and more soft and $\omega_{tr} \sim 1/\sqrt{N} \rightarrow 0$. Indeed, we want to keep the distance between ions in the center to be independent of N and thus size of the chain $x_{chain} \sim N/\nu$ is growing since it is approximately determined by the condition at the chain end $F_{chain} \sim \omega_{tr}^2 x_{chain} \sim \nu^2$ that gives the above dependence $\omega_{tr} \sim 1/\sqrt{N}$.

Thus for $K < K_c$ the lowest excitation frequency goes to zero with the increase of number of ions in the trap. Hence the Cirac-Zoller proposal is not really scalable. In contrast for $K > K_c$ the lowest frequency excitation is independent of N as it is well seen in Fig. 2. Thus this Aubry pinned phase has certain chances to represent a scalable architecture for a quantum computer with cold ions.

Indeed, for the quantum case the energy of lowest phonon excitation is $E_0 = \hbar\omega_0(K) = \hbar\omega_g$ being independent of N . For a temperature $T \ll \hbar\omega_0(K)$ the phonon excitations become frozen and should not perturb the accuracy of quantum gates operations.

There are also another type of quantum excitations in the quantum ion chain inside the Aubry pinned phase. In fact the Aubry theorem [16], which guaranties that the Aubry cantori ground state has the minimal energy E_A of the classical ion chain is mathematically correct but it is wrong from the physical view point. Indeed, in the classical chain there are exponentially many static configurations of ions which number

N_s grows exponentially with the number of trapped ions N . In addition the energies of these configurations are approaching exponentially close to the Aubry ground state energy E_A with increase of N (see Fig. 4 in [17] where this feature is clearly illustrated). In fact this property is similar to the random spin glass systems [30]. However, in our case the randomness is absent and the system is described by a rather simple deterministic Hamiltonian (1). Thus the Aubry pinned phase represents the dynamical spin glass system with an enormous amount of quasi-degenerate configurations in a vicinity of the Aubry ground state.

In the quantum case there is quantum tunneling between these quasi-degenerate configurations that can be viewed as an instanton excitations. However, for small dimensionless Planck constant \hbar_{eff} the gas of instantons is very dilute and the tunneling times are enormously long [17]. Thus on a scale of typical tunneling time $t_{tun} \propto \exp(A/\hbar_{eff})$ we can consider the ions to be frozen at their positions (here $A \propto K$ is a typical action between energy minima coupled by tunneling). The dimensionless Planck constant is $\hbar_{eff} = \hbar / (e\sqrt{m\ell}/2\pi)$ and for a typical lattice period $\ell \approx 1\mu m$, ion density $\nu \sim 1$ and $^{40}Ca^+$ ions we have very small $\hbar_{eff} \approx 10^{-5}$. Thus the quantum ions can be considered as frozen at their configuration positions for the whole time scale of quantum computations.

3 Quantum gates

As in the proposal of Cirac-Zoller [2] I assume that the qubit is formed by two internal levels $S_{1/2}$ and $D_{5/2}$ of $^{40}Ca^+$ ion with a radiative life-time of more than one second. All single-qubit gates can be realized by laser pulses as described in [2,13]. At present these gates are routinely performed with the fidelity exceeding 0.99 [13]. The individual accessing of ions is also available in experiments with ion spacing of about $5\mu m$ [13].

Since single-qubit gates with ions are reliable the most important for quantum computations become two-qubit gates which in combination with single-qubit gates allow to perform universal quantum computations [1]. There are three types of two-qubit gates usually discussed for cold ions (see e.g. review [13]): the Cirac-Zoller gate [2], the Molmer-Sorensen gate [31] and the geometric phase gate [7] closely related to the Molmer-Sorensen gate.

In all these gates the motional oscillator states of ions (sideband) with frequency ω_0 are coupled by a tuned laser pulse with internal $S - D$ levels of ions. Usually as an example one considers two ions with two internal levels and their sideband modes [13]. The laser pulse duration is selected in a way allowing to perform two-qubit gate. In the case of long ion chain in an oscillator trap the operational frequency of the Cirac-Zoller gate is proportional to the strength of coupling of internal levels with the whole chain oscillator state (the bus mode) which decreases with the number of ions as $1/\sqrt{N}$.

For the Molmer-Sorensen gate both ions are irradiated with a bichromatic laser field with frequencies $\omega_0 \pm (\omega_{qubit} + \delta)$ tuned close to the red and the blue sideband of a collective mode (see Fig.14 in [13]). This approach allowed to create experimentally Bell states with a fidelity 99.3% [13]

The same gates can be implemented for ions in the Aubry pinned phase. In this case the interaction of ions is well approx-

imated by the nearest neighbor interactions as is discussed in the previous Section with the map (2) description of equilibrium ion configurations. The oscillations of ions in a vicinity of equilibrium positions is harmonic and we can consider them as sideband transitions for laser pulses as for the two-qubit gates considered above. Since the interactions are dominated by nearest neighbors the coupling between internal $S - D$ qubit levels and ion oscillator mode is independent of the number of ions in the chain. The frequency of this oscillator or phonon mode gap is $\omega_g = \omega_0(K)$ being also independent of the chain length as it is shown in Fig. 2.

The construction of two-qubit gates should also take into account that when cold ions are cooled and loaded in the Aubry pinned phase it is most probable that they will be located in one of quasi-degenerate static configurations. Thus the distances between nearby ions will be somehow irregular that will affect the interactions between specific pairs of ions. However, it is possible to determine experimentally the actual ion positions and then to adapt the laser pulses of two-qubit gates to these experimentally found ion positions. In a sense for a good work of a piano each string should be checked and adapted. Here, for quantum gates with ions in the Aubry phase we have a similar situation.

However, there are certain points that require additional investigations. The low energy phonon excitations with the lowest phonon frequency $\omega_0(K)$ are excited by a tuned laser pulse which acts mainly on one or two nearby ions. Thus there is a question how this excitation will propagate along the chain of ions in the Aubry pinned phase. This propagation or spreading along the chain depends on two main factors: the localization properties of phonon modes in the pinned phase and the rate of decomposition of local ion oscillations into these phonon modes. At present very little analysis has been performed for these important properties of ionic phonon modes in the Aubry phase. Examples of a few phonon eigenmodes are given in [17] (see Figs.9,10 there). Some of modes look to be localized some of them have spreading over several ions. The spreading rate of one or two ion oscillations has not been studied and require further investigations.

Thus there are open questions on the possible fidelity and accuracy of two-qubit gates for cold ions in the Aubry phase.

4 Discussion

In this work I analyzed the properties of cold ion chain in a periodic potential which amplitude locates ions inside the Aubry pinned phase. The emergence of Aubry transition from KAM sliding phase to Aubry pinned phase takes place when the potential exceeds a critical value $V_A = K_c(v)e^2/(\ell/2\pi)$. For a typical lattice period $\ell = 1\mu\text{m}$ and dimensionless ion density per period $v = 1.618$ this corresponds to $V_A \approx 3k_B\text{Kelvin}$. Apparently this amplitude significantly exceeds the amplitudes reachable with presently available laser power for optical lattices. Usually the optical lattice amplitude is assumed to be able to reach the value $V_A \approx 10^{-3}k_B\text{Kelvin}$ (see e.g. review [32]).

In contrast to optical lattices the radio-frequency (RF) traps provide the potential depth $V_{RF} \approx 10^4k_B\text{Kelvin}$ that is significantly above the estimated Aubry transition potential amplitude [32]. At present there is a significant miniaturization of these

RF traps with sizes going down to tens of microns [32]. Thus such microtrap linear arrays can model the periodic potential considered here with high amplitudes of periodic potential allowing to place ions in the Aubry pinned phase. There is also progress with the Penning microtraps of about 10 micron size [33]. Thus the linear array of RF or Pinned microtraps would allow to observe the Aubry transition and hopefully to perform scalable quantum computations with cold ions in the Aubry pinned phase.

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