Linear response theory for Google matrix

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Abstract. We develop the linear response theory for the Google matrix PageRank algorithm with respect to a general weak perturbation and a numerical efficient and accurate algorithm, called LIRGOMAX algorithm, to compute the linear response of the PageRank with respect to this perturbation. We illustrate its efficiency on the example of the English Wikipedia network with more than 5 millions of articles (nodes). For a group of initial nodes (or simply a pair of nodes) this algorithm allows to identify the effective pathway between initial nodes thus selecting a particular subset of nodes which are most sensitive to the weak perturbation applied to them (injection or pumping at one node and absorption of probability at another node). The further application of the reduced Google matrix algorithm (REGOMAX) allows to determine the effective interactions between the nodes of this subset. General linear response theory already found numerous applications in various areas of science including statistical and mesoscopic physics. Based on these grounds we argue that the developed LIRGOMAX algorithm will find broad applications in the analysis of complex directed networks.

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1 Introduction

Linear response theory finds a great variety of applications in statistical physics, stochastic processes, electron transport, current density correlations and dynamical systems (see e.g. [1,2,3,4,5]). In this work we apply the approach of linear response to Google matrices of directed networks with the aim to characterize nontrivial interactions between nodes.

The concept of Google matrix and the related PageRank algorithm for the World Wide Web (WWW) has been proposed by Brin and Page in 1998 [6]. A detailed description of the Google matrix construction and its properties is given in [7]. This approach can be applied to numerous situations and various directed networks [8].

Here we develop the LInear Response algorithm for GOogle MAtriX (LIRGOMAX) which applies to a very general model of a weakly perturbed Google matrix or the related PageRank algorithm. As a particular application we consider a model of injection and absorption at a small number of nodes of the networks and test its efficiency on examples of the English Wikipedia network of 2017 [9]. However, the scope of LIRGOMAX algorithm is more general. Thus, for example, it can be also applied to compute efficiently and accurately the PageRank sensitivity with respect to small modifications of individual elements of the Google matrix or its reduced version [10, 11,12,13].

From a physical viewpoint the approach of injection/absorption corresponds to a small pumping probability at a certain network node (or group of nodes) and absorbing probability at another specific node (or group of nodes). In a certain sense such a procedure reminds lasing in random media where a laser pumping at a certain frequency generates a response in complex absorbing media [14].

More specifically we select two particular nodes, one for injection and one for absorption, for which we use the LIRGOMAX algorithm to determine a subset of most sensitive nodes involved in a pathway between these two nodes. Furthermore we apply to this subset of nodes the REduced GOogle MAtriX (REGOMAX) algorithm developed in [10, 11] and obtain in this way an effective Google matrix description between nodes of the found pathway.

In general the REGOMAX algorithm determines effective interactions between selected nodes of a certain relatively small subset embedded in a global huge network. Its efficiency was recently demonstrated for the Wikipedia networks of politicians [11] and world universities [12], SIGNOR network of protein-protein interactions [13] and multiproduct world trade network of UN COMTRADE [15].

In this work our aim is to provide a first illustration of the efficiency of the LIRGOMAX algorithm combined with the reduced Google matrix analysis. Due to this we restrict in this work our considerations to the analytical description of the LIRGOMAX algorithm and the illustration of its application to two cases from the English Wikipedia network of 2017. The paper is constructed as follows: in Section 2 we provide the analytical description of the LIRGOMAX algorithm complemented by a brief description of the Google matrix construction and the REGOMAX algorithm, in Section 3 we present certain results for two examples of the Wikipedia network, the discussion of results is given in Section 4. Additional data are also available at [16].

2 Theory of a weakly perturbed Google matrix

2.1 Google matrix construction

We first briefly remind the general construction of the Google matrix G from a direct network of N nodes. For this one first computes the adjacency matrix A_{ij} with elements 1 if node j points to node i and zero otherwise. The matrix elements of G have the usual form G_{ij} = $\alpha S_{ij} + (1-\alpha)/N$ [6,7,8], where S is the matrix of Markov transitions with elements $S_{ij} = A_{ij}/k_{out}(j)$ and $k_{out}(j) =$ $\sum_{i=1}^{N} A_{ij} \neq 0$ being the out-degree of node j (number of outgoing links) or $S_{ij} = 1/N$ if j has no outgoing links (dangling node). The parameter $0 < \alpha < 1$ is the damping factor with the usual value $\alpha = 0.85$ [7] used here. We note that for the range $0.5 \le \alpha \le 0.95$ the results are not sensitive to α [7,8]. This corresponds to a model of a random surfer who follows with probability α at random one of the links available from the actual node or jumps with probability $(1 - \alpha)$ to an arbitrary other node in the network.

The right PageRank eigenvector of G is the solution of the equation $GP = \lambda P$ for the unit eigenvalue $\lambda = 1$ [6,7]. The PageRank P(j) values represent positive probabilities to find a random surfer on a node j ($\sum_j P(j) = 1$). All nodes can be ordered by decreasing probability P numbered by the PageRank index K = 1, 2, ...N with a maximal probability at K = 1 and minimal at K = N. The numerical computation of P(j) is done efficiently with the PageRank iteration algorithm described in [6,7].

It is also useful to consider the original network with inverted direction of links. After inversion the Google matrix G^* is constructed via the same procedure (using the transposed adjacency matrix) and its leading eigenvector P^* , determined by $G^*P^* = P^*$, is called CheiRank [17] (see also [8]). Its values $P^*(j)$ can be again ordered in decreasing order resulting in the CheiRank index K^* with highest value of P^* at $K^* = 1$ and smallest values at $K^* = N$. On average, the high values of $P(P^*)$ correspond to nodes with many ingoing (outgoing) links [7,8].

2.2 Reduced Google matrix algorithm

The REGOMAX method is described in detail in [10, 11,13,12]. For a given relatively small subset of $N_r \ll N$ nodes it allows to compute efficiently a "reduced Google matrix" $G_{\rm R}$ of size $N_r \times N_r$ that captures the full contributions of direct and indirect pathways appearing in the

full Google matrix G between the N_r selected nodes of interest. The PageRank vector P_r of G_R coincides with the full PageRank vector projected on the subset of nodes, up to a constant multiplicative factor due to the sum normalization. The mathematical computation of G_R provides a decomposition of G_R into matrix components that clearly distinguish direct from indirect interactions: $G_R =$ $G_{rr} + G_{pr} + G_{qr}$ [11]. Here G_{rr} is given by the direct links between the selected N_r nodes in the global G matrix with N nodes. G_{pr} is a rank one matrix whose columns are rather close (up to constant factor) to the reduced PageRank vector P_r . Even though the numerical weight of G_{pr} is typically quite large it does not give much new interesting information about the reduced effective network structure.

The most interesting role is played by G_{qr} , which takes into account all indirect links between selected nodes happening due to multiple pathways via the global network nodes N (see [10,11]). The matrix $G_{qr} = G_{qrd} + G_{qr}^{(nd)}$ has diagonal (G_{qrd}) and non-diagonal ($G_{qr}^{(nd)}$) parts with $G_{qr}^{(nd)}$ describing indirect interactions between selected nodes. The exact formulas and the numerical algorithm for an efficient numerical computation of all three components of G_R are given in [10,11]. It is also useful to compute the weights W_R , W_{pr} , W_{rr} , W_{qr} of G_R and its 3 matrix components G_{pr} , G_{rr} , G_{qr} given by the sum of all its elements divided by the matrix size N_r . Due to the column sum normalization of G_R we obviously have $W_R = W_{rr} + W_{pr} + W_{qr} = 1$.

2.3 General model of linear response

We consider a Google matrix $G(\varepsilon)$ (with non-negative matrix elements satisfying the usual column sum normalization) depending on a small parameter ε and a general stochastic process $P(t + 1) = G(\varepsilon) F(\varepsilon, P(t))$ where $F(\varepsilon, P)$ is a general function on ε and P which does NOT need to be linear in P. (Here P(t) denotes a time dependence of the vector P; below and for the rest of this paper we will use the notation P(j) for the *j*th component of the vector P).

Let $E^T = (1, ..., 1)$ be the usual vector with unit entries. Then the condition of column sum normalization of $G(\varepsilon)$ reads $E^T G(\varepsilon) = E^T$. The function $F(\varepsilon, P)$ should satisfy the condition $E^T F(\varepsilon, P) = 1$ if $E^T P = 1$. At $\varepsilon = 0$ we also require that F(0, P) = P, i.e. F(0, P) is the identity operation on P. We denote by $G_0 = G(0)$ the Google matrix at $\varepsilon = 0$ and by P_0 its PageRank vector such that $G_0 P_0 = P_0$ with $E^T P_0 = 1$. We denote by P the more general, ε -dependent, solution of

$$P = G(\varepsilon) F(\varepsilon, P) \quad , \quad E^T P = 1 \; . \tag{1}$$

2.3.1 Pump model

As a first example we present the *Pump model* to model an injection- and absorption scheme. For this we choose for

the Google matrix simply $G(\varepsilon)=G_0$ (i.e. no $\varepsilon\text{-dependence}$ for G) and

$$F(\varepsilon, P) = \frac{(\mathbf{1} + \varepsilon D)P}{E^T[(\mathbf{1} + \varepsilon D)P]} = \frac{(\mathbf{1} + \varepsilon D)P}{1 + \varepsilon e(P)}$$
(2)

with $e(P) = E^T DP$ and D being a diagonal matrix with entries D_j which are mostly zero and with few positive values $D_j > 0$ for nodes j with injection and few negative values $D_j < 0$ for nodes j with absorption. The non-vanishing diagonal entries of D_j should be comparable (a global scaling factor can be absorbed in a redefinition of the parameter ε) and we have $e(P) = \sum_{D_j \neq 0} D_j P(j)$. Physically, we multiply each entry P(j) by the factor $1 + \varepsilon D_j$ (which is unity for most nodes j) and then we sum-normalize this vector to unity before we apply the Google matrix G_0 to it.

2.3.2 PageRank sensitivity

As a second example we consider the PageRank sensitivity. For this we fix a pair (i, j) of indices and multiply the matrix element $(G_0)_{ij}$ by $(1 + \varepsilon)$ and then we sumnormalize the column j to unity which provides the ε dependent Google matrix $G(\varepsilon)$. For the function $F(\varepsilon, P)$ we simply choose the identity operation: $F(\varepsilon, P) = P$. In a more explicit formula we have:

$$\forall_{k,l} \qquad G_{kl}(\varepsilon) = \frac{\left(1 + \varepsilon \,\delta_{ki}\delta_{lj}\right)(G_0)_{kl}}{1 + \varepsilon \,\delta_{lj}\left(G_0\right)_{ij}} \tag{3}$$

where $\delta_{ki} = 1$ (or 0) if k = i (or $k \neq i$). Note that the denominator is either 1 if $l \neq j$ or the modified column sum $1 + \varepsilon (G_0)_{ij}$ of column j if l = j. Then the sensitivity $D_{(j \to i)}(k)$ is defined as:

$$D_{(j \to i)}(k) = \frac{P(k) - P_0(k)}{\varepsilon P_0(k)}$$
(4)

where P is the ε -dependent PageRank of $G(\varepsilon)$ computed in the usual way. We expect that this quantity has a well defined limit if $\varepsilon \to 0$ but equation (4) is numerically not very precise for very small values of ε due to the effect of loss of precision. Below we present a method to compute the sensitivity in a precise way in the limit $\varepsilon \to 0$.

Examples of the sensitivity analysis, using directly (4), were considered for the reduced Google matrix of sets of Wikipedia and other networks (see e.g. [12, 15]).

2.4 Linear response

2.4.1 General scheme

One can directly numerically determine the ε -dependent solution $P(\varepsilon)$ of (1) for some small but finite value of ε (by iterating $P^{(n+1)} = G(\varepsilon), F(\varepsilon, P^{(n)})$ with some suitable initial vector $P^{(0)}$) and compute the quantity

$$\Delta P(\varepsilon) = \frac{P(\varepsilon) - P(0)}{\varepsilon} .$$
 (5)

We expect that $\Delta P(\varepsilon)$ has a finite well defined limit if $\varepsilon \to 0$. However, its direct numerical computation by (5) is subject to numerical loss of precision if ε is too small. In the following, we will present a different scheme to compute ΔP which is numerically more accurate and stable and that we call linear response of Google matrix. For this we expand $G(\varepsilon)$ and $F(\varepsilon, P)$ up to order ε^1 (neglecting terms $\sim \varepsilon^2$ or higher):

$$G(\varepsilon) = G_0 + \varepsilon G_1 + \dots , \quad F(\varepsilon, P) = P + \varepsilon F_1(P) + \dots$$
(6)

Furthermore we write

$$P(\varepsilon) = P_0 + \varepsilon P_1 + \dots$$
 (7)

The usual sum-normalization conditions for the first order corrections read as :

$$E^T G_1 = 0$$
 , $E^T F_1(P) = 0$, $E^T P_1 = 0$ (8)

if $E^T P = E^T P_0 = 1$. These conditions imply that P_1 and also $F_1(P)$ belong to the subspace "bi-orthogonal" to the PageRank, i.e. orthogonal to the left leading eigenvector of G_0 which is just the vector E^T .

Inserting (6) and (7) into (1) we obtain (up to order ε^1):

$$P = P_0 + \varepsilon P_1 = G_0 P_0 + \varepsilon \Big[G_0 P_1 + G_1 P_0 + G_0 F_1(P_0) \Big] .$$
(9)

Comparing the terms of order ε^0 one obtains the usual unperturbed PageRank equation $P_0 = G_0 P_0$. The terms of order ε^1 provide an inhomogeneous PageRank equation of the type :

$$P_1 = G_0 P_1 + V_0$$
, $V_0 = G_1 P_0 + G_0 F_1(P_0)$. (10)

The solution P_1 of this equation is just the limit of (5):

$$P_1 = \lim_{\varepsilon \to 0} \Delta P(\varepsilon) = \lim_{\varepsilon \to 0} \frac{P(\varepsilon) - P(0)}{\varepsilon} .$$
 (11)

To solve numerically (10) we first determine the unperturbed PageRank P_0 of G_0 in the usual way and compute V_0 which depends on P_0 . Then we iterate the equation:

$$P_1^{(n+1)} = G_0 P_1^{(n)} + V_0 \tag{12}$$

where for the initial vector we simply choose $P_1^{(0)} = 0$. This iteration converges with the same speed as the usual PageRank algorithm versus the vector P_1 and it is numerically more accurate than the finite difference $\Delta P(\varepsilon)$ at some finite value of ε .

We remind that $E^T V_0 = \sum_j V_0(j) = 0$ and also $E^T G_0 = E^T$. Therefore if at a given iteration step the vector $P_1^{(n)}$ satisfies the condition $E^T P_1^{(n)} = 0$ we also have $E^T P_1^{(n+1)} = E^T G_0 P_1^{(n)} + E^T V_0 = E^T P_1^{(n)} = 0$.

Therefore the conditions (8) are satisfied by the iteration equation (12) at least on a theoretical/mathematical level. However, rounding errors may produce slight errors in the conditions (8) and since such numerical errors contain a contribution in the direction of the unperturbed PageRank vector P_0 , corresponding to the eigenvector of G_0 with maximal eigenvalue, they do not disappear during the iteration and might even (slightly) increase with n. Therefore, due to purely numerical reasons, it is useful to remove such contributions by a projection after each iteration step of the vector $P_1^{(n+1)}$ on the subspace biorthogonal to the PageRank by:

$$P_1^{(n+1)} \to Q\left(P_1^{(n+1)}\right) \quad , \quad Q(X) = X - (E^T X) P_0$$
(13)

where Q(X) is the projection operator applied on a vector X. It turns out that such an additional projection step indeed increases the quality and accuracy of the convergence of (12) but even without it the iteration (12) converges numerically, however with a less accurate result.

It is interesting to note that one can "formally" solve (12) by:

$$P_1 = \sum_{n=0}^{\infty} G_0^n V_0 = \frac{1}{1 - G_0} V_0 \tag{14}$$

which can also be found directly from the first equation in (10). Strictly speaking the matrix inverse $(\mathbf{1} - G_0)^{-1}$ does not exist since G_0 has always one eigenvalue $\lambda = 1$. However, since $E^T V_0 = 0$, the vector V_0 , when expanded in the basis of (generalized) eigenvectors of G_0 , does not contain a contribution of P_0 which is the eigenvector for $\lambda = 1$ such that the expression (14) is actually well defined. From a numerical point of view a different scheme to compute P_1 would be to solve directly the linear system of equations $(\mathbf{1} - G_0) P_1 = V_0$ where the first (or any other suitable) equation of this system is replaced by the condition $E^T P_1 = 0$ resulting in a linear system with a well defined unique solution. Of course, such a direct computation is limited to modest matrix dimensions N such that a full matrix inversion is possible (typically N being a few multiples of 10^4) while the iterative scheme (12) is possible for rather large matrix dimensions such that the usual PageRank computation by the power method is possible. For example for the English Wikipedia edition of 2017 with $N \approx 5 \times 10^6$ the iterative computation of P_1 using (12) takes typically 2-5 minutes on a recent single processor core (e.g.: Intel i5-3570K CPU) without any use of parallelization once the PageRank P_0 is known. (The computation of P_0 by the usual power method takes roughly the same time.)

2.4.2 Application to the pump model

For the injection- and absorption scheme we can compute $F_1(P)$ from (2) as:

$$F(\varepsilon, P) = (\mathbf{1} - \varepsilon e(P) + \ldots)(P + \varepsilon DP)$$
(15)
= $P + \varepsilon [P - (E^T DP)P] + \ldots$.

Here the term $\sim \varepsilon^1$ is just the projection of DP to the subspace bi-orthogonal to P. This projection is the manifestation in first order in ε of the renormalization used in (2).

Furthermore, since for the injection- and absorption scheme we also have $G_1 = 0$, the vector V_0 in (10) and (12) becomes:

$$V_{0} = G_{0} F_{1}(P_{0}) = G_{0} Q(D P_{0})$$

= $G_{0} D P_{0} - (E^{T} D P_{0}) G_{0} P_{0}$ (16)
= $G_{0} D P_{0} - (E^{T} G_{0} D P_{0}) P_{0} = Q(G_{0} D P_{0})$

with Q being the projector given in (13). Here we have used that $E^T G_0 = E^T$ and $G_0 P_0 = P_0$. This small calculation also shows that the projection operator can be applied before or after multiplying G_0 to $D P_0$.

2.4.3 Application to the sensitivity

In this case we have $F(\varepsilon, P) = P$ such that $F_1(P) = 0$ and we have to determine G_1 from the expansion $G(\varepsilon) = G_0 + \varepsilon G_1 + \ldots$ Expanding (3) up to first order in ε we obtain:

$$\forall_{kl} \qquad (G_1)_{kl} = \delta_{ki}\delta_{lj} (G_0)_{kl} - \delta_{lj} (G_0)_{ij} \qquad (17)$$

where (i, j) is the pair of indices for which we want to compute the sensitivity. Using G_1 we compute $V_0 = G_1 P_0$ and solve the inhomogeneous PageRank equation (10) iteratively as described above to obtain P_1 . Once P_1 is know we can compute the sensitivity from :

$$D_{(j \to i)}(k) = \frac{P_1(k)}{P_0(k)} .$$
(18)

We note that equation (18) is numerically accurate and corresponds to the exact limit $\varepsilon \to 0$ while (4) is numerically not very precise and requires a finite small value of ε .

2.5 LIRGOMAX combined with REGOMAX

We consider the pump model described above and we take two particular nodes i with injection and j with absorption. For the diagonal matrix D we choose $D_i = 1/P_0(i)$ and $D_i = -1/P_0(j)$ where P_0 is the PageRank of the unperturbed network and all other values $D_k = 0$. In this way we have $e(P_0) = E^T D P_0 = D_i P_0(i) + D_j P_0(j) = 0.$ Due to this the renormalization denominator in (2) is simply unity and all excess probability provided by the injection at node i will be exactly absorbed by the absorption at node *j*. We insist that this is only due to our particular choice for the matrix D and concerning the numerical procedure one can also choose different values of D_i or D_i with $e(P_0) \neq 0$ (which would result in some global excess probability which would be artificially injected or absorbed due the normalization denominator in (2) being different from unity).

Using the above values of D_i and D_j we compute the vector $V_0 = G_0 D P_0 = G_0 W_0$ (since $E^T D P_0 = 0$) where $W_0 = D P_0$ is a vector with only two non-zero components $W_0(i) = 1$ and $W_0(j) = -1$. Therefore for

Table 1. Top 20 nodes of strongest negative values of P_1 (index number i = 1, ..., 20) and top 20 nodes of strongest positive values of P_1 (index number i = 21, ..., 40) with P_1 being created as the linear response of PageRank of English Wikipedia 2017 PageRank with injection (or pumping) at University of Cambridge and absorption at Harvard University; K_L is the ranking index obtained by ordering $|P_1|$ and K is the usual PageRank index obtained by ordering the PageRank probability P_0 of the global network with N nodes.

i	K_L	K	Node name
1	1	129	Harvard University
2	2	1608	Cambridge, Massachusetts
3	4	1	United States
4	5	296	Yale University
5	6	4617	Harvard College
6	7	62115	Harvard Yard
7	10	104359	Harvard Museum of Natural History
8	11	415	National Collegiate Athletic Ass.
9	12	52	The New York Times
10	13	75	American Civil War
11	14	7433	Harvard Medical School
12	15	73	American football
13	16	20994	Charles River
14	17	50	Washington, D.C.
15	18	23901	Harvard Divinity School
16	19	436	Massachusetts Institute of Tech.
17	20	88022	President and Fellows of Harvard Col.
18	21	128	Boston
19	22	42608	The Harvard Crimson
20	23	42259	Harvard Square
21	3	229	University of Cambridge
22	8	15	England
23	9	1414	Cambridge
24	69	1842	Trinity College, Cambridge
25	253	6591	St John's College, Cambridge
26	254	7022	King's College, Cambridge
27	256	285	Order of the British Empire
28	257	6	United Kingdom
29	258	33256	Newnham College, Cambridge
30	260	316	Church of England
31	262	238	The Guardian
32	263	21569	Clare College, Cambridge
33	264	4656	Durham University
34	265	191614	Regent House
35	266	3814	Chancellor (education)
36	267	16193	Gonville and Caius Col. Cambridge
37	270	25165	E. M. Forster
38	274	1650	Archbishop of Canterbury
39	277	2076	Fellow
40	278	73538	Colleges of the Univ. of Cambridge

all k we have $V_0(k) = (G_0)_{ki} - (G_0)_{kj}$. According to the above theory we know that V_0 and W_0 are orthogonal to E^T , i.e. $E^T V_0 = E^T W_0 = 0$ or more explicitely $\sum_k V_0(k) = \sum_k W_0(k) = 0$. For W_0 the last equality is obvious and the first one is due to the column sum normalization of G_0 meaning that $\sum_k (G_0)_{ki} = \sum_k (G_0)_{kj} = 1$. Using the expression $V_0(k) = (G_0)_{ki} - (G_0)_{kj}$ we determine the solution of the linear response correction to the PageRank P_1 by solving iteratively the inhomogeneous PageRank equation (10) as described above. The vector P_1 has real positive and negative entries also satisfying the condition $\sum_k P_1(k) = 0$. Then we determine the 20 top nodes with strongest negative values of P_1 and further 20 top nodes with strongest positive values of P_1 which constitute a subset of 40 nodes which are the most significant nodes participating in the pathway between the pumping node i and absorbing node j.

Using this subset we then apply the REGOMAX algorithm to compute the reduced Google matrix and its components which are analyzed in a similar way as in [11]. The advantage of the application of LIRGOMAX at the initial stage is that it provides an automatic procedure to determine an interesting subset of nodes related to the pumping between nodes i and j instead of using an arbitrary heuristic choice for such a subset.

The question arises if the initial two nodes i and jbelong themselves to the subset of nodes with largest P_1 entries (in modulus). From a physical point of view we indeed expect that this is generically the case but there is no simple mathematical argument for this. In particular for nodes with a low PageRank ranking and zero or few incoming links this is probably not the case. However, concerning the two examples which we will present in the next section both initial nodes i and j are indeed present in the selected subset and even with rather top positions in the ranking (provided by ordering $|P_1|$).

3 LIRGOMAX for Wikipedia network

As a concrete example we illustrate the application of LIR-GOMAX algorithm to the English Wikipedia network of 2017 (network data available at [9]). This network contains N = 5416537 nodes, corresponding to article titles, and $N_i = 122232932$ directed hyperlinks between nodes. Previous applications of the REGOMAX algorithm for the Wikipedia networks of years 2013 and 2017 are described in [11, 12].

3.1 Case of pathway Cambridge - Harvard Universities

As a first example of the application of the combined LIR-GOMAX and REGOMAX algorithms we select two articles (nodes) of the Wikipedia network with pumping at University of Cambridge and absorption at Harvard University. The global PageRank indices of these two nodes are K = 229 (PageRank probability $P_0(229) = 0.0001078$) and K = 129 (PageRank probability $P_0(129) = 0.0001524$). As described above we chose the diagonal matrix D as $D(229) = 1/P_0(229)$ and $D(129) = -1/P_0(129)$ (other diagonal entries of D are chosen as zero) and determine the vector V_0 used for the computation of P_1 (see (12)) by $V_0 = G_0 W_0$ where the vector $W_0 = D P_0$ has the nonzero components $W_0(229) = 1$ and $W_0(129) = -1$. Both W_0 and V_0 are orthogonal to the left leading eigenvector $E^T = (1, \ldots, 1)$ of G_0 according to the theory described in the last section.

The subset of 40 most affected nodes with 20 strongest negative and 20 strongest positive values of the linear response correction P_1 to the initial PageRank P_0 are given in Table 1. We order these 40 nodes by the index i = 1, ..., 20 for the first 20 most negative P_1 values and then i = 21, ..., 40 for the most positive P_1 values. The index K_L is obtained by ordering $|P_1|$ for all $N \approx 5 \times 10^6$



Fig. 1. Linear response vector P_1 of PageRank for the English Wikipedia 2017 with injection (or pumping) at University of Cambridge and absorption at Harvard University. Here K_L is the ranking index obtained by ordering $|P_1|$ from maximal value at $K_L = 1$ down to its minimal value. Top panel shows $|P_1|$ versus K_L in a double logarithmic representation for all N nodes. Bottom panel shows a zoom of P_1 versus K_L for $K_L \leq 10^3$ in a double logarithmic representation with sign; blue data points correspond to $P_1 > 0$ and red data points to $P_1 < 0$.

network nodes. The table also gives the PageRank index K obtained by ordering P_0 . The first 4 positions in K_L are taken by Harvard University; Cambridge, Massachusetts; University of Cambridge; United States. Thus, even if the injection is made for University of Cambridge the strongest response appears for Harvard University; Cambridge, Massachusetts and only then for University of Cambridge ($K_L = 1, 2, 3$). We attribute this to nontrivial flows existing in the global directed network. This shows that the linear response approach provides rather interesting information about the sensitivity and interactions of nodes on directed networks. We will see below for other examples that the top nodes of the linear response vector P_1 can have rather unexpected features.

In general the most sensitive nodes of Table 1 are rather natural. They represent countries, cities and other administrative structures related to the two universities. Other type of nodes are *Yale University*, *The New York*



Fig. 2. Reduced Google matrix components $G_{\rm R}$, $G_{\rm pr}$, G_{rr} and $G_{\rm qr}$ for the English Wikipedia 2017 network and the subgroup of nodes given in Table 1 corresponding to injection at University of Cambridge and absorption at Harvard University (see text for explanations). The axis labels correspond to the index number i used in Table 1. The relative weights of these components are $W_{\rm pr} = 0.920$, $W_{\rm rr} = 0.036$, and $W_{\rm qr} = 0.044$. Note that elements of G_{qr} may be negative. The values of the color bar correspond to $\operatorname{sgn}(g)(|g|/\max|g|)^{1/4}$ where g is the shown matrix element value. The exponent 1/4 amplifies small values of g for a better visibility.

Times, American Civil War for Harvard U and Church of England, The Guardian, Durham University for U Cambridge (in addition to many Colleges presented in the list) corresponding to closest other universities and also newspapers appearing on the pathway between the pair of selected nodes.

Of course, the linear response vector P_1 extends on all N nodes of the global network. We show its dependence on the ordering index K_L in Figure 1. Here the top panel represents the decay of $|P_1|$ with K_L and the bottom panel shows the decay of negative and and positive P_1 values for $K_L \leq 10^3$. We note that among top 100 values of K_L there are only 4 nodes related to U Cambridge with positive P_1 values while all other values of P_1 are negative being related to Harvard U. This demonstrates a rather different structural influence between these two universities.

After the selection of 40 most significant nodes of the pathway between both universities (see Table 1) we apply the REGOMAX algorithm which determines all matrix elements of Markov transitions between these 40 nodes including all direct and indirect pathways via the huge global Wikipedia network with 5 million nodes.

The reduced Google matrix $G_{\rm R}$ and its three components $G_{\rm pr}$, G_{rr} , $G_{\rm qr}$ are shown in Figure 2. As discussed above the weight $W_{\rm pr} = 0.920$ of $G_{\rm pr}$ is close to unity and its matrix structure is rather similar to the one of $G_{\rm R}$ with strong transition lines of matrix elements corresponding to



Fig. 3. Same as in Fig. 2 but for the matrix $G_{rr} + G_{qr}^{(nd)}$, where $G_{qr}^{(nd)}$ is obtained from G_{qr} by putting its diagonal elements at zero; the weight of these two components is $W_{rr+qrnd} = 0.066$.

United States at top PageRank index K = 1 $(i = 3, K_L = 4)$ and United Kingdom at K = 6 $(i = 28, K_L = 257)$. The weights $W_{\rm rr} = 0.036$, $W_{\rm qr} = 0.044$ of G_{rr} , $G_{\rm qr}$ are significantly smaller. These values are similar to those obtained in the REGOMAX analysis of politicians and universities in Wikipedia networks [11,12]. Even if the weights of these matrix components are not large they represent the most interesting and nontrivial direct (G_{rr}) and indirect $(G_{\rm qr})$ interactions between the selected 40 nodes. The image of G_{rr} in Figure 2 shows that the direct links between the U Cambridge block of nodes (with index $21 \le i \le 40$ in Table 1) and the Harvard U block of nodes (with index $1 \le i \le 20$ in Table 1) are rather rare and relatively weak while the links within each block are multiple and relatively strong. This confirms the appropriate selection of nodes in each block provided by the LIRGOMAX algorithm.

The matrix elements of the sum of two components $G_{rr}+G_{\rm qr}{}^{(nd)}$ (component $G_{\rm qr}$ is taken without diagonal elements) are shown in Figure 3. We note that some elements are negative which is not forbidden since only the sum of all three components given by $G_{\rm R}$ should have positive matrix elements. However, the negative values are rare and relatively small compared to the values of positive matrix elements. Thus the minimal value is -0.00216 for the transition from Church of England to United States while other typical negative values are smaller by a factor 5-10. For comparison, the maximal value of positive element is 0.1135 from Regent House to University of Cambridge and there are many other positive elements of the order of 0.03. Thus we consider that the negative elements play no significant role. A similar conclusion was also obtained for the interactions of politicians and universities in [11, 12].



Fig. 4. Network of friends for the subgroup of nodes given in Table 1 corresponding to injection at University of Cambridge and absorption at Harvard University constructed from the matrix $G_{rr} + G_{qr}^{(nd)}$ using 4 top (friends) links per column (see text for explanations). The numbers used as labels for the different nodes correspond to the index *i* of Table 1.

From Figure 3 we see that for $G_{rr} + G_{qr}^{(nd)}$ the strongest interactions are also inside each university block. However, there are still some significant links between blocks with strongest matrix elements being 0.0120 from *Fellow* to *United States* and 0.0050 from *Harvard University* to *University of Cambridge* (for both directions between blocks). The link *Fellow* to *United States* is also the strongest indirect link in its off diagonal sub-block (for G_{qr}) while the strongest direct link (for G_{rr}) is *Fellow* to *Harvard University of Cambridge* is also the strongest indirect link in its off diagonal sub-block (for G_{qr}) while the strongest direct link (for G_{rr}) is *Harvard University* to *University of Cambridge* is also the strongest indirect link in its off diagonal sub-block (for G_{qr}) while the strongest direct link (for G_{rr}) is *Harvard College* to *University of Cambridge*.

Using the transition matrix elements of $G_{rr} + G_{qr}^{(nd)}$ we construct a network of effective friends shown in Figure 4. First, we select five initial nodes which are placed on a (large) circle: the two nodes with injection/absorption (University of Cambridge and Harvard University) and three other nodes with a rather top position in the K_L ranking (England, (Town of) Cambridge and United States). For each of these five initial nodes we determine four friends by the criterion of largest matrix elements (in modulus) in the same column, i.e. corresponding to the four strongest links from the initial node to the potential friends. The friend nodes found in this way are added to the network and drawn on circles of medium size around their initial node (if they do not already belong to the initial set of 5 top nodes). The links from the initial nodes to their friends are drawn as thick black arrows. For each of the newly added nodes (level 1 friends) we continue to determine the four strongest friends (level 2 friends) which are drawn on small circles and added to the network (if there are not already present from a previous level). The corresponding links from level 1 friends to level 2 friends are drawn as thin red arrows.

Each node is marked by the index i from the first column of Table 1. The colors of the nodes are essentially red for nodes with strong negative values of P_1 (corresponding to the index i = 1, ..., 20) and blue for nodes with strong positive values of P_1 (for i = 21, ..., 40). Only for three of the initial nodes we choose different colors which are olive for US, green for England and cyan for (the town of) Cambridge.

The network of Figure 4 shows a quite clear separation of network nodes in two blocks associated to the two universities with a rather small number of links between the two blocks (e.g. US is a friend of England but not vice-versa).

3.2 Case of pathway Napoleon - Alexander I of Russia

We illustrate the application of the LIRGOMAX and RE-GOMAX algorithms on two other nodes of the Wikipedia network with injection (pumping) at Napoleon and absorption at Alexander I of Russia. The global PageRank indices of these two nodes are K = 201 (PageRank probability $P_0(201) = 0.0001188$) and K = 5822 (PageRank probability $P_0(5822) = 1.389 \times 10^{-5}$) respectively. In contrast to the the previous example the two PageRank probabilities are rather different. However, this difference is compensated by our choice of the diagonal matrix Dwith $D(201) = 1/P_0(201)$ and $D(5822) = -1/P_0(5822)$ (other diagonal entries of D begin zero). Again we determine the vector V_0 used for the computation of P_1 (see (12)) by $V_0 = G_0 W_0$ where the vector $W_0 = D P_0$ has the nonzero components $W_0(201) = 1$ and $W_0(5822) = -1$. Furthermore, both W_0 and V_0 are orthogonal to the left leading eigenvector $E^T = (1, \ldots, 1)$ of G_0 .

The top nodes of P_1 , noted by index *i*, with 20 strongest negative and 20 strongest positive values are presented in Table 2. The ranking of nodes in decreasing order of $|P_1|$ given by the index K_L is shown in the second column of Table 2. It is interesting to note that the injection node Napoleon is only at position $K_L = 29$ with a significantly smaller value of $|P_1|$ compared to Alexander I of Russia at $K_L = 3$, Russian Empire at $K_L = 1$ and Saint Petersburg at $K_L = 2$. But among the positive P_1 values Napoleon is still at the first position. We attribute this relatively small $|P_1|$ value of Napoleon compared to the nodes of the other block to significant complex directed flows in the global Wikipedia network. Also Napoleon has a significantly stronger PageRank probability and thus this node produces a stronger influence on Alexander I of Russia than vice-versa.

In contrast to the previous case of universities Table 2 contains mainly countries, a few towns and islands, and

Table 2. Same as in Table 1 for English V	Vikipedia	2017 with
injection (pumping) at Napoleon and abso	rption at	Alexander
I of Russia.		

i	K_L	K	Node name
1	1	181	Russian Empire
2	2	216	Saint Petersburg
3	3	5822	Alexander I of Russia
4	4	15753	Paul I of Russia
5	5	3409	Catherine the Great
6	6	158	Moscow
7	7	17	Russia
8	8	153	Azerbaijan
9	9	9035	Nicholas I of Russia
10	10	203707	Elizabeth Alexeievna (Louise of Baden)
11	11	92	Ottoman Empire
12	12	7	Iran
13	13	177854	Government reform of Alexander I
14	14	889	Caucasus
15	15	31784	Russo-Persian War (1804–13)
16	16	8213	Alexander II of Russia
17	17	475	Prussia
18	18	5764	Dagestan
19	19	32966	Serfdom in Russia
20	20	131205	Adam Jerzy Czartoryski
21	29	201	Napoleon
22	52	192	French Revolution
23	144	4	France
24	149	12	Italy
25	167	10611	French Directory
26	180	24236	Joséphine de Beauharnais
27	188	7361	National Convention
28	189	21727	French campaign in Egypt and Syria
29	195	2237	Corsica
30	198	3166	Louis XVI of France
31	199	7875	Saint Helena
32	200	40542	André Masséna
33	201	3353	French Revolutionary Wars
34	203	11916	Maximilien Robespierre
35	204	1241	Louvre
36	205	69382	Lucien Bonaparte
37	206	21754	Coup of 18 Brumaire
38	207	15926	French Republican Calendar
39	208	14931	Jacobin
40	209	7509	Napoleonic Code

historical figures related in some manner to Napoleon or Alexander I of Russia.

The dependence of the linear response vector P_1 on the index K_L is shown in Figure 5 (analogous to Figure 1). The decay of $|P_1|$ with K_L is shown in the top panel, being similar to the top panel of Figure 1. The values of P_1 with sign are shown in the bottom panel. The difference of the $|P_1|$ values for Napoleon and Alexander I of Russia is not so significant but many nodes (28) from the block of Alexander I of Russia have larger $|P_1|$ values than $|P_1|$ of Napoleon.

The results for the reduced Google matrix of 40 nodes of Table 2 are shown in Figure 6. The strongest lines of transitions in $G_{\rm R}$ and $G_{\rm pr}$ correspond to nodes with top PageRank positions of the global Wikipedia network being *France* at K = 4 (i = 23, $K_L = 144$), *Iran* at K = 7 (i =12, $K_L = 12$), *Italy* at K = 12 (i = 24, $K_L = 149$) and *Russia* at K = 17 (i = 7, $K_L = 7$). As explained above the structure of transitions appears rather similar between $G_{\rm R}$ and $G_{\rm pr}$. The weights of all three components $G_{\rm pr}$, $G_{\rm rr}$, $G_{\rm qr}$ are similar to those of the two universities (see caption of Figure 6).



Fig. 5. Same as in Fig. 1 for the subgroup of Table 2 corresponding to injection at *Napoleon* and absorption at *Alexander I of Russia*.



Fig. 6. As Fig. 2 for the subgroup of Table 2 corresponding to injection at *Napoleon* and absorption at *Alexander I of Russia*. The relative weights of the different matrix components are $W_{\rm pr} = 0.900$, $W_{\rm rr} = 0.042$ and $W_{\rm qr} = 0.058$.



Fig. 7. Same as in Fig. 3 for the subgroup of Table 2 corresponding to injection at *Napoleon* and absorption at *Alexander I of Russia*. The weight of $G_{rr} + G_{qr}^{(nd)}$ is $W_{rr+qrnd} = 0.087$.

The components G_{rr} and G_{qr} , shown in Figure 6, are also dominated by the two diagonal blocks related to the two initial nodes Napoleon and Alexander I of Russia. There are only a few direct links between the two blocks but the number of indirect links is substantially increased. The sum of these two components $G_{rr} + G_{qr}^{(nd)}$ is shown in Figure 7, where the diagonal elements of G_{qr} are omitted. The strongest couplings between the two blocks in $G_{rr} + G_{qr}^{(nd)}$ are 0.009879 for the link from French campaign in Egypt and Syria to Ottoman Empire and 0.02439 for the link from Elizabeth Alexeievna (Louise of Baden) to Napoleon (for both directions between the diagonal blocks).

In analogy to Figure 4 we construct the network of friends for the subset of Table 2 shown in Figure 8. As in Figure 4, we use the four strongest transition matrix elements of $G_{rr} + G_{qr}^{(nd)}$ per column to construct links from the five top nodes to level 1 friends (thick black arrows) and from level 1 to level 2 friends (thin red arrows). As the five initial top nodes we choose *France* (cyan), *Russian Empire* (olive), *Saint Petersburg* (green), *Napoleon* (blue) and *Alexander I of Russia* (red); all other nodes of the *Napoleon* block ($21 \le i \le 40$ in Table 2) are shown in blue, and all other nodes of the *Alexander I of Russia* block ($1 \le i \le 20$ in Table 2)) are shown in red; numbers inside the points correspond to the index *i* of Table 2.

The network of Figure 8 also shows a clear two block structure with relatively rare links between the two blocks. The coupling between two blocks appears due to one link from *Alexander I of Russia* to *Prussia*, which even being red is more closely related to the blued block of nodes.

For both examples network figures constructed in the same way using the other matrix components $G_{\rm R}$, G_{rr} or $G_{\rm qr}$ (instead of $G_{rr} + G_{qr}^{(nd)}$) or using strongest matrix el-



Fig. 8. Same as in Fig. 4 for the subgroup of Table 2 corresponding to injection at *Napoleon* and absorption at *Alexander I of Russia*.

ements in rows (instead of columns) to determine follower networks are available at [16].

This type of friend/follower effective network schemes constructed from the reduced Google matrix (or one of its components) were already presented in [11] in the context of Wikipedia networks of politicians.

4 Discussion

We introduced here a linear response theory for a very generic model where either the Google matrix or the associated Markov process depends on a small parameter and we developed the LIRGOMAX algorithm to compute efficiently and accurately the linear response vector P_1 to the PageRank P_0 with respect to this parameter for large directed networks. As a particular application of this approach it is in particular possible to identify the most important and sensitive nodes of the pathway connecting two initial groups of nodes (or simply a pair of nodes) with injection or absorption of probability. This group of most sensitive nodes can then be analyzed with the reduced Google matrix approach by the related REGOMAX algorithm which allows to determine effective indirect network interactions for this set of nodes. We illustrated the efficiency of the combined LIRGOMAX and REGOMAX algorithm for the English Wikipedia network of 2017 with two very interesting examples. In these examples, we use two initial nodes (articles) for injection/absorption, corresponding either to two important universities or to two related historical figures. As a result we obtain associated sets for most sensitive Wikipedia articles given in Tables 1 and 2 with effective friend networks shown in Figures 4 and 8.

As a further independent application the LIRGOMAX algorithm allows also to compute more accurately the Page-Rank sensitivity with respect to variations of matrix elements of the (reduced) Google matrix as already studied in [12, 15].

It is known that the linear response theory finds a variety of applications in statistical and mesoscopic physics [1,3], current density correlations [4], stochastic processes and dynamical chaotic systems [2,5]. The matrix properties and their concepts, like Random Matrix Theory (RMT), find important applications for various physical systems (see e.g. [18]). However, in physics one usually works with unitary or Hermitian matrices, like in RMT. In contrast the Google matrices belong to another class of matricies rarely appearing in physical systems but being very natural to the communication networks developed by modern societies (WWW, Wikipedia, Twitter ...). Thus we hope that the linear response theory for the Google matrix developed here will also find useful applications in the analysis of real directed networks.

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