Shnirelman theorem

This article has not yet been published; it may contain inaccuracies, unapproved changes, or be unfinished.

- **Prof. Alexander Shnirelman**, Dept. of Mathematics & Statistics, Concordia University, Montreal, Canada

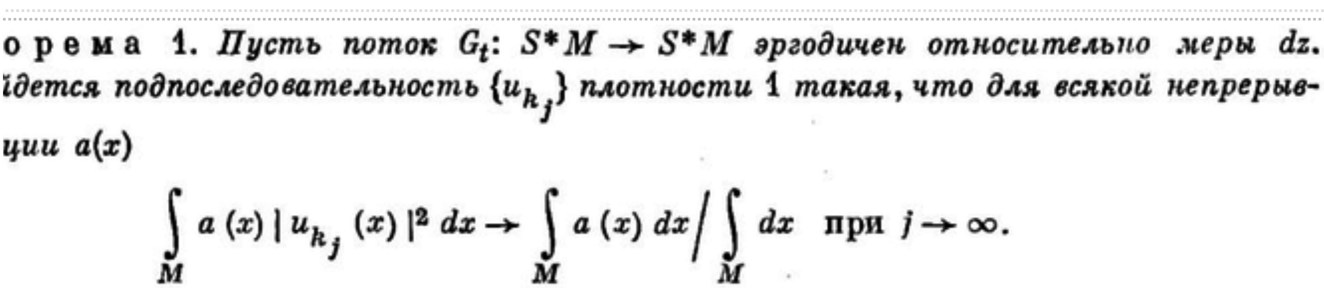
Prof. Alexander Shnirelman accepted the invitation on 19 July 2009 (self-imposed deadline: 19 July 2010).

Shnirelman theorem refers to the asymptotic properties of eigenfunctions of the Schroedinger operator in case of a classically chaotic system. It says that for almost all eigenvalues the probability of finding the system in a vicinity of a given classical state is uniformly distributed along the surface of constant energy in the phase space.

The text below is written by Scholarpedia Editor Dima Shepelyansky in May 2020.

Contents

Theorem formulation



орема 1. *Пусть поток* G_t: $S^*M \to S^*M$ *эргодичен относительно меры* dx. *ідется подпоследовательность* $\{u_{k_j}\}$ *плотности* 1 *такая, что для всякой непреры- ции* $a(x)$

$$\int_M a(x)\,|u_{k_j}(x)|^2\,dx \to \int_M a(x)\,dx \Big/ \int_M dx \quad \text{при } j \to \infty.$$

al Shnirelman theorem from the related article in Russian (from Shnirelman, 1974).

Shnirelman theorem: Assume that the geodesic flow G_t on volume M is ergodic with measure dx. Let u_{k_j} be an orthonomal basis of eigenfunctions of Laplace operator Δ on M. Then there exists some sub-sequence u_{k_j} of unit density that for any continuous function $a(x)$ we have $\int_M a(x)|u_{k_j}(x)|^2 dx \to \int_M a(x)dx / \int_M dx$ for $j \to \infty$ (from Shnirelman, 1974, see also Fig.1).

The details of theorem proof are available at Shnirelman, 1993. This theorem generated several quantum ergodicity theorems. The arguments are made more precise in Colin de Verdiere, 1985,Zeldich, 1987. They are then extended to the class of manifolds with boundary by Gerard and Leichtnam, 1993, to the semi-classical regime by Zeldich and Zworski, 1996 and to the case of discontinuous metrics by Jakobson et al., 2015.

Various mathematical extentions, applications and links with other theorems (e.g the Egorov theorem Egorov, 2015) are analyzed and discussed by Colin de Verdiere et al., 2018, Anantharaman, 2019,Toulouse lecture notes, 2015.

Physical interpretation

From a physical view point, in the limit of small effective values of Planck constant $\hbar$, the Bohr correspondence principle (Bohr, 1920) and the Ehrenfest theorem (Ehrenfest, 1927) implies that a quantum narrow wave packet will follow a chaotic classical trajectory, or better to say a Liouville packet corresponding to initial quantum packet, during a long time scale. Thus it can be expected that the quantum evolution, and thus quantum eigenstates, will be ergodic, in agreement with the Shnirelman theorem, as pointed by Chirikov et al., 1981. However, due to exponential instability of chaotic dynamics the quantum packet follows the classical one only during a relatively short Ehrenfest time $t_E \sim |\ln \hbar|/\Lambda$ where Λ is the Lyapunov exponent of chaotic classical motion (see Chirikov et al., 1981,Chirikov et al., 1988 and discussion of the Ehrenfest time scale at Chirikov and Shepelyansky, 2008). It should be pointed that the problem of semiclassical quantization of nonintegrable systems had been rosen by Einstein (Einstein, 1917) in view of the Poincare theorem (Poincare, 1890) showing that generic classical Hamiltonian systems are not integrable.

From the Shnirelman theorem it follows that the eigenstates of chaotic billiards are ergodic and it is possible to expect that in such quantum billiard the level spacing statistics is described by the Random Matrix Theory (RMT). This is confirmed by the Bohigas-Giannoni-Schmit conjecture which demonstrated the validity of RMT for generic chaotic billiards with detailed numerical simulations of a quantum Sinai billiard (see Bohigas et al., 1984 and Ullmo, 2016).

Below we consider an example of rough billiards for which we establish the quantum chaos (or ergodicity) border from which the Shnirelman theorem becomes valid. Other physical properties of eigenstates are also analysed.

Physical example of rough billiards

We discuss here an example of quantum rough billiard introduced and analysed by Frahm and Shepelyansky, 1997a,Frahm and Shepelyansky, 1997b. The rough billiard is obtained by a deformation of an elastic circle of radius R_0. The deformed boundary is $R(\theta) = R_0 + \Delta R(\theta)$ with $\Delta R(\theta)/R_0 = \mathrm{Re}\ \sum_{m=2}^{M} \gamma_m\, e^{im\theta}$. Here γ_m are random complex coefficients, M is large but finite, θ is circle angle. Then the surface roughness is given by $\kappa(\theta) = (dR/d\theta)/R_0$. The analysis is done for the case of weak roughness with $\kappa \ll 1$ with $\gamma_m \sim 1/m$. The domain of strong chaos with the classical diffusion and quantum localization in orbital momentum space is determined by the average roughness $\tilde{\kappa}^2 = \langle \kappa^2(\theta) \rangle_\theta \sim M(\Delta R/R_0)^2$. In this regime the ray dynamics is approximately described by the symplectic rough map $\bar{l} = l + 2\sqrt{l_{max}^2 - l_r^2}\,\kappa(\theta),\ \bar{\theta} = \theta + \pi - 2\arcsin(\bar{l}/l_{max})$. Here the first equation gives the change of orbital momentum from l to $\bar{l}$ due to collision with boundary and the second one gives the angle change between collisions (also mass and Planck constant are $m = \hbar = 1$). The map describes the dynamics in a vicinity of a resonant value l_r defined by the condition $\bar{\theta} = \theta + 2\pi r$ with integer r and l_{max} is the maximal momenum at a given particle velocity (or energy). The global chaos on the energy surface sets in above some critical value roughness $\tilde{\kappa} > \kappa_c$. Below κ_c the Kolmogorov-Arnold-Moser theory (KAM) is valid and the phase space is divided by isolated invariant curves. The chaos border can be estimated on the basis of Chirikov criterion of overlapping resonances (Chirikov, 1979) which gives $\kappa_c \sim 4M^{-5/2}$ (the numerical coefficient is extracted from the data for $M = 20$). This border drops strongly with M and therefore the analysis is done for the regime of strong chaos without visible islands of stability. In this regime the spreading in angular momentum space is diffusive with the diffusion constant $D = (\Delta l)^2/\Delta t = 4\,({l_{max}}^2 - {l_r}^2)\,\tilde{\kappa}^2$ where time is measured in number of collisions. The physical time of diffusive spreading over the whole energy surface is $\tau_D \approx \tau_c\, l^2_{max}/D$ with $\tau_c \sim R_0/v$ where the particle velocity v at energy $E = v^2/2 = l^2_{max}/2$. According to the Weyl law the level number at energy E is $N \approx mR_0^2E/2\hbar^2 = {l_{max}}^2/4$.

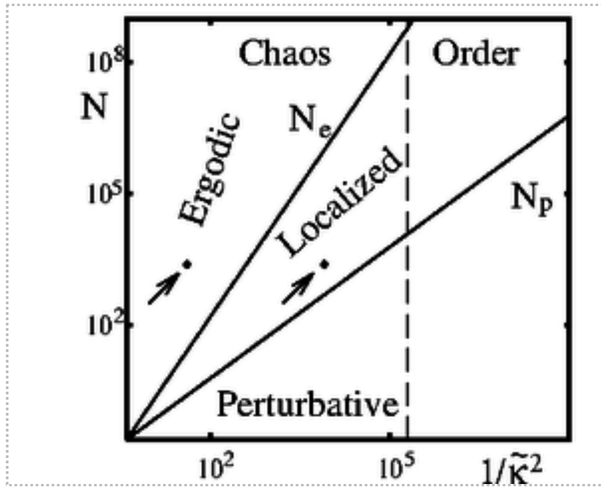


Figure 2: Global diagram showing the eigenstate properties for different values of level number N and roughness $\tilde{\kappa}$ for high diffusion case ($D \gg M$). The two full lines give the ergodic (N_e) and perturbative (N_p) borders, the dashed line is the classical chaos border $\kappa_c \approx 0.002$ for $M = 20$; arrows show cases tested numerically (from Frahm and Shepelyansky, 1997a).

Dynamical localization of eigenstates

The effects of quantum interference can lead to localization of the diffusive spreading over the energy surface. This phenomenon is similar to the Anderson localization of diffusion in disordered solid state systems (see Anderson, 1958, the review of this phenomenon at Akkermans and Montambaux, 2007 and Anderson localization and quantum chaos maps, Chirikov standard map). For the rough billiards this phenomenon leads to the exponensial localization of eigenstates with the exponential decay of orbital harmonics a_l on the energy surface with $|a_l| \propto \exp(-|l - l_0|/\ell)$ where ℓ is the localization length and l_0 certain value of orbital momentum.

According to the results obtained by Chirikov et al., 1981,Shepelyansky, 1987,Chirikov et al., 1988,Frahm and Shepelyansky, 1997a the localization length is $\ell = D = 4\,(l^2_{max} - l^2_r)\,\tilde{\kappa}^2$ corresponding to the case of local unitary symmetry at $D > M$, $1 < D \ll l_{max}$. The quantum ergodicity is established for $\ell > l_{max}$. Thus the Shnirelman theorem and quantum ergodicity of eigenstates take place for quantum level numbers with $N > N_e \approx \frac{1}{64\,\tilde{\kappa}^4}$. This border is much higher than the perturbative border $N < N_p \approx 1/(16\tilde{\kappa}^2)$ where the diffusion mixes less than one state ($D \approx 1$). The phase diagram of eigenstate properties is shown in Fig.2.

The properties of eigenstates can be extracted from their expansion coefficients $C^{(\alpha)}_{nl}$ in the basis of eigenstates of a circular billiard with orbital and radial quantum numbers l, n. The typical structure of eigenstates is shown in Fig.3 for the localized regime $N_p < N < N_e$ (a), the Wigner ergodicity regime (b) and the quantum ergodicity of Shnirelman theorem $N > N_e$ (c). The eigenstates are mainly located on the energy surface being close to those of a circle and determined from the the Bohr-Sommerfeld quantization

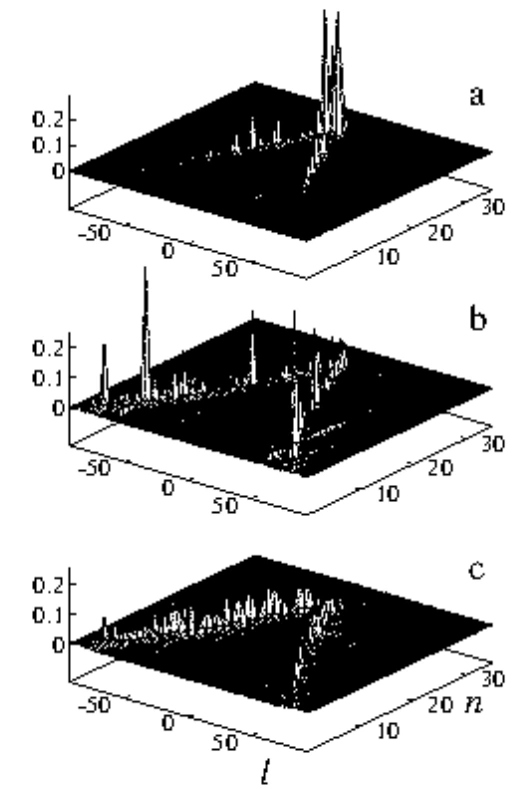


Figure 3: Transition from localization to Shnirelman ergodicity on energy surface for level number $N \approx 2250$, $l_{max} \approx 95$ and $M = 20$; shown are the absolute amplitudes $|C_{nl}^{(\alpha)}|$ of one eigenstate: (a) localization for $D(l_r = 0) = 20$; (b) Wigner ergodicity for $D = 80$; (c) Shnirelman ergodicity for $D = 1000 >$ (from Frahm and Shepelyansky, 1997b).

$\mu_l(E) = \sqrt{l_{max}^2 - l^2} - l \arctan(l^{-1}\sqrt{l_{max}^2 - l^2}) + \pi/4 = \pi(n+1)$. The energy surface $H(n,l) = E_\alpha$ is shown in Fig.4. For the localized regime eigenstates are exponentially localized on the energy surface. For the case of Shnirelman ergodicity they are distributed over the whole energy surface with fluctuations. In this regime the level spacing statisitics corresponds to the RMT case (see below). The interesting intermediate regime of Wigner ergodicity has amplitudes speading over the whole energy surface but having very peaked structure. It appears due to a finite Breit-Wigner width $\Gamma_\mu = 2\pi\rho_\mu < (V(\theta)/2)^2 > \approx 3D/(2M^2)$ which allows to populate only those integer quantum numbers n, l which are sufficiently close to the energy surface curve (see Fig.4). Here $V(\theta) = 2\sqrt{l_{max}^2 - l_r^2}\,\Delta R(\theta)/R_0$ appears from the effective kick potential of the ray map. In the regime of Wigner ergodicity the Breit-Wigner width is small and only specific integers n, l, which are especially close to the energy surface, contribute to the eigenstate leading to delocalized but peaked structure of eigenstates. Of course at larger values of Γ_μ eigenstates are homogeneously (but with fluctuations) distributed over the whole energy surface corresponding to the Shnirelman ergodicity. More details are given in Frahm and Shepelyansky, 1997b.

Shnirelman peak in level spacing statistics

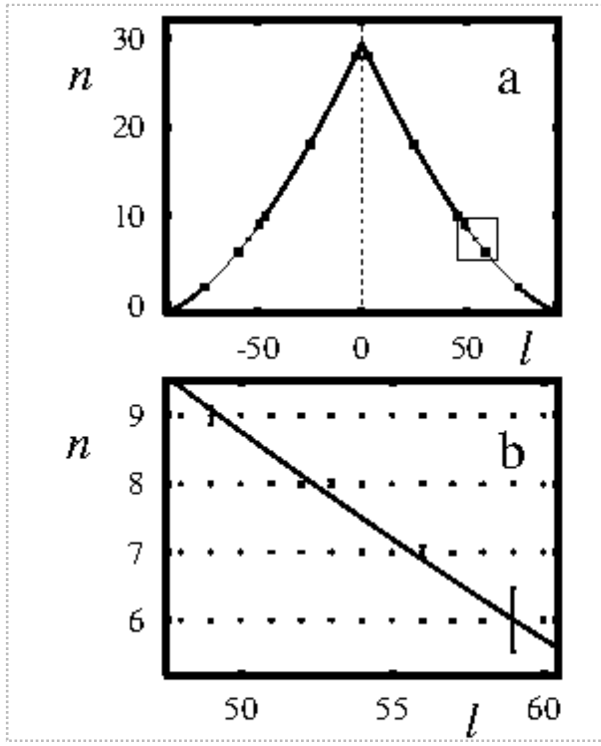


Figure 4: (a) Main peaks of eigenstate in Fig. 2b (squares for $|C_{nl}^{(\alpha)}| \geq 0.1$) shown on the energy surface $H(n,l) = E_\alpha$; (b) rescaled part of (a): diamonds show the integer (n,l)-lattice, the error-bar size is $2|C_{nl}^{(\alpha)}|$ (from Frahm and Shepelyansky, 1997b).

The energy level statistics is one of the most important and well studied characteristics of quantum systems. Particularly, it is commonly assumed (Bleher et al., 1993) that in the limit of classically completely integrable systems the distribution of nearest-neighbor level spacings is Poissonian as for independent levels (Berry and Tabor, 1977). In the opposite limit of classically chaotic systems this distribution is characterized by level repulsion and the RMT law (Bohigas et al., 1984,Haake, 2010).

However, this picture is in a sharp contradiction with the Shnirelman theorem of 1975 for integrable flows (Shnirelman, 1975). This theorem states that for a classically nearly integrable system at least each second level spacing in the corresponding quantum system becomes exponentially small in the quasiclassical domain. This would imply a big narrow peak in the distribution of nearest-neighbor level spacings (level clustering). This result is especially surprising as no special symmetry was assumed in a particular model considered by Shnirelman. However, the time reversal symmetry holds in such a model. Formaly the theorem states that the spectrum λ_k is asymptotically multiple, i.e. for each $L > 0$ there exists $C_L > 0$ such that $min(\lambda_k - \lambda_{k-1}, \lambda_{k+1} - \lambda_k) < C_L \lambda_k{}^{-M}$. In the first formulation the theorem had been proved for a geodesic flow on a two-dimensional torus (some nearly integrable billiards) while in the second formulation its applicability had been extended to a broader class of two-dimensional nearly integrable systems with at least 4 invariant tori (see Shnirelman, 1975 with details in Shnirelman, 1993). The physical interpretation of this theorem was given by Chirikov and Shepelyansky, 1995. It is based on the conception of quasiclassical degeneracy destroyed by tunneling. Similar phenomena in presence of spatial symmetry were studied in many papers (see e.g. Bohigas et al., 1993 and Refs. therein) but the effect of time reversibility on level statistics in absence of spatial symmetry was not considered. In some sense the degeneracy between the states connected by time reversal symmetry is destroyed by **tunneling between the future and the past**. Such situation corresponds to a double well in the momentum space.

An example of the Shnirelman peak was illustrated for the kicked rotator model with asymmetric kick potential $V(\theta) = k(\cos\theta - 0.5\sin 2\theta)$ (Chirikov and Shepelyansky, 1995). The peak in $p(s)$ statistics exists in the KAM phase of the corresponding classical map. However, even in the classically chaotic regime the peak still exists if the quantum eigenstates are localized (so that the localization length is small compared to the system size

$\ell \sim D \sim k^2 \ll N$, see above). Thus more general conditions for the appearance of the Shnirelman peak have been proposed by Chirikov and Shepelyansky, 1995: first, the quantum system must have a discrete symmetry, e.g. time reversibility, and second, the states with opposite symmetry must be separated in the phase space either classically (as for the KAM case) or quantum mechanically (as for the case of quantum localization).

Here we show in Fig.5 the level spacing statistics $p(s)$ for rough billiard. In the case of Shnirelman ergodicity (a) the statistics corresponds to the RMT or Wigner-Dyson distribution while for the case of quantum localization of eigenstates at $\ell \ll l_{max}$ the time reversal symmetry leads to the Shnirelman peak due to quasi-degeneracy of eigenstates with opposite values of orbital momentum. The measure of spacings in peak is $\alpha \approx 0.33$ and the fraction of the nondegenerate levels is $1 - 2\alpha \approx 0.34$ due to states localized near $l_0 \approx 0$ (Frahm and Shepelyansky, 1997a).

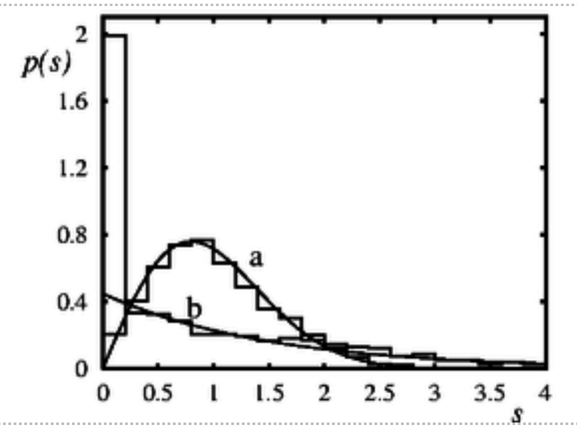


Figure 5: Level spacing distribution $p(s)$ for a rough billiard with $l_{max} \approx 100$, $M = 20$ and average roughness: a) $\tilde{\kappa} \approx 0.16$ (the case of Shnirelman ergodicity with $\ell \approx 1000$) and b) $\tilde{\kappa} \approx 0.012$ (the case of Shnirelman peak appearing due to quantum localization with $\ell \approx 6$). The total statistics is 2000 levels $2150 < N < 2350$) from 10 different realizations of the rough boundary. Also shown are the Wigner-Dyson and rescaled Poisson distributions with effective measure peak spacings $\alpha = 0.33$ (from Frahm and Shepelyansky, 1997a).

Related experiments

The transition from dynamical localization to the Shnirelman ergodicity was observed in experiments with rough billiards. These billiards were realized with rough microwave cavities by Sirko et al., 2000 and with chaotic microlares of microdisk resonator with rough boundary by Podolskiy et al., 2004, Fang et al., 2005. For microlasers it was shown that localized modes lead to a good lasing action. A direct experimental detection of the Shnirelman peak represents an experimental challange due to quasi-degeneracy of eigenstates inside the peak.

References

- Akkermans E. and Montambaux G. (2007). *Mesoscopic physics of electrons and photons*, Cambridge University Press, Cambridge, UK.
- Anantharaman N. (2019). *Delocalization of Schrodinger eigenfunctions*, Proc. Int. Congress Math., Rio de Janeiro 2018, Eds. Sirakov Boyan, Souza Paulo Ney De, Viana Marcelo, World Sci., Singapore **1**, 341.
- Anderson P.W. (1958). *Absence of diffusion in certain random lattices*, Phys. Rev. **109**, 1492.
- Berry M. and Tabor M. (1977). *Level clustering in the regular spectrum*, Proc. R. Soc. Lond. A **356**, 375.
- Bleher P.M., Dyson F.J. and Lebowitz J.L. (1993). *Non-Gaussian energy level statistics for some integrable systems*, Phys. Rev. Lett. **71**, 3047.
- Bohigas O., Giannoni M.-J. and Schmit C. (1984). *Characterization of chaotic quantum spectra and*

universality of level fluctuation laws, Phys. Rev. Lett. **52**, 1.

- Bohigas O., Tomsovic S. and Ullmo D. (1993). *Manifestations of classical phase space structures in quantum mechanics*, Physics Reports **223**, 43.
- Bohr N. (1920). *Uber die Serienspektra der Element*, Zeitschrift fur Physik **2**, 423.
- Chirikov B.V. (1979). *A universal instability of many-dimensional oscillator systems*, Physics Reports **52**, 263.
- Chirikov B.V., Izrailev F.M. and Shepelyansky D.L. (1981). *Dynamical stochasticity in classical and quantum mechanics*, Math. Phys. Rev **2C**, 209 (Sov. Scient. Rev. Ed. Novikov S.P., Harwood Acad. Publ., Chur, Switzerland).
- Chirikov B.V., Izrailev F.M. and Shepelyansky D.L. (1988). *Quantum chaos: localization vs. ergodicity*, Physica D **33**, 77.
- Chirikov B.V. and Shepelyansky D.L. (1995). *Shnirelman peak in level spacing statistics*, Phys. Rev. Lett. **74**, 518.
- Chirikov B.V.and Shepelyansky D. (2008). *Chirikov standard map*, Scholarpedia **3(3)**, 3550.
- Colin de Verdiere Y. (1985). *Ergodicite et fonctions propres du laplacien*, Commun. Math. Phys. **102**, 497.
- Colin de Verdiere Y., Hillairet L. and Trelat E. (2018). *Spectral asymptotics for sub-Riemannian Laplacians. I: Quantum ergodicity and quantum limits in the 3D contact case*, Duke Math. J. **167(1)**, 109; also hal-01144257v4.
- Einstein A. (1917). *Zum quantensatz von Sommerfeld und Epstein*, Verh. Dt. Phys. Ges. **19**, 82 (English translation: *On the quantum theorem of Sommerfeld and Epstein*, The Collected Papers of Albert Einstein, Princeton U. Press, Princeton, NJ (1997) **6**, 434).
- Egoroff D.Th. (1911). *Sur les suites des fonctions mesurables* (On sequences of measurable functions), Comptes rendus hebdomadaires des seances de l'Academie des sciences (in French), **152**, 244 (see also Wikipedia article Egorov's theorem (https://en.wikipedia.org/wiki/Egorov's_theorem)).
- Ehrenfest P. (1927). *Bemerkung über die angenaherte Gultigkeit der klassischen Mechanik innerhalb der Quantenmechanik*, Zeitschrift fur Physik **45**, 455.
- Fang W., Cao H., Podolskiy V.A. and Narimanov E. (2005). *Dynamical localization in microdisk lasers*, Opt. Express **13(15)**, 5641.
- Frahm K.M. and Shepelyansky D.L. (1997)a. *Quantum localization in rough billiards*, Phys. Rev. Lett. **78**, 1440.
- Frahm K.M. and Shepelyansky D.L. (1997)b. *Emergence of quantum ergodicity in rough billiards*, Phys. Rev. Lett. **79**, 1833.
- Gerard P. and Leichtnam E. (1993). *Ergodic properties of eigenfunctions for the Dirichlet problem*, Duke Math. J. **71(2)**, 559.
- Haake F. (2010). *Quantum signatures of chaos*, Springer, Berlin.
- Jakobson D., Safarov Y. and Strohmaier A. (2015). *The semiclassical theory of discontinuous systems and ray-splitting billiards* , American J. Math **137(4)**, 859 (with an appendix by Y. Colin de Verdiere).
- Podolskiy V.A., Narimanov E., Fang W. and Cao H. (2004). *Chaotic microlasers based on dynamical localization*, Proc. Nat. Acad. Sci. **101(29)**, 10498.
- Poincare H. (1890). *Sur le probleme des trois corps et les equations de la dynamique*, Acta Math. **13**, 1

(DOI:10.1007/BF02392506).

- Shepelyansky D.L. (1987). *Localization of diffusive excitation in multi-level systems*, Physica D **28**, 103.
- Sirko L., Bauch Sz., Hlushchuk Y., Kich P.M., Blumel R., Barth M., Kuhl U. and Stockmann H.-J. (2000). *Observation of dynamical localization in a rough microwave cavity*, Phys. Lett. A **266**, 331.
- Shnirelman A.I. (1974). *Ergodic properties of eigenfunctions* (in Russian), Uspekhi Mat. Nauk. **29(6(180))**, 181 (http://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=rm&paperid=4463&option_lang=eng) .
- Shnirelman A.I. (1975). *The asymptotic multiplicity of the spectrum of the Laplace operator* (in Russian), Uspekhi Mat. Nauk. **30(4(184))**, 265 (http://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=rm&paperid=4279&option_lang=rus) .
- Shnirelman A.I. (1993). *On the asymptotic properties of eigenfunctions in the regions of chaotic motion*, in: V. Lazutkin, KAM theory and semiclassical approximations to eigenfunctions, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 24, Springer-Verlag, Berlin.
- Toulouse lecture notes (2015). *A first course in quantum chaos*, Universite de Toulouse, Math. Dept. preliminary lecture notes (https://www.google.com/url?sa=t&rct=j&q=&esrc=s&source=webcd=4ved=2ahUKEwjR_9as8a3pAhUH8xQKHRFDDOIQFjADegQIBhAB&url=https%3A%2F%2Fwww.math.univ-toulouse.fr%2F~bouclet%2FNotes-de-cours-exo-exam%2FM2%2FShnirelman.pdf&usg=AOvVaw1Dx1LVpV4xWqOe-ZdlJ-nc) .
- Ullmo D. (2016). *Bohigas-Giannoni-Schmit conjecture*, Scholarpedia **11(9)**, 31721.
- Zeldich S. (1987). *Uniform distribution of eigenfunctions on compact hyperbolic surfaces*, Duke Math. J. **55**, 919.
- Zeldich S. and Zworski M. (1996). *Ergodicity of eigenfunctions for ergodic billiards*, Commun. Math. Phys. **175(3)**, 673.

See also at Scholarpedia

Dynamical billiards, Random Matrix Theory, Bohigas-Giannoni-Schmit conjecture, Microwave billiards and quantum chaos, Chirikov standard map, Anderson localization and quantum chaos maps

Sponsored by: Dima Shepelyansky, Laboratoire de Physique Théorique, CNRS, Université Paul Sabatier, Toulouse

Category: Quantum Chaos