Ehrenfest time and chaos

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The **Ehrenfest time** τ_E gives the scale of time on which the Bohr correspondence principle (Bohr, 1920) remains valid for a quantum evolution of an initial state at high characteristic quantum numbers q (or small effective Planck constant $\hbar \sim 1/q$) closely following the corresponding classical distribution. For a narrow initial wave packet the **Ehrenfest theorem** (Ehrenfest, 1927) guaranties that the average values of quantum operators are close to the corresponding classical averages. For systems with integrable classical dynamics the Ehrenfest time is rather long being generally inversely proportional to the Planck constant $\tau_E \propto q \sim 1/\hbar$ (or another power of it). The new nontrivial situation appears for classically chaotic dynamics when nearby trajectories diverge exponentially with time due to exponential instability of motion characterized by the positive Kolmogorov-Sinai entropy $h > 0$. Thus in such semiclassical systems the Ehrenfest time is logarithmically short $\tau_E \sim (\ln q)/h \sim |\ln \hbar|/h$. The properties of the Ehrenfest time of quantum dynamics of such chaotic systems, with related examples, are discussed below.

Contents

Bohr correspondence principle and chaos

The Bohr correspondence principle (Bohr, 1920) states that the quantum evolution reproduces the classical behavior of corresponding classical system in the limit of small Planck constant $\hbar$, expressed in some dimensional units (e.g. an inverse quantum level number). Thus the important question is till what time scales holds such a correspondence. The Ehrenfest theorem (Ehrenfest, 1927) shows that this correspondence remains valid until the quantum wave packet remains narrow and compact in a system phase space. In a first approximation the spreading of a packet follows a spreading of classical trajectories in the phase space as it is described by the Liouville equation. For an integrable classical dynamics the separation of classical trajectories grows polynomially with time (e.g. linearly with time for a free motion with $\Delta x(t) \propto \Delta p t$). Thus, for an integrable dynamics, the Ehrenfest time τ_E is growing polynomially with a decrease of Planck constant, e.g. $\tau_E \propto 1/\sqrt{\hbar}$ for a free type propagation and an initial coherent wave packet width $\Delta x \sim \Delta p \sim \sqrt{\hbar}$.

The situation is quantitatively different for a case of chaotic dynamical systems. For them there is an exponential separation of classical trajectories with time characterized by the Kolmogorov-Sinai entropy h, given by a sum of positive Lyapunov exponents of motion (Lichtenberg and Lieberman, 1992). As a result the separation of trajectories of initial coherent wave packet grows as $\Delta x_t \sim \sqrt{\hbar}\exp(ht) \sim 1$ and the Ehrenfest time becomes logarithmically short for the case of chaotic dynamics with $\tau_E \sim |\ln \hbar|/h$ (due to the logarithmic dependence on Planck constant a numerical coefficient is not important in this estimate, even if we keep it for chaotic maps discussed below). Thus for chaotic systems the Ehrenfest theorem guaranties the validity of the correspondence principle only on rather short time scale. Hence, the important questions remain open: are there still certain variables which quantum averages remain close to their classical values beyond the Ehrenfest time $\tau_E \sim |\ln \hbar|/h$?, are there those which become very different from their classical values at this short time scale ?, what does happen for initial broad wave packets for which the classical evolution is described by the Liouville equation?

The answers on these questions had been proposed by Chirikov et al., 1981,Chirikov et al., 1988 on the basis of a concept of two time scales. The first one is the logarithmically short time scale given by the Ehrenfest time $\tau_E \sim |\ln \hbar|/h$, after which the wave packet spreads on almost all phase space (or its significant part, e.g over phase). After this time the Ehrenfest theorem looses its validity. The second time scale is a much longer time determined by a discreteness of levels and inverse level spacing between effectively coupled states $\tau_H \propto 1/\hbar^2$ (now this time scale is usually called the Heisenberg time scale). It was also shown (Shepelyanskii, 1981b,Shepelyansky, 1983) that the quantum correlators become drastically different from their classical values (decreasing exponentially with time) after the short Ehrenfest time. However, their values are rather small, being proportional to a certain power of small Planck constant, and thus their influence on energy growth and diffusion appears only on much larger Heisenberg times $\tau_H \gg \tau_E$. The physical origin, due to which an exponential decay of classical correlations is replaced by constant fluctuations or slow polynomial decay in time for the quantum case, is related to the Heisenberg uncertainty principle with $\Delta x \Delta p \geq \hbar/2$. Indeed, for a classical chaotic dynamics a mixing in the phase space goes exponentially fast on smaller and smaller space scales $\Delta x \sim \exp(-ht)$ that would require exponentially large values of momentum $\Delta p \sim 1/\Delta x \sim \exp(ht)$ that is usually not possible due to system phase space restrictions. This argument is valid not only for a coherent initial state but also for a broad initial distribution which classical evolution is described by the Liouville equation (an example of a broad initial distribution is a line in a phase space with a fixed action and homogeneous distribution in phase). At the same time the semiclassical expansion of the wavefunction over the classical orbits (see Maslov, 1961,Maslov and Fedoriuk, 1981) remains valid much beyond the Ehrenfest time scale, up to the Heisenberg time scale τ_H, when an exponential number of orbits contribute to the expansion (Shepelyanskii, 1981a).

Below the above aspects of quantum properties of classically chaotic systems are presented and discussed on examples of symplectic dynamical maps.

System examples

We consider examples of different systems illustrating for them the Ehrenfest time dependence on parameters.

Chirikov standard map

The classical dynamics of this system, also known as kicked rotator, is described by a simple symplectic map $p_{t+1} = p_t + K\sin x_t, x_{t+1} = x_t + p_{t+1}$ for canonical conjugated momentum and coordinate variables (p, x) taken at time moments $t, t+1$. The corresponding quantum evolution is described by the quantum map for the wave function $\psi_{t+1} = \exp(-i\hat{p}^2/2\hbar)\exp(-iK/\hbar(\cos\hat{x}))\psi_t$ with the commutator of momentum and coordinate operators being $[p_t, x_t] = -i\hbar$. Here the new variables p_{t+1}, x_{t+1} are determined after one map iteration composed by kick and rotation on a circle of period 2π (or free propagation in space for the case of cold atoms in kicked optical lattice). The time interval between kicks, mass are unity and $K, \hbar$ are dimensionless. The detailed system description is given in (Chirikov, 1979,Lichtenberg and Lieberman, 1992) for classical dynamics and (Chirikov et al., 1981,Chirikov et al., 1988) for quantum evolution; see also Chirikov standard map at Scholarpedia. The quantum system was experimentally realized with cold atoms in kicked optical lattice by the Raizen group (Moore et al., 1995, see also Cold atom experiments in quantum chaos).

Integrable dynamics

For $K \ll 1$ the main part of the classical phase space is covered by the Kolmogorov-Arnold-Moser (KAM) curves and the dynamics is intergable (except exponentially narrow chaotic layers near separatrix of resonances). Then an initial coherent wavepacket with a width $\Delta p \approx \Delta x \sim \sqrt{\hbar}$ at a position with $p_0 \sim x_0 \sim 1$ spreads on a space width $\Delta x \sim 2\pi$ after the Ehrenfest time $\tau_E \sim 1/\sqrt{\hbar}$ since trajectories with momentum difference Δp diverge in space linearly with time (measured in number of map iterations). Thus in such an integrable case the Ehrenfest time is polynomially large at small values of Planck constant.

Chaotic dynamics

At $K \gg 1$ the phase space is mainly chaotic and the measure of stability island is very small (e.g. about 2 percent for at $K = 5$). In this regime the Kolmogorov-Sinai entropy is $h \approx \ln(K/2)$ (Chirikov, 1979) and the distance between two trajectories grows exponentially with time $\Delta x_t \approx \exp(ht)\Delta x_0$. It is natural to consider that the initial quantum wavepacket is spread completely when it has $\Delta x \sim \Delta p \sim 1$. Thus for the initial coherent wavepacket with $\Delta x_0 \approx \Delta p_0 \sim \sqrt{\hbar}$ the Ehrenfest time is

$$\tau_E \sim |\ln \hbar|/2h \ ,$$

as obtained by Chirikov et al., 1981. Thus for the case of chaotic dynamics this time scale is logarithmically short being relatively small even for very small values of the Planck constant (or related very high quantum numbers). According to the Ehrenfest theorem during this Ehrenfest time the quantum wavepacket follows closely a chaotic classical trajectory, or a corresponding classical Liouville packet in the phase space with a probability distribution being close to the initial quantum one, and thus the quantum evolution is chaotic on this time scale $\tau_E \sim |\ln \hbar|/2h$. For typical parameter values used in (Chirikov et al., 1981) $K = 5, \hbar = 1/4$ (see Fig.5 in this Ref.) we have $h \approx 1$ and the Ehrenfest time is about one map iteration $\tau_E \sim 1$. However, the quantum diffusion in momentum follows the classical one on a significantly longer times $t_d \approx 40$ corresponding to an effective discreteness of the spectrum of motion. This time scale t_d (or t_H) is also often called the Heisenberg time to mark the fact that after this time the uncertainty relation between energy level spacing and time allows to resolve the discreetness of the spectrum. After this time the quantum interference effects lead to a suppression of quantum diffusion and its dynamical localization being similar to the Anderson localization in disordered solid state systems (see Chirikov et al., 1981,Fishman et al., 1982,Chirikov et al., 1988 and Anderson localization and quantum chaos maps). It is shown by Chirikov et al., 1981,Shepelyansky, 1986,Chirikov and Shepelyanskii, 1986,Chirikov et al., 1988 that the localization time scale t_H and the localization length ℓ are

$$t_H \sim \ell \sim K^2/\hbar^2 \gg \tau_E \ .$$

Thus there is a question if there are quantum averages which are different from classical ones at a short Ehrenfest time scale. We discuss this point below.

Chirikov typical map

For the Chirikov standard map the Kolmogorov-Sinai entropy is rather large being of the order of one map iteration. Due to that the wavepacket spreading is very fast and very small values of Planck constant are required to obtain high values to the Ehrenfest time to be able to follow the spreading. Thus it is useful to consider another map proposed by Chirikov, 1969 for which the quantum evolution was analyzed by Frahm and Shepelyansky, 2009. In this case $h \ll 1$ and thus $\tau_E \gg 1$ at moderate $\hbar$ values.

The typical map is obtained from the standard map by a finite-number T of random phase-shift angles at each map iteration. The map has the form

$$p_{t+1} = p_t + k\sin(x_t + \alpha_t)\ ,\ x_{t+1} = x + p_{t+1}\ ,$$

where independent random phase shifts α_t are uniformly distributed and are repeated periodically after T map iterations. The detailed study of classical and quantum map dynamics are reported in Frahm and Shepelyansky, 2009 (see also Refs. therein for properties of classical dynamics). The global chaos with unbounded diffusion appears for $k > k_c = \pi^2/(4T^{3/2})$ with $k_c << 1$ at $T >> 1$. Thus the typical map describes a quasi-continuous chaotic flow. The Lyapunov exponent and Kolmogorov-Sinai entropy are $\lambda = h \approx 0.29k^{2/3}$ (for two-dimensional maps there is only one positive Lyapunov exponent and hence $h = \lambda$). The diffusion rate in the regime of global chaos at $k > k_c$ is $D = \langle p^2 \rangle/t = k^2/2 \sim \lambda^3$. In the quantum case this diffusion is localized due to quantum interference effects (similar to the Anderson localization in disordered solid-state systems) with the localization length $\ell = k^2T/(2\hbar^2)$. The Ehrenfest time scale in this system is $\tau_E \approx \ln(2\pi/\sqrt{\hbar})/h$. The exponentially fast spearing of initial wavepacket of coherent state is illustrated Fig.1. For the figure parameters, e.g. $\hbar = 2\pi/2^{16}$ this gives $\tau_E \approx 103$ that is in agreement with the numerical data showing that the spearing in phase reaches 2π approximately at 100 iterations.

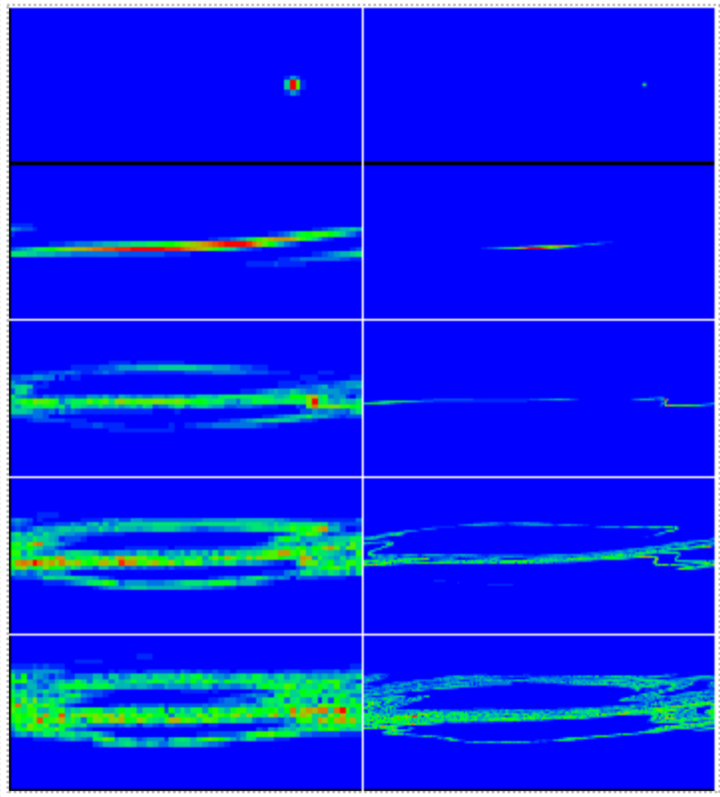

Figure 1: Quantum evolution of the Chirikov typical map at $k = 0.1, T = 10$ in the semiclassical limit with $\hbar = 2\pi/N$ and with the Hilbert space dimension $N = 2^{12}; 2^{16}$ (left; right column) and at times $t = 0, 20, 60, 100, 150$ (first, second, third, fourth, fifth row). Shown are Husimi functions (smoothed Wigner functions) with maximum values at red, intermediate values at green and minimum values at blue at the lower half of one elementary cell $x \in [0, 2\pi[; p \in [0, \pi[$. The initial condition is a coherent Gaussian state centered at $x_0 = 0.8 \cdot 2\pi; p_0 = 0.25 \cdot 2\pi$ with a momentum variance being $\Delta p = 2\pi/\sqrt{12N}$. The resolution corresponds to $\sqrt{N}$ (64 or 256) squares in one line. At the considered value $k = 0.1$ the global classical dynamic is diffusive but requires iteration times of $t \sim 10000$ to fill one elementary cell (Figure is taken from Frahm and Shepelyansky, 2009).

Quantum correlators at Ehrenfest time

The properties of quantum correlators on Ehrenfest time and beyond are analyzed by Shepelyanskii, 1981b. The consideration is done for the quantum standard map of the Heisenberg operators $\hat{p}_{t+1} = \hat{p}_t + K \sin \hat{x}_t$, $\hat{x}_{t+1} = \hat{x}_t + \hat{p}_{t+1}$ with the standard commutator $[\hat{p}_t, \hat{x}_t] = -i\hbar$ at time moments t.

It is shown that the average of quantum correlator $R(t) = \langle n|(\exp(-im\hat{x}_0) \exp(i\hat{x}_t) + \exp(i\hat{x}_t) \exp(-im\hat{x}_0))/2|n\rangle$, performed over an initial sate, decays with time not faster than $|R(t)| > 1/\sqrt{Kt/\hbar}$ (here an initial state has a momentum n being homogeneous in coordinate $0 \le x \le 2\pi$; m is an integer). This bound for $|R(t)|$ appears due to the fact that in the quantum case the momentum cannot grow faster than linearly with time. Thus, due to the Heisenberg uncertainty relation, the mixing scale in coordinate space cannot decrease faster than linearly with time. In contrast, due to the exponential instability of classical dynamics the mixing scale decreases exponentially with time. Hence, for the classical dynamics such a correlator decays usually exponentially with time at high values of chaos parameter $K \gg 1$ with an expected decay $|R(t)| \sim \exp(-ht)$. Thus at the Ehrenfest time $\tau_E \sim |\ln \hbar|/h$ the quantum correlator becomes much larger than its classical values. However, the averages of quantum energy, or square of momentum, remain close to their classical values on much longer time scale $t_H \propto 1/\hbar^2$ since the small values of quantum correlator are accumulated with time (like sum of correlators) leading to deviations between quantum and classical energy values only on time scale of the order of t_H.

Examples of the time dependence of quantum correlator $R(\tau) = \langle 0| \cos \hat{x}_t \cos \hat{x}_{t+\tau} + \cos \hat{x}_{t+\tau} \cos \hat{x}_t |0\rangle$ are shown in Fig.2,Fig.3 (here the initial state has zero momentum and averaging is done over all times t at fixed τ). The results clearly show a stationary level of correlator fluctuations starting from the Ehrenfest time of 1 to 3 map iterations. The direct computation of the ratio of quantum correlator to its classical value shows that this ratio becomes more than 100 percent different from its classical value at $\tau = 2 (K = 5, \hbar = 1, \hbar = K/40)$; $\tau = 5 (K = 5, \hbar = K/100)$; $\tau = 3 (K = 5 + 2\pi, \hbar = K/100)$ (see Table 1 in (Shepelyansky, 1983)). These time values are in agreement with the above estimate for the Ehrenfest time.

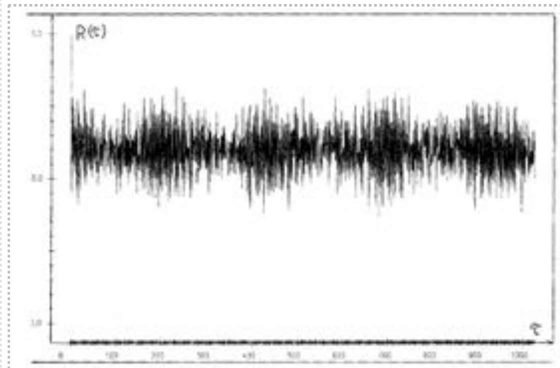

Figure 2: Quantum correlator $R(\tau)$ (see text) for the Chirikov standard map at $K = 5, \hbar = 1$ and initial state with zero momentum (Figure is taken from Shepelyansky, 1983(a)).

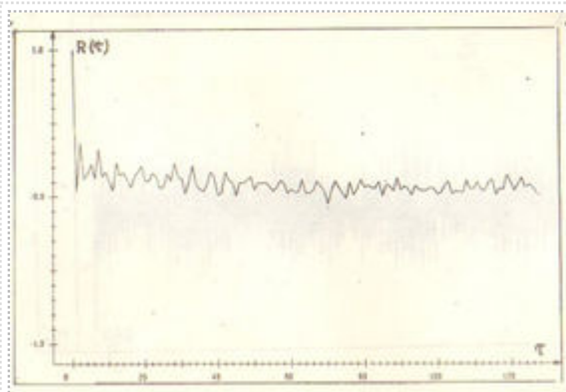

Figure 3: Same as Fig.2 at $K = 5, \hbar = 1/8$ (Figure is taken from Shepelyansky, 1983(b)).

The dependence of Heisenberg operators at a given time $\hat{p}_t, \hat{x}_t$ on the initial operators $\hat{p}_0, \hat{x}_0$ at zero time, presented in the ordered form, was obtained by Shepelyanskii, 1981b. In this way it was shown that at times up to short Ehrenfest times $t < t_E$ the sensitivity of operators $\hat{p}_t, \hat{x}_t$ to initial operators $\hat{p}_0, \hat{x}_0$ is growing with time exponentially as in the classical case while at large times $t \gg t_E$ the sensitivity cannot grow faster than linearly with time since the number of momentum harmonics grows not faster than linearly with time. Thus there is no exponential sensitivity of Heisenberg operators to their initial values and thus the quantum Kolmogorov-Sinai entropy is zero (Shepelyanskii, 1981a).

Recently there is a growing interest to so called out-of-time-ordered correlators (OTOC) (see e.g. Hamazaki et al., 2018,,Jalabert et al., 2018,) which illustrate on short times an exponential growth and saturation on large times (e.g. $\langle 0|\hat{p}_0 \cos \hat{x}_t|0\rangle$) in agreement with the results obtained by Shepelyanskii, 1981a.

The computation of quantum correlations for $\hbar > 1$ allows to determine the quantum diffusion coefficient D in this regime showing that in the classical dependence $D(K)$ the replacement $K \to K_q = 2(K/\hbar)\sin(\hbar/2)$ should be done for the quantum case Shepelyansky, 1987. This leads to characteristic oscillations of the diffusion coefficient and localization length of quantum eigenstates $\ell = D(K_q)/2$ with $K_q = 2(K/\hbar)\sin(\hbar/2)$ (see also Fig.5 at

Chirikov standard map).

Time reversibility

Even if the equation of motion describing classical dynamics can be reversible in time (e.g. Chirikov standard map, chaotic Dynamical billiards) the exponential instability of chaos leads to a practical irreversibility of motion due to exponential growth of errors with time. However, the corresponding quantum evolution of such systems has no exponential instability (see above) and thus the quantum evolution shows the time reversibility even at a rather high level of quantum errors. The demonstration of this difference for classical and quantum dynamics of the Chirikov standard map was given by Shepelyansky, 1983 (see also Figs.6,7 at Chirikov standard map). The effect of time reversibility is also discussed at the Scholarpedia article Loschmidt echo by Gusev et al., 2012. The absence of instability of quantum evolution beyond the Ehrenfest time, as discussed above, is at the origin of stability of time reversibility of quantum motion in a drastic difference from the classical chaotic dynamics with its exponentially growing errors during the whole time reversibility interval. The stability of quantum evolution becomes especially evident for a case of a quantum computer simulating classical chaotic dynamics of the Arnold cat map as discussed by Georgeot and Shepelyansky, 2001, Georgeot and Shepelyansky, 2002.

Semiclassical expansion beyond Ehrenfest time

For quantum evolution of systems with classical chaotic dynamics it was shown by Shepelyanskii, 1981a that even beyond the Ehrenfest time the semiclassical expression for the wavefunction still can be presented as a sum over classical trajectories (see Maslov, 1961,Maslov and Fedoriuk, 1981). The expansion has the form

$$\psi(x,t) = \sum_{k=1}^{N} |J_k|^{-1/2} \exp(iS_k(x,t)/\hbar - i\pi\mu_k/2) \times [\sum_{m=0}^{\infty} [\hat{L}_k^{\,m} \varphi_0(x_0)]|_{x_0={x_0}^k(x,k)}] + O(\hbar^\infty)$$

where the k-summation runs over all classical trajectories which arrive at the point x at time t and satisfy the initial conditions $x_0(x,t) = {x_0}^k, p_0(x_0) = \partial S/\partial x_0|_{x_0={x_0}^k}, J_k = \partial x(x_0,t)/\partial x_0|_{x_0={x_0}^k}$ where $S_k(x,t)$ is the action along the classical trajectory which connects ${x_0}^k$ and x, and μ_k is the Morse index. At initial time $t=0$ we have $\psi(x,0) = \varphi_0(x_0)\exp(iS_0(x_0))$. The sum over m is essentially an expansion in powers of $\hbar$. Since the classical dynamics is chaotic the Jacobian J_k and the number N in the sum increases exponentially with time, so that $J_k \propto \exp(ht), N \propto \exp(ht)$. Indeed, there are exponentially many classical trajectories from initial distribution at zero time arriving to a giving coordinate at time t (see Fig.4). Fig.4 directly shows that there are exponentially many trajectories, from initial distribution, arriving at time t to a given point x with different momentum values. Nonetheless, the semiclassical expansion remains valid if $\hat{L}_k\varphi_0 \ll \varphi_0$ that is satisfied on long times $t \sim t_q \propto O(1/\hbar) \gg \tau_E$ which greatly exceed the Ehrenfest time on which a coherent wave packet spreads over almost a whole phase space. During these times $t_q \propto O(1/\hbar) \gg \tau_E$ the quantum diffusion coefficient is close to its classical values as shown by Shepelyanskii, 1981a.

A similar type of semiclassical wavefunction expressed by a sum over periodic orbits is used in the Gutzwiller trace formula for the semiclassical quantization of systems being chaotic in the classical limit (see Gutzwiller, 1990). This gave the answer on the problem of semiclassical quantization of nonintegrable systems pointed by Einstein, 1917 in view of the existence of nonintegrable dynamics proved by Poincare, 1890.

Quantum evolution beyond Ehrenfest time

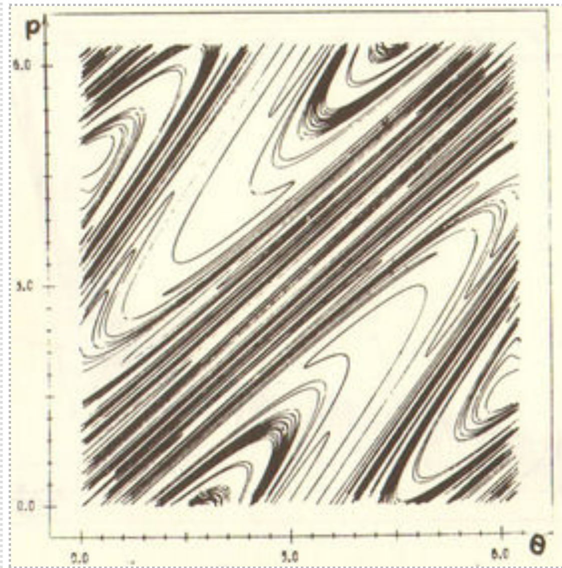

Figure 4: Classical distribution of initial line $p = 0$ at the Chirikov standard map at $K = 5$ after 4 map iterations showing that exponentially many orbits contribute to the semiclassical expansion of wavefunction, here $\theta = x$ (Figure is taken from Shepelyanskii, 1981a(preprint)).

The distributions of quantum states at large times, obtained from an initial narrow wave packet, are illustrated for the quantum Chirikov standard map (4 vertical panels of Fig.5) and typical map (Fig.6).

The top 3 panels of 4 vertical panels of Fig.5 show the quantum Poincare sections represented by the Husimi function (the smoothed Wigner function) for the quantum Chirikov standard map at $K = 1.1$ and $\hbar = 2\pi/N$ with $N = 2^8, 2^{12}, 2^{16}$ quantum states inside the phase space cell $(x, p) = 2\pi \times 2\pi$. Here the initial state is a minimal coherent state of one quantum cell taken in the chaotic component at $x = p = 0.1 \times 2\pi$; the section is shown after $t = 2 \times 10^4$, map iterations (red is for maximum density, blue is for minimum). The fourth bottom panel shows the classical Poincare section for the same parameters, initially $N = 2^{16}$ classical orbits are homogeneously distributed inside the classical area corresponding to the effective Planck constant $\hbar = 2\pi/2^{16}$ (density is averaged over 1000 last iterations).

The similar quantum evolution for the Chirikov typical map is shown in Fig.6 on large times (corresponding to the short time evolution in Fig.1).

In both examples at moderate values of Planck constant the wave packet spreading over the whole space is suppressed by quantum interference effects leading to a quantum localization of slow diffusion through a weakly destroyed KAM curves. With the further decrease of Planck constant the spreading goes over the whole phase space domain occupied by the connected chaotic component. The comparison with the classical distribution shows that the quantum wavefunction becomes ergodic, over the phase space of classical chaotic component, in the limit of large times and small Planck constant. This corresponds to the Bohr correspondence principle which implies that at small values of Planck constant the quantum distribution follows the classical one at least on some time scale. Since the classical distribution for the standard map is ergodic on a chaotic component (see bottom panel of Fig.5) we can expect that the quasi-energy eigenstates of quantum standard map will be ergodic on the chaotic component (surrounding stability islands). This corresponds to the regime of Shnirelman ergodicity of quantum eigenstates of chaotic billiard as discussed at Shnirelman theorem. For chaotic billiards with energy conservation the quantum ergodicity of eigenstates takes place on the energy surface while for the quantum standard map the ergodicity of quasi-energy eigenstates takes place on the chaotic component of the map.

Ehrenfest explosion and collapse

The method of quantum trajectories is applied for the investigation of quantum dynamics of the Chirikov standard map with dissipation by Carlo et al., 2005. The classical map has the form $p_{t+1} = (1-\gamma)p + K \sin x_t, x_{t+1} = x_t + p_{t+1}$ where $0 < \gamma < 1$ describes dissipation (at $\gamma = 0$ we have the standard map). At large chaos parameter and moderate dissipation the classical dynamics has a strange attractor. The quantum dissipative evolution is described by a master equation in the Lindblad form (see e.g. Weiss, 1999). The numerical simulations are performed with the method of quantum trajectories (Brun et al., 1996). They clearly show a transition from the wave packet collapse to explostion as shown in Fig.7. The detailed studies show that the dissipation leads to collapse or localization of the wave packet on a time scale $t_\gamma \sim 1/\gamma$. On the other side the classical chaos instability leads to the spreading of the wave packet on the Ehrenfest time $\tau \sim |\ln \hbar|/2h \approx |\ln \hbar|/2\ln(K/2)$. Hence the wave packet collapse takes place at $1/\gamma < \tau_E$ and the Ehrenfest explosion takes place at weak dissipation with $1/\gamma > \tau_E$. For the regime of integrable dynamics the Ehrenfest time is rather large at small values of Planck constant and thus the explosion practically disappears leaving place to collapse.

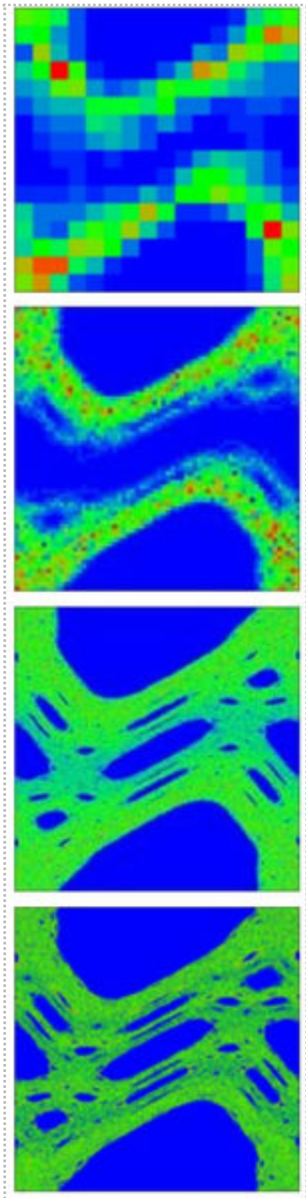

Figure 5: Quantum Poincare section of the standard map represented by the Husimi function at $K = 1.1$ (see text for details and Category:Quantum_Chaos).

Other aspects of Ehrenfest time

Other aspects and applications of Ehrenfest time concept to mesoscopic transport and Loschmidt echo are described at Scholarpedia articles Mesoscopic transport and quantum chaos by Jalabert, 2016 and Loschmidt echo by Gusev et al., 2012 . An interested reader can also find additional discussions of this topic by Silvestrov and Beenakker, 2032, Faure, 2007, Schubert et al., 2012 .

Historical notes

It should be noted that the first attempt to compare the quantum and classical averages for systems with chaotic dynamics was done by Berman and Zaslavsky, 1978. The expansion of average of a quantum operator in powers of Planck constant was used there for initial coherent state. The comparison was done between a quantum average and a classical value given by the classical trajectory started from the center of coherent packet. It was shown that the quantum corrections of the order of Planck constant have coefficients growing exponentially with time (roughly with the rate given by Lyapunov exponent of classical dynamics). On this basis it was concluded that for classically chaotic system the quantum corrections grow exponentially fast. However, in fact the comparison of quantum averages should be done not with the classical variable value of one central trajectory but with the average obtained from a classical Liouville distribution modeling the probability distribution of initial quantum state. Indeed, it is clear that, even only for classical evolution, there are exponentially growing differences between the values of one trajectory and those obtained from the Liouville distribution due to exponential divergence in time of classical trajectories of chaotic dynamics. Such a comparison was not done by Berman and Zaslavsky, 1978 and thus their results are not conclusive. Also no links with the Ehrenfest theorem were given by Berman and Zaslavsky, 1978.

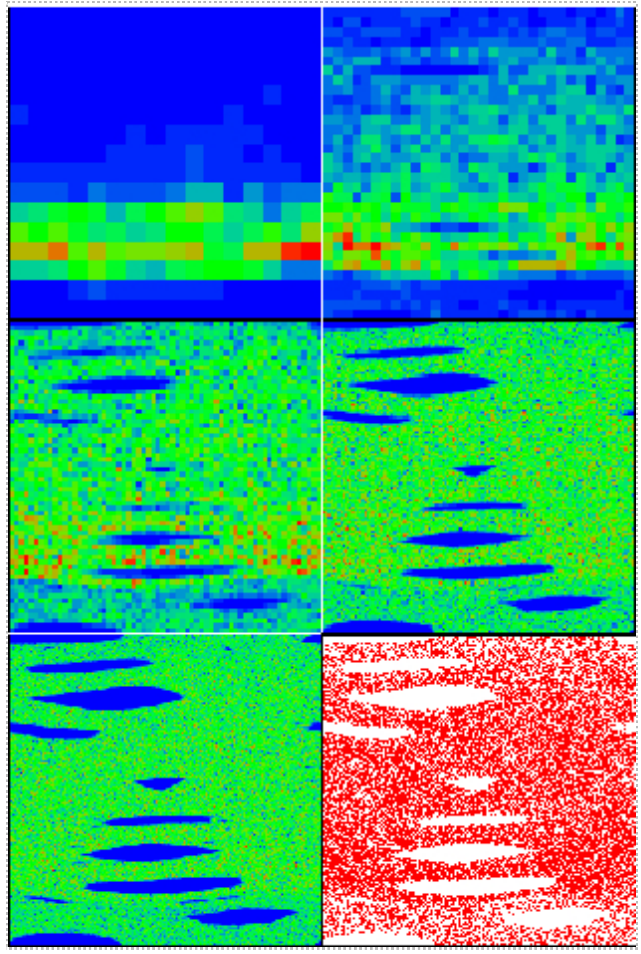

Figure 6: Quantum evolution of the Chirikov typical map as in Fig.1 with the same coherent Gaussian state as initial condition, same values $k = 0.1, T = 10$, $\hbar = 2\pi/N$ but at the iteration time $t = 20000$ and at different values of $N = 2^8, 2^{10}$ (first row), $N = 2^{12}, 2^{14}$ (second row), $N = 2^{16}$ and classical simulation (third row). For the classical map 20000 trajectories have been iterated up to the same time $t = 20000$ with random initials conditions very close to the initial position at $x_0 = 0.8 \cdot 2\pi \pm 0.002$ and $p_0 = 0.25 \cdot 2\pi \pm 0.002$ (colors are as in Fig.1).(Figure is taken from Frahm and Shepelyansky, 2009).

The links with the Ehrenfest theorem and estimates of the Ehrenfest time for systems with chaotic dynamics were first done by Chirikov et al., 1981 and the term Ehrenfest time was coined by Chirikov et al., 1988. The strong difference between classical and quantum correlators was shown to appear on the Ehrenfest time by Shepelyanskii, 1981b with numerical confirmations given in Shepelyansky, 1983. The validity of the semiclassical wavefunction expansion beyond the Ehrenfest time was shown by Shepelyanskii, 1981a.

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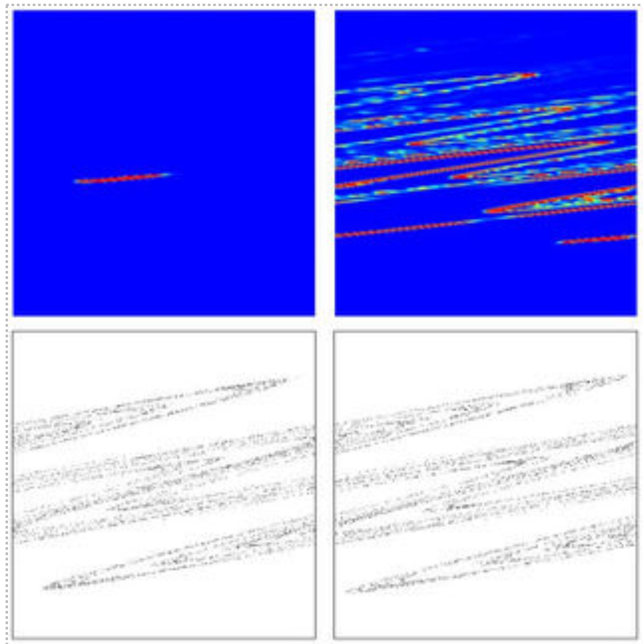

Figure 7: Transition from the wave packet collapse (top left) to explosion (top right). Top: Husimi functions in phase space for a single quantum trajectory taken after $t = 300$ kicks, at $K = 7, \hbar = 0.012$, and dissipation $\gamma = 0.5$ (left) and $\gamma = 0.01$ (right). Here x (horizontal coordinate axis) and p (vertical momentum axis) vary in the intervals $0 \le x < 2\pi$, $-25 \le p \le 25$ (left) and $-100 \le p \le 50$ (right); the width of the p-interval is the same in both cases for comparison purposes. The initial Gaussian wave packet is located at $(\langle x \rangle, \langle p \rangle) = (5\pi/4, 0)$. The color is proportional to density: blue for zero and red for maximum. Bottom: quantum Poincare section (left), obtained from average quantum x, p values for the case of top left panel and its classical counterpart (right); here $0 \le x < 2\pi$ and $-15 \le p \le 15$. (Figure is taken from Carlo et al., 2005).

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