QUANTUM CHAOS IN QUANTUM NETWORKS$^{(*)}$

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SHORT DESCRIPTION OF THE RESULTS

Quantum chaos in a quantum small world

We introduce and study a quantum small world model. It is obtained from the one-dimensional Anderson model with diagonal disorder taken on a ring of L sites with an addition of randomly chosen shortcut links typical of the small world models. The density p of shortcuts is very small so that the total number of shortcuts $pL \ll L$. Without shortcuts the eigenstates of the model are exponentially localized with the length $\xi \approx 100V^2/W^2$, where V is a hopping matrix element and W is a diagonal disorder amplitude. Using the Lanczos diagonalization we study numerically localization-delocalization transition with up to $L = 32000$. The localized phase is characterized by the Poisson level spacing statistics while the delocalized phase has the Wigner-Dyson statistics typical of the random matrix theory. We show that the localized phase exists at $p < p_c$ and the delocalized phase appears at $p > p_c$ with $p_c \approx 1/(4\xi)$.

CLASSICAL SMALL WORLD NETWORKS

Six degrees of separation

It is believed that a randomly chosen pair of people among the six billion available individuals can be related by a short chain of immediate acquaintances.

This property, called the “small world effect” was first investigated by Stanley Milgram in the late 1960.

He distributed to a random selection of people in Nebraska letters addressed to some acquaintances of his in Boston. The instructions were to transmit them from person to person on a basis of immediate relationship.

A reasonable number of these letters did eventually reach their destinations and Milgram found that in average it had only taken six steps to get from Nebraska to Boston. He conjectured therefore that any two people in the world are separated by just “six degrees of separation”.

Small world networks in practice

The small-world effect is in fact quite widespread, and appears in social networks modeling for example collaboration between scientists, artists, power grids and neural networks (Tables from Refs.[1-4]).

Network	N	ℓ	C	C_{rand}
movie actors	225 226	3.65	0.79	0.00027
neural network	282	2.65	0.28	0.05
power grid	4941	18.7	0.08	0.0005

		Los Alamos e-Print Archive			
	MEDLINE	complete	astro-ph	cond-mat	hep-th
total papers	2156769	98502	22029	22016	19085
total authors	1388989	52909	16706	16726	8361
first initial only	1006412	45685	14303	15451	7676
mean papers per author	5.5(4)	5.1(2)	4.8(2)	3.65(7)	4.8(1)
mean authors per paper	2.966(2)	2.530(7)	3.35(2)	2.66(1)	1.99(1)
collaborators per author	14.8(1.1)	9.7(2)	15.1(3)	5.86(9)	3.87(5)
cutoff z_c	7300(2700)	52.9(4.7)	49.0(4.3)	15.7(2.4)	9.4(1.3)
exponent τ	2.5(1)	1.3(1)	0.91(10)	1.1(2)	1.1(2)
size of giant component	1193488	44337	14845	13861	5835
first initial only	892193	39709	12874	13324	5593
as a percentage	87.3(7)%	85.4(8)%	89.4(3)	84.6(8)%	71.4(8)%
2nd largest component	56	18	19	16	24
mean distance	4.4(2)	5.9(2)	4.66(7)	6.4(1)	6.91(6)
maximum distance	21	20	14	18	19
clustering coefficient C	0.072(8)	0.43(1)	0.414(6)	0.348(6)	0.327(2)

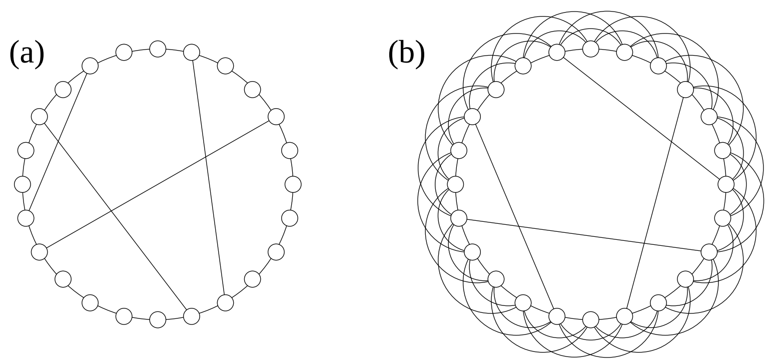
MODEL FOR CLASSICAL SMALL WORLDS

The Watts - Strogatz Model

Nature, Vol. 393, June 1998.

The Newman - Watts Model (1999)

In a circular graph with L vertexes each vertex is linked to $z = 2k$ of its nearest neighbors, then with some probability p, pLk shortcuts are added randomly to the network.



(a) An example of a small-world graph with $L = 24$, $k = 1$ and, in this case, four shortcuts (b) An example with $k = 3$ (from Ref.[2]).

Our quantum network model corresponds to case (a)

This model reflects the so called clustering we observe in real networks, our neighbors are likely to be neighbors of each other.

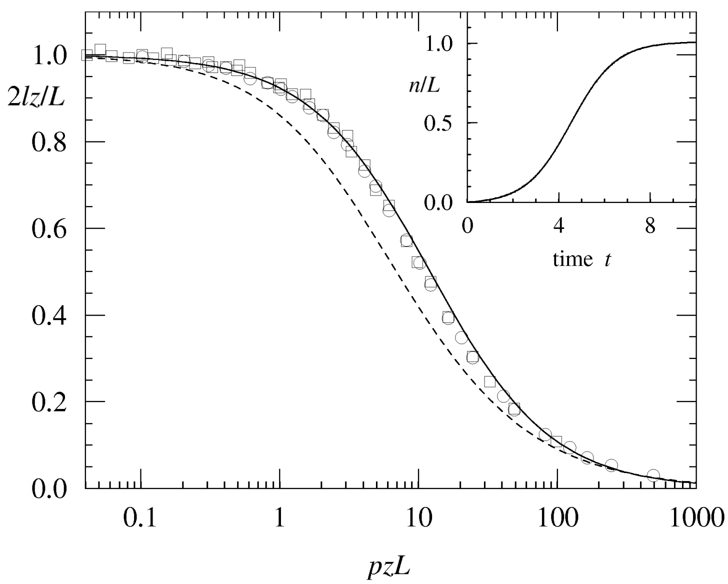
RECENT RESULTS IN SMALL WORLD THEORY

The Average vertex separation l of the graph

It is found that l obeys the following equation :

$$l = L\ F(pzL)/2z \tag{1}$$

where $F(x)$ is an universal scaling function depending only on its argument x.



Graph of the Scaling function $F = 2zl/L$ as a function of its argument $x = pzL$ (from Ref.[2]).

It is shown that F has the following asymptotic forms

$$F(x) \approx \frac{\log x}{x} \qquad \text{as } x \to \infty, \tag{2}$$

and

$$F(x) \approx 1 \sim \text{as } x \to 0. \tag{3}$$

THE QUANTUM SMALL WORLD MODEL

Schroedinger equation

The Schroedinger equation of the small world network reads

$$\epsilon_n \psi_n + V(\psi_{n+1} + \psi_{n-1}) + \sum_s V(\psi_{n+s} + \psi_{n-s}) = E\psi_n \quad (4)$$

Here ϵ_i are random variables homogeneously distributed within $[-W/2; W/2]$ and V is the hopping matrix element, with periodic boundary conditions applied. The summation is taken over all randomly established shortcuts from the site n to any other site in the network. The number of such shortcuts in the entire network is $s_{total} = pL$. We take $V = 1$.

We note that at $p = 0$ our model is in fact the (1D) Anderson Model. Here the eigenstates are exponentially localized:

$$\psi_n \sim \frac{1}{\sqrt{\xi}} \exp^{-|n-n_0|/\xi} \quad (5)$$

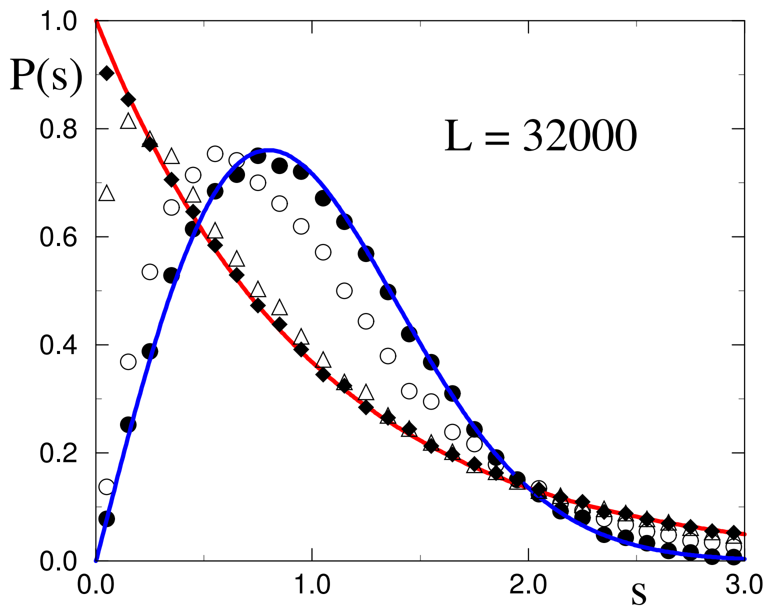
with the localization length

$$\xi \approx 100(V/W)^2 \quad (6)$$

in the middle of the energy band $E \approx 0$ and for weak disorder W/V (when $1 < \xi < L$).

Level spacing statistics

In the localized regime at $p = 0$ the level spacing s obeys the Poisson probability distribution $P_P(s) = \exp(-s)$.
For $p > 0$ our numerical data show that $P(s)$ evolves from the Poisson distribution for strong disorder to the Wigner-Dyson distribution $P_{WD}(s) = (\pi/2)\exp(-\pi s^2/4)$ typical for random matrices, which appears in our case at weak disorder. This next figure gives an example of such an evolution at $L = 32000$.

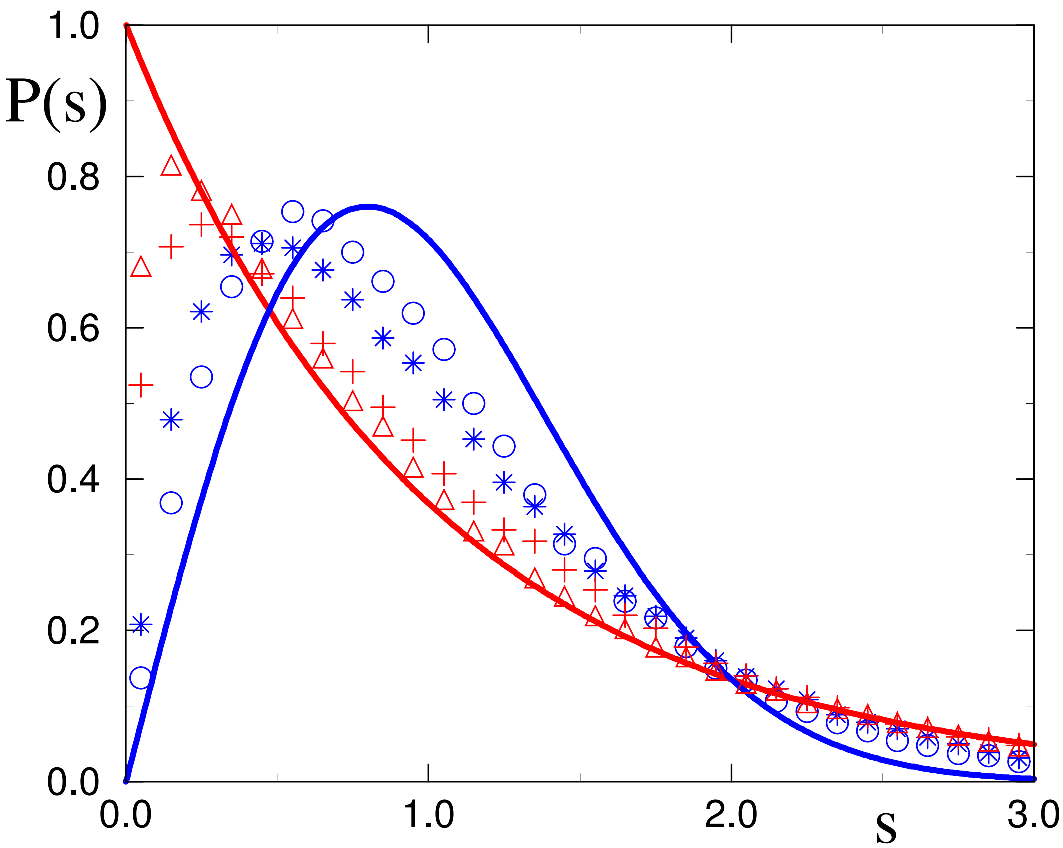


The red and blue curves represent the Poisson and Wigner - Dyson distributions. Diamonds, triangles, circles and black disks do respectively represent the level spacing statistics at $W/V = 4, 3, 2, 1$; $p = 0.02, L = 32000$. Averaging was done over 60 network realizations.

These data were obtained by Lanczos diagonalization of the sparse Hamiltonian matrices.

Dependence of P(s) on the system size L

The data show that for $W > W_c$: the level spacing statistics tends to the Poisson distribution when the size of the network L increases. On the opposite for $W < W_c$ the level spacing statistics tends to the Wigner - Dyson distribution.



The red and blue full curves represent the Poisson and Wigner - Dyson distributions. Blue symbols represent $P(s)$ for $W = 2$ at $L = 1000$ (stars) and $L = 32000$ (circles). Red pluses and triangles show $P(s)$ respectively for $L = 1000$ and $L = 32000$ at $W/V = 3$. Averaging was done for 1000 realizations of the network at $L = 1000$ and for 60 realizations for $L = 32000$; $p = 0.02$.

η a level spacing distribution criteria

A convenient criteria to determine whether $P(s)$ is closer to the Poisson or to the Wigner - Dyson distribution is η defined as:

$$\boxed{\eta = \frac{\int_0^{s_0}(P(s) - P_{WD}(s))ds}{\int_0^{s_0}(P_P(s) - P_{WD}(s))ds} \qquad (7)}$$

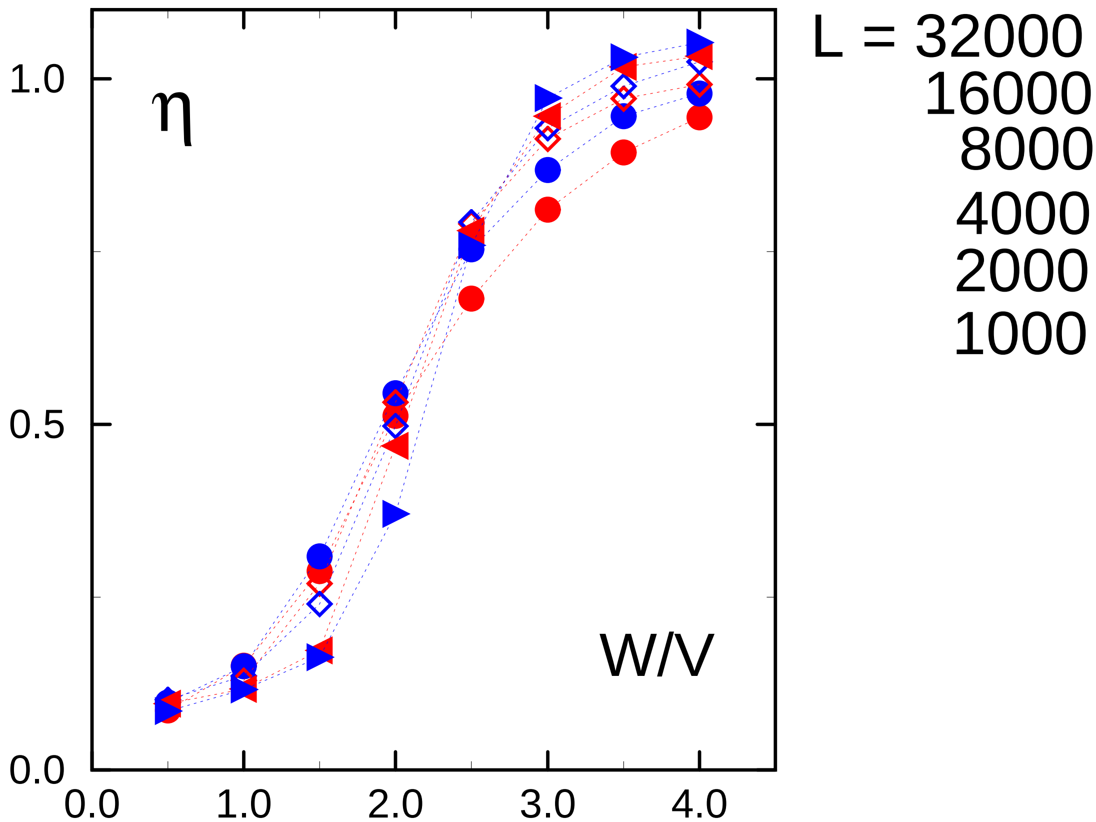
Where $s_0 \approx 0.47$ is the lowest s coordinate of the two intersections of $P_P(s)$ and $P_{WD}(s)$. For $\eta = 0$ the level spacing statistics obeys a Wigner - Dyson distribution on the contrary for $\eta = 1$ one has the Poisson distribution.

η and localization properties

For $\eta = 0$ eigenstates are delocalized in the whole system, as in the random matrix case. On the contrary for $\eta = 1$ eigenstates are localized on localization length ξ.

RANDOM MATRICES AND ANDERSON LOCALIZATION
IN QUANTUM SMALL WORLDS

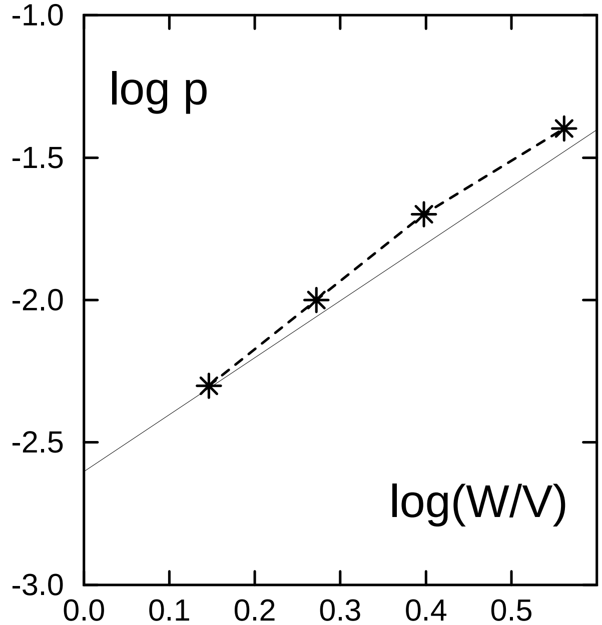
η dependence on W/V and L The dependence of η on W/V and L is investigated numerically.



η as a function of W/V for different values of L ranging from 1000 to 32000, respectively for red circles, blue circles, red diamonds, blue diamonds, red triangles, blue triangles. Averaging was done over 60 network realizations for $L = 32000$ and over 1000 networks for $L = 1000$; $p = 0.02$.

The above figure shows that the transition from random matrices to Anderson localization takes place approximately at $W_c \approx 2.6$ and $\eta \approx 0.83$.

Dependence of transition on density p and disorder W



Stars are points of the curve $\eta(W, p) = 0.8$ for $p = 0.005, 0.01, 0.02, 0.04$ on the (W, p) plane with logarithmic scale. $L = 8000$ is fixed. The line represents $p = 1/4\xi \approx \frac{1}{400}(W/V)^2$ Logarithms are decimal.

For η fixed to the transition value $\eta \approx 0.8$ p and ξ obey the relation:

$$\boxed{p \approx \frac{1}{4\xi}} \tag{8}$$

Indeed when the double of localization length 2ξ is lower than the average distance between shortcuts: $1/2p$ the eigenstates are localized. In the opposite case eigenstates become delocalized since particles can pass from one short link to another.

RESULTS AND CONCLUSIONS

Using the efficient Lanczos algorithm we studied numerically the level spacing statistics $P(s)$ in the quantum small world model up to matrix size $L = 32000$.

We found a transition from Poisson statistics (localized phase) to Wigner - Dyson statistics (delocalized phase). The transition takes place at the density of shortcuts $p_c \approx 1/4\xi$ where ξ is the localization length in the 1D Anderson model.

For $p < p_c$ the quantum network becomes effectively disconnected, even if classically all sites are connected. In the opposite case quantum information spreads rapidly over the whole network by the shortcuts.

References

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