Classical and quantum aspects of higher-dimensional systems

Arnd Bäcker, Technische Universität Dresden

Joint work with: S. Lange, F. Onken, M. Richter, R. Ketzmerick

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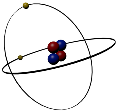
Motivation

- Helium atom



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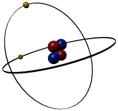
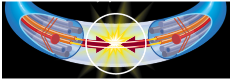


- Particle accelerators



CERN

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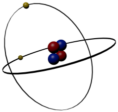

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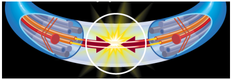
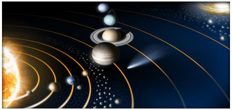


CERN

- Solar system



NASA

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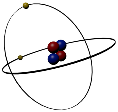

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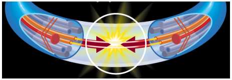

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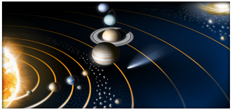
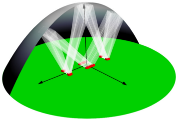


NASA

- 3D billiards



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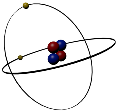

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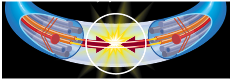

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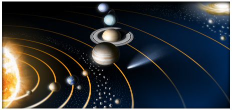

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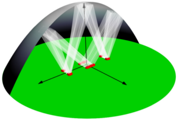

Simplest case: **4D area-preserving maps**

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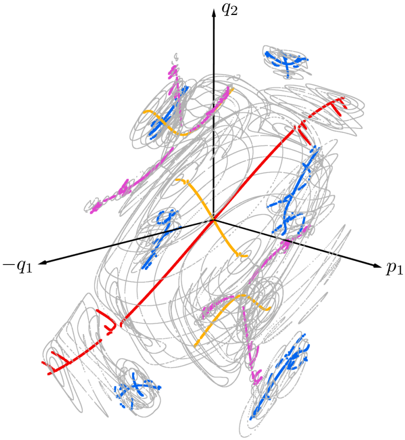

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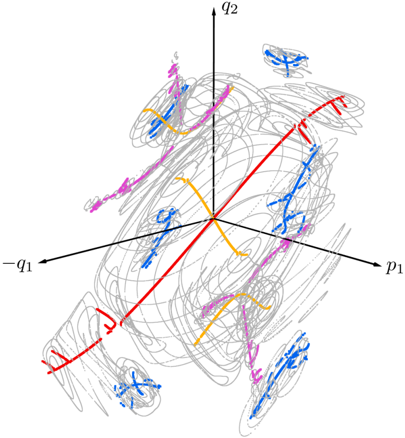

- Visualization
  - 3D Phase-Space Slice
  - Organization

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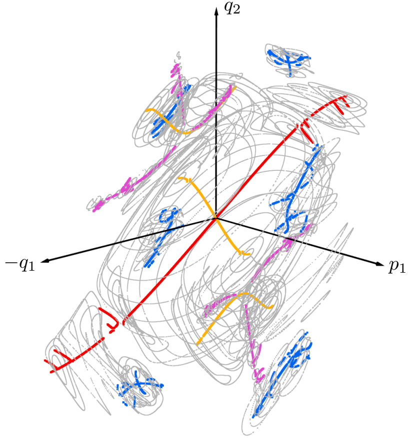

- Visualization
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  - Organization
- Applications
  - Power-law trapping
  - Dynamical tunneling
  - Spectral statistics
  - Entanglement
  - Husimi functions

2D map: Standard map

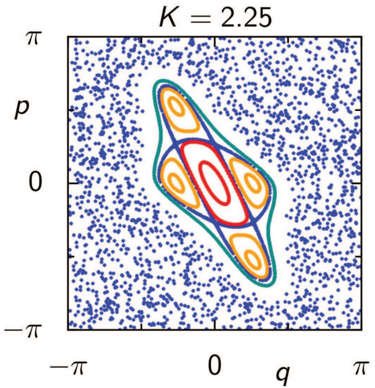
[Chirikov 1969]

Mapping: $\mathbb{T}^2 \to \mathbb{T}^2$

$$q' = q + p$$

$$p' = p - \frac{\partial V}{\partial q}(q')$$

with $V(q) = K \cos(q)$



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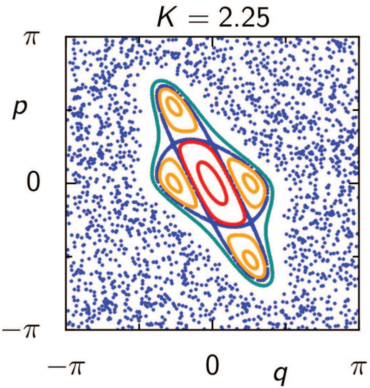

Structures

- Periodic orbits

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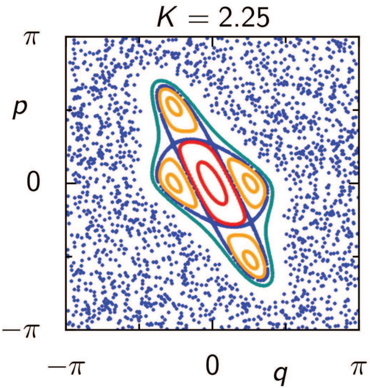

Structures

- Periodic orbits
- Regular tori (KAM)

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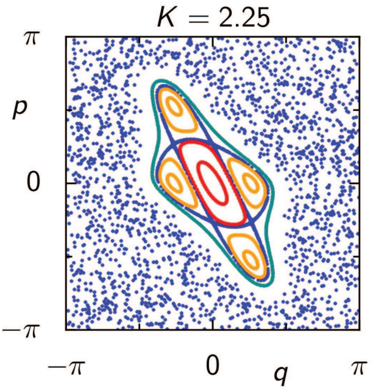


Structures

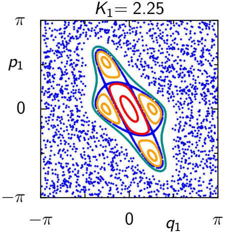
- Periodic orbits
- Regular tori (KAM)
- Chaotic motion

4D map: Coupled standard maps

[Froeschlé 1972]

$$q_1' = q_1 + p_1$$
$$p_1' = p_1 - \frac{\partial V_1}{\partial q_1}(q_1')$$

Uncoupled:



$$V_1(q_1) = K_1 \cos(q_1)$$

4D map: Coupled standard maps

[Froeschlé 1972]

Mapping: $\mathbb{T}^4 \to \mathbb{T}^4$

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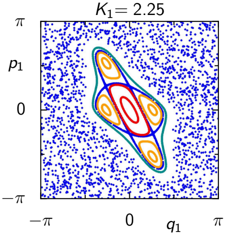
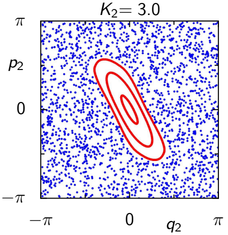
Uncoupled:





4D map: Coupled standard maps

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$$q_1' = q_1 + p_1$$
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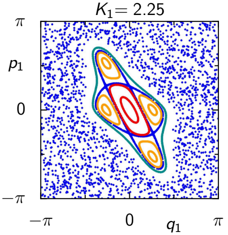
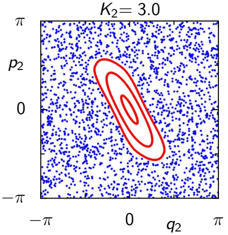
$$V_2(q_2) = K_2 \cos(q_2)$$
$$V_{12}(q_1, q_2) = \xi_{12} \cos(q_1 + q_2)$$

Uncoupled:





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Strong coupling: $\xi_{12} = 1.0$

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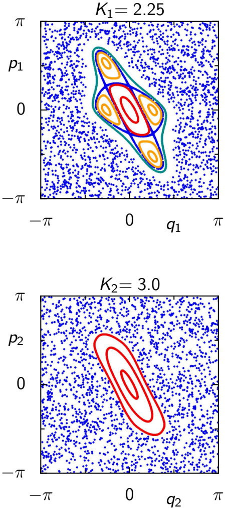

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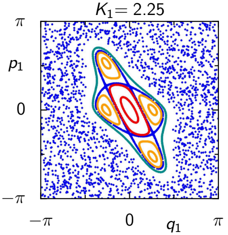
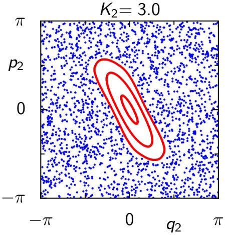
Uncoupled:





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Regular region in 4D map?

Organization of phase space? How to visualize?

4D map: Visualization?

[Richter, Lange, AB, Ketzmerick: Phys. Rev. E 2014]

PHYSICAL REVIEW E **89**, 022902 (2014)

Visualization and comparison of classical structures and quantum states of four-dimensional maps

Martin Richter, Steffen Lange, Arnd Bäcker, and Roland Ketzmerick

Technische Universität Dresden, Institut für Theoretische Physik and Center for Dynamics, 01062 Dresden, Germany and Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Straße 38, 01187 Dresden, Germany

(Received 7 November 2013; published 3 February 2014)

For generic 4D symplectic maps we propose the use of 3D phase-space slices, which allow for the global visualization of the geometrical organization and coexistence of regular and chaotic motion. As an example, we consider two coupled standard maps. The advantages of the 3D phase-space slices are presented in comparison to standard methods, such as 3D projections of orbits, the frequency analysis, and a chaos indicator. Quantum mechanically, the 3D phase-space slices allow for the comparison of Husimi functions of eigenstates of 4D maps with classical phase-space structures. This confirms the semiclassical eigenfunction hypothesis for 4D maps.

DOI: 10.1103/PhysRevE.89.022902 PACS number(s): 05.45.Mt, 03.65.Sq, 05.45.Jn

I. INTRODUCTION

Understanding higher-dimensional, dynamical systems, even with just a few particles, is a challenging task. Such systems are relevant in many areas of physics and chemistry [1], ranging from the dynamics of the solar system [2–4], dynamics of particle accelerators [5], to atoms and molecules [6–9]. Particular topics of interest concern the quantum signatures of Arnold diffusion [10–12] and quantum-classical correspondence in higher-dimensional mixed systems [13,14].

A standard example would be autonomous Hamiltonian systems with three degrees of freedom, which have a 6D phase space, which can be reduced to a 5D manifold due to energy conservation. Introducing a Poincaré section leads to a 4D symplectic map. This type of map also arises from time-periodically driven, Hamiltonian systems with two degrees of freedom, where a stroboscopic Poincaré section leads to a symplectic map acting on the 4D phase space. Such 4D maps are prototypical for the behavior of higher-dimensional systems, as they have the smallest possible dimension showing the essential difference to 2D symplectic maps: In 2D maps tori are one-dimensional and thus lead to absolute barriers of motion. In contrast, in 4D maps regular tori are two-dimensional manifolds in the 4D phase space, which cannot divide the phase space into dynamically distinct regions. One of the most fundamental consequences of this topological structure of 4D (and higher-dimensional) maps is that generically chaotic orbits can get arbitrarily close to any point in phase space, even if regular tori are present. One possible mechanism was constructed by Arnold [15], leading to the so-called Arnold diffusion [1,4,16,17]. Another striking phenomenon is the occurrence of power-law trapping of chaotic orbits in higher-dimensional systems with a mixed phase space [18–21], for which the mechanism is still an open question. While most of the analytical and numerical results about higher-dimensional systems have been obtained for the near-integrable case, many practical applications are concerned with generic systems, which cannot be described by perturbative methods; see, e.g., Refs. [22–24] and references therein.

For the lowest-dimensional Hamiltonian systems with regular and chaotic dynamics, such as 2D billiards or time-periodically driven 1D systems, the dynamics can be reduced to 2D symplectic maps. Their phase space can be easily visualized, providing substantial insight and intuition of the dynamics in phase space, such as chaotic motion, regular regions formed by 1D regular tori around stable periodic orbits, stable and unstable manifolds of unstable periodic orbits, nonlinear resonances, and hierarchical regions due to partial barriers at the border of the regular island. Also the time-evolution of trajectories can be visualized straightforwardly.

Such a direct visualization of the classical dynamics in phase space is also very useful when trying to understand the properties of the corresponding quantum mechanical system. According to the semiclassical eigenfunction hypothesis [25–27], one expects that eigenstates semiclassically concentrate on those regions in phase space that a generic orbit explores in the long-time limit. For ergodic systems this is proven by the quantum ergodicity theorem stating that almost all eigenfunctions become equidistributed in the semiclassical limit [28–30]. For systems with a mixed phase space one thus expects that (almost all) eigenstates either concentrate in the chaotic region or within the regular regions on the invariant tori. Away from the semiclassical limit this can be violated due to dynamical tunneling [31–34] and partial transport barriers [35]. The semiclassical eigenfunction hypothesis has been confirmed in several studies for 2D billiard systems and maps; see, e.g., Refs. [36–40] and references therein. In order to study this question for 4D maps, it is necessary to visualize the organization of phase space and in particular to display sets of individual tori.

For higher-dimensional systems a direct visualization of phase space is not possible.[1] Starting with the pioneering work of Froeschlé [43,44], several methods have been introduced to obtain a reduction to understand the dynamics. For example, two-dimensional plots of multisections [44,45] or projections to two [2,43,46,47] or three [48–50] dimensions, also including color to indicate the projected coordinate [51,52], frequency analysis [53–56], and action-space plots [57] . Further tools to investigate higher-dimensional phase spaces are chaos indicators to distinguish regular from chaotic motion, like finite-time Lyapunov exponents [58–60], fast Lyapunov indicator (FLI) [45,61,62], and many more; see, e.g., Refs. [63–65] and references therein.

[1] *You're just not thinking fourth dimensionally! – Right, right. I have a real problem with that.* (Quote from *Back to the Future Part III*).

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PHYSICAL REVIEW E **89**, 022902 (2014)

Visualization and comparison of classical structures and quantum states of four-dimensional maps

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For generic 4D symplectic maps we propose the use of 3D phase-space slices, which allow for the global visualization of the geometrical organization and coexistence of regular and chaotic motion. As an example, we consider two coupled standard maps. The advantages of the 3D phase-space slices are presented in comparison to standard methods, such as 3D projections of orbits, the frequency analysis, and a chaos indicator. Quantum mechanically, the 3D phase-space slices allow for the comparison of Husimi functions of eigenstates of 4D maps with classical phase-space structures. This confirms the semiclassical eigenfunction hypothesis for 4D maps.

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I. INTRODUCTION

Understanding higher-dimensional, dynamical systems, even with just a few particles, is a challenging task. Such systems are relevant in many areas of physics and chemistry [1], ranging from the dynamics of the solar system [2–4], dynamics of particle accelerators [5], to atoms and molecules [6–9]. Particular topics of interest concern the quantum signatures of Arnold diffusion [10–12] and quantum-classical correspondence in higher-dimensional mixed systems [13,14].

A standard example would be autonomous Hamiltonian systems with three degrees of freedom, which have a 6D phase space, which can be reduced to a 5D manifold due to energy conservation. Introducing a Poincaré section leads to a 4D symplectic map. This type of map also arises from time-periodically driven, Hamiltonian systems with two degrees of freedom, where a stroboscopic Poincaré section leads to a symplectic map acting on the 4D phase space. Such 4D maps are prototypical for the behavior of higher-dimensional systems, as they have the smallest possible dimension showing the essential difference to 2D symplectic maps: In 2D maps tori are one-dimensional and thus lead to absolute barriers of motion. In contrast, in 4D maps regular tori are two-dimensional manifolds in the 4D phase space, which cannot divide the phase space into dynamically distinct regions. One of the most fundamental consequences of this topological structure of 4D (and higher-dimensional) maps is that generically chaotic orbits can get arbitrarily close to any point in phase space, even if regular tori are present. One possible mechanism was constructed by Arnold [15], leading to the so-called Arnold diffusion [1,4,16,17]. Another striking phenomenon is the occurrence of power-law trapping of chaotic orbits in higher-dimensional systems with a mixed phase space [18–21], for which the mechanism is still an open question. While most of the analytical and numerical results about higher-dimensional systems have been obtained for the near-integrable case, many practical applications are concerned with generic systems, which cannot be described by perturbative methods; see, e.g., Refs. [22–24] and references therein.

For the lowest-dimensional Hamiltonian systems with regular and chaotic dynamics, such as 2D billiards or time-periodically driven 1D systems, the dynamics can be reduced to 2D symplectic maps. Their phase space can be easily visualized, providing substantial insight and intuition of the dynamics in phase space, such as chaotic motion, regular regions formed by 1D regular tori around stable periodic orbits, stable and unstable manifolds of unstable periodic orbits, nonlinear resonances, and hierarchical regions due to partial barriers at the border of the regular island. Also the time-evolution of trajectories can be visualized straightforwardly.

Such a direct visualization of the classical dynamics in phase space is also very useful when trying to understand the properties of the corresponding quantum mechanical system. According to the semiclassical eigenfunction hypothesis [25–27], one expects that eigenstates semiclassically concentrate on those regions in phase space that a generic orbit explores in the long-time limit. For ergodic systems this is proven by the quantum ergodicity theorem stating that almost all eigenfunctions become equidistributed in the semiclassical limit [28–30]. For systems with a mixed phase space one thus expects that (almost all) eigenstates either concentrate in the chaotic region or within the regular regions on the invariant tori. Away from the semiclassical limit this can be violated due to dynamical tunneling [31–34] and partial transport barriers [35]. The semiclassical eigenfunction hypothesis has been confirmed in several studies for 2D billiard systems and maps; see, e.g., Refs. [36–40] and references therein. In order to study this question for 4D maps, it is necessary to visualize the organization of phase space and in particular to display sets of individual tori.

For higher-dimensional systems a direct visualization of phase space is not possible.[1] Starting with the pioneering work of Froeschlé [43,44], several methods have been introduced to obtain a reduction to understand the dynamics. For example, two-dimensional plots of multisections [44,45] or projections to two [2,43,46,47] or three [48–50] dimensions, also including color to indicate the projected coordinate [51,52], frequency analysis [53–56], and action-space plots [57] . Further tools to investigate higher-dimensional phase spaces are chaos indicators to distinguish regular from chaotic motion, like finite-time Lyapunov exponents [58–60], fast Lyapunov indicator (FLI) [45,61,62], and many more; see, e.g., Refs. [63–65] and references therein.

[1] *You're just not thinking fourth dimensionally! – Right, right. I have a real problem with that.* (Quote from *Back to the Future Part III*).

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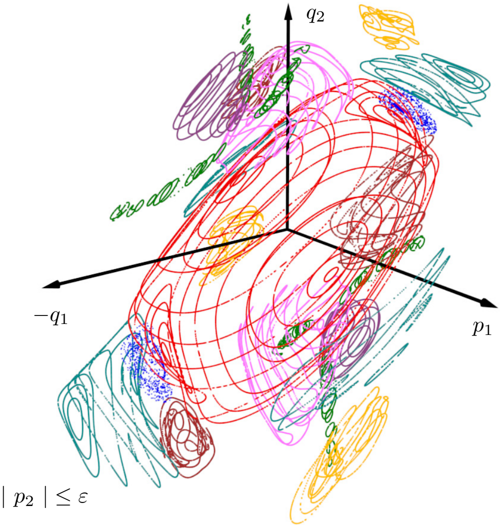
3D phase-space slices

[Richter, Lange, AB, Ketzmerick: Phys. Rev. E 2014]

Orbit: (p_1, p_2, q_1, q_2). Plot (p_1, q_1, q_2) when $|p_2| \leq \varepsilon = 10^{-4}$



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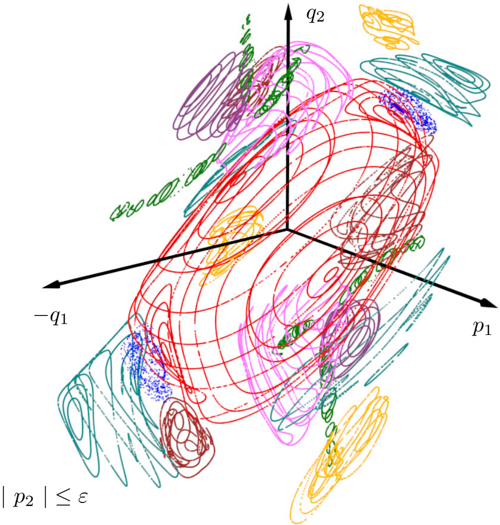
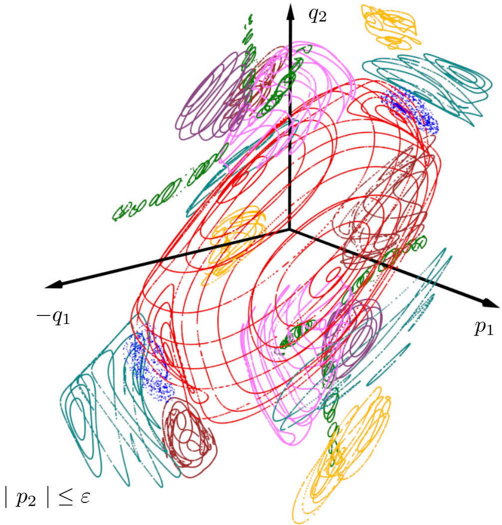


- Elliptic-elliptic fixed point

3D phase-space slices

[Richter, Lange, AB, Ketzmerick: Phys. Rev. E 2014]

Orbit: (p_1, p_2, q_1, q_2). Plot (p_1, q_1, q_2) when $|p_2| \leq \varepsilon = 10^{-4}$



- Elliptic-elliptic fixed point
- Regular tori
  - 2D in 4D
  - 1D lines in slice

3D phase-space slices

[Richter, Lange, AB, Ketzmerick: Phys. Rev. E 2014]

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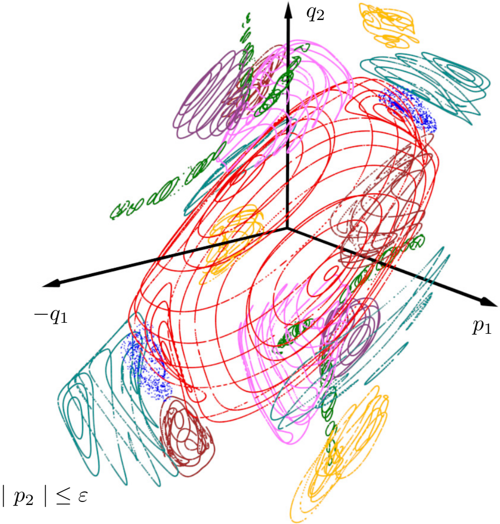

- Elliptic-elliptic fixed point
- Regular tori
  - 2D in 4D
  - 1D lines in slice
  - inner tori not shown

3D phase-space slices

[Richter, Lange, AB, Ketzmerick: Phys. Rev. E 2014]

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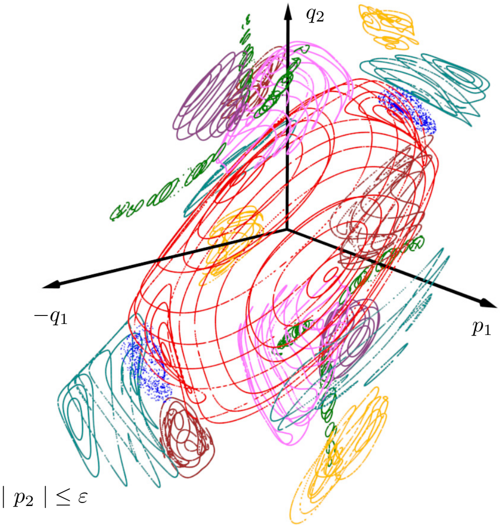

- Elliptic-elliptic fixed point
- Regular tori
  - 2D in 4D
  - 1D lines in slice
  - inner tori not shown
- smaller ε:
  - sharper image
  - more iterations

3D phase-space slices

[Richter, Lange, AB, Ketzmerick: Phys. Rev. E 2014]

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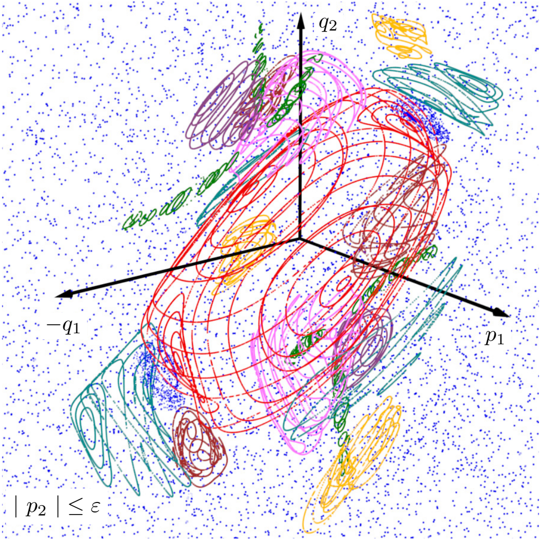
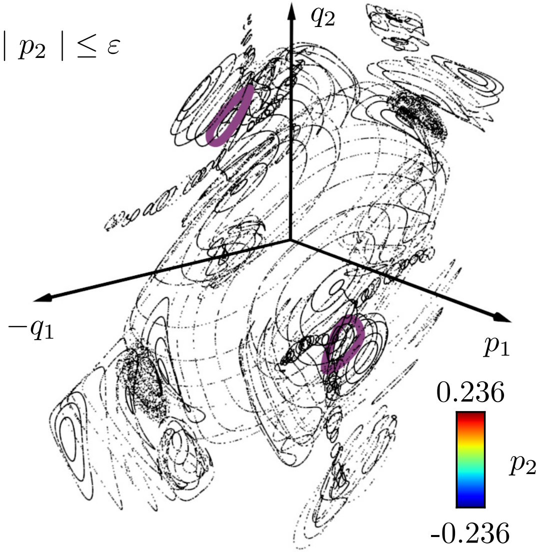


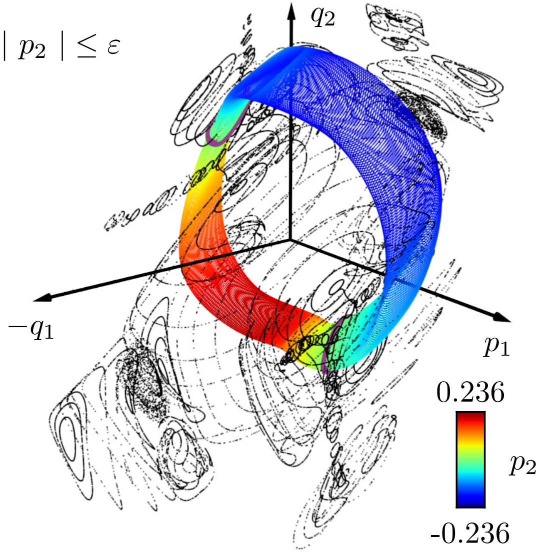
- Elliptic-elliptic fixed point
- Regular tori
  - 2D in 4D
  - 1D lines in slice
  - inner tori not shown
- smaller ε:
  - sharper image
  - more iterations
- chaotic motion “outside”, can go “inside”

2D tori in the 3D Phase-Space Slice



Use color for p_2 [Patsis, Zachilas 1994]

2D tori in the 3D Phase-Space Slice



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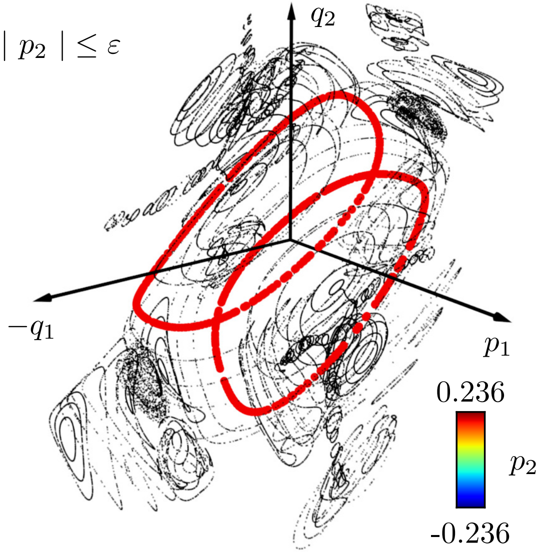

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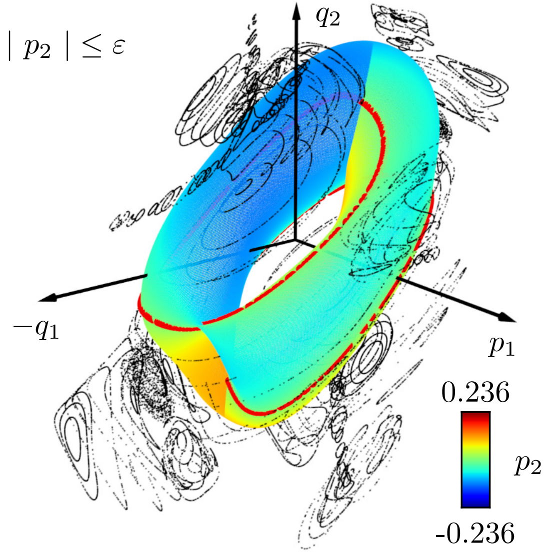

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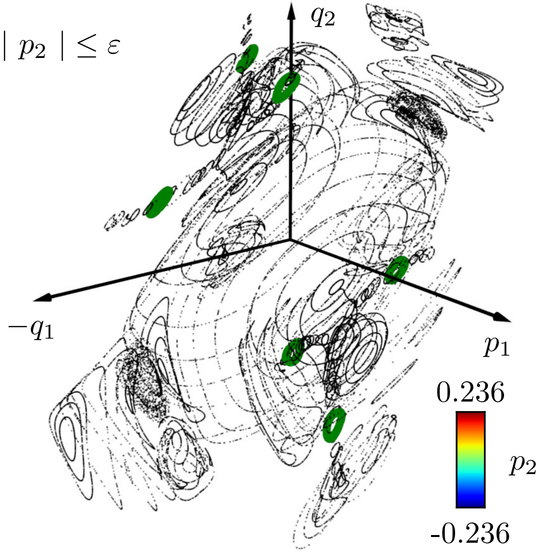

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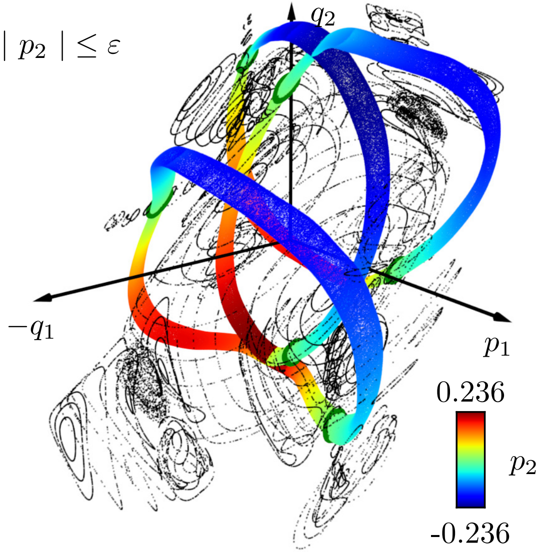
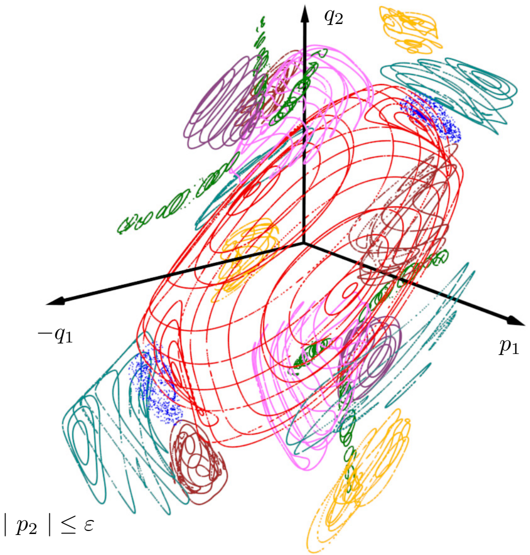
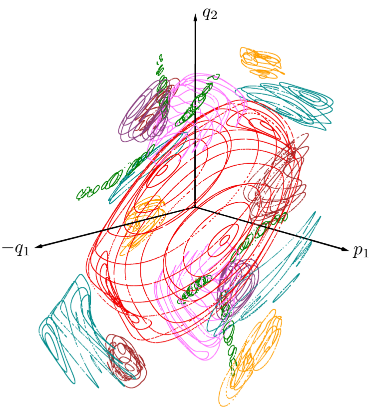


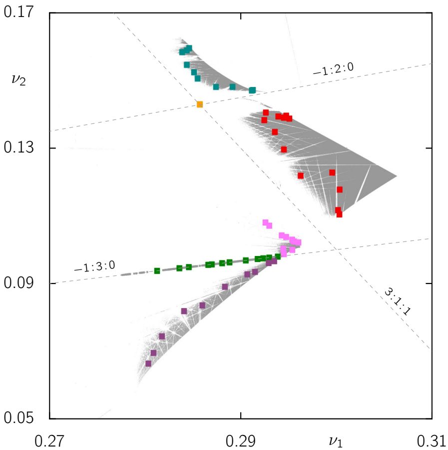
Use color for p_2 [Patsis, Zachilas 1994]

3D Phase-Space Slices for $|p_2 - p_2^*| \leq \varepsilon$, variation of p_2^*



3D Phase-Space Slices and Frequency Analysis





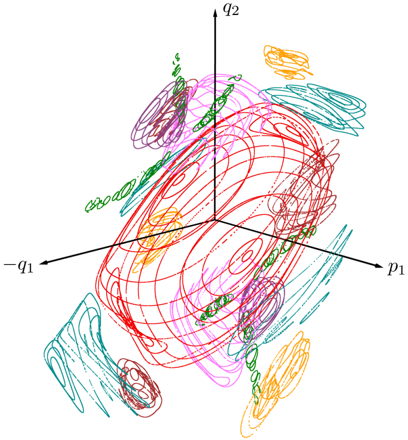
Resonance lines $m_1 : m_2 : n$

$$m_1 \cdot \nu_1 + m_2 \cdot \nu_2 = n$$

Origin of gaps!

Organization of phase space

[Lange, Richter, Onken, AB, Ketzmerick: Chaos 2014]



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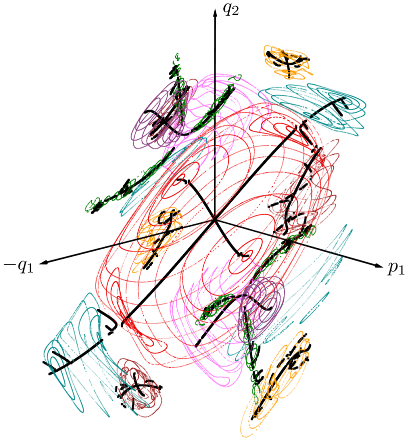
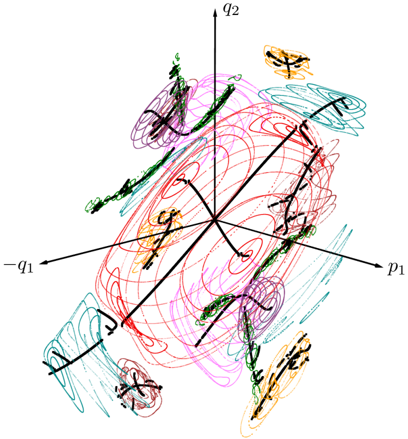


Skeleton of 1D tori

Organization of phase space

[Lange, Richter, Onken, AB, Ketzmerick: Chaos 2014]



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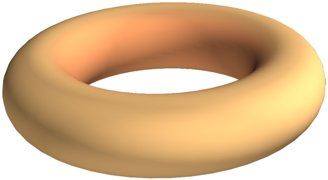

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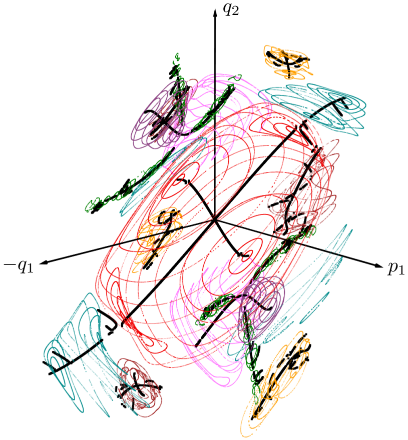
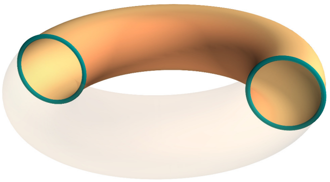


Skeleton of 1D tori



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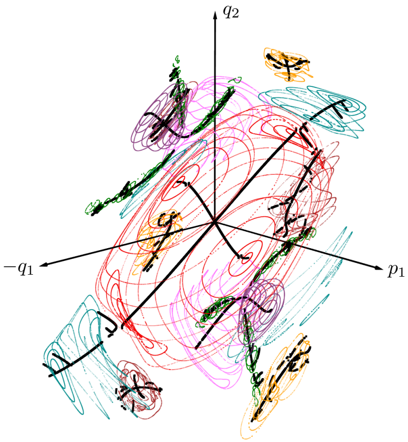
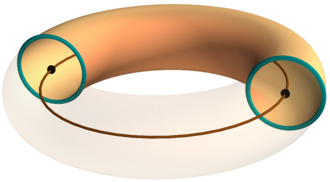


Skeleton of 1D tori



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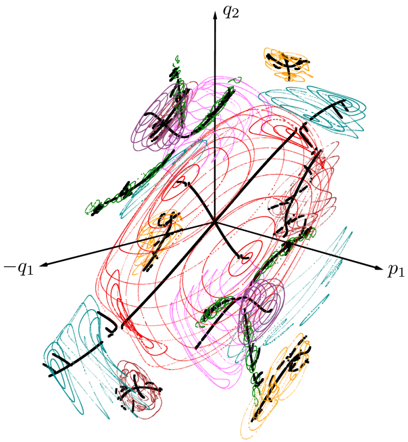

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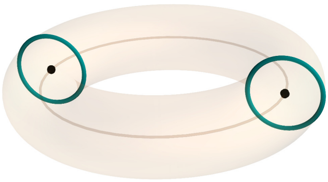

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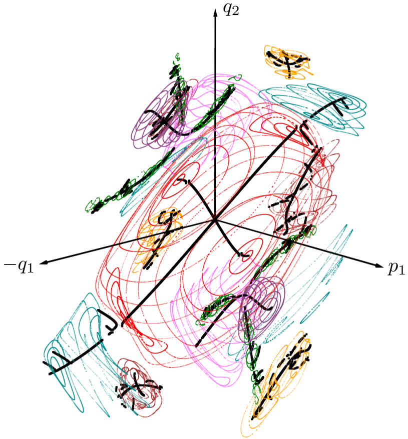
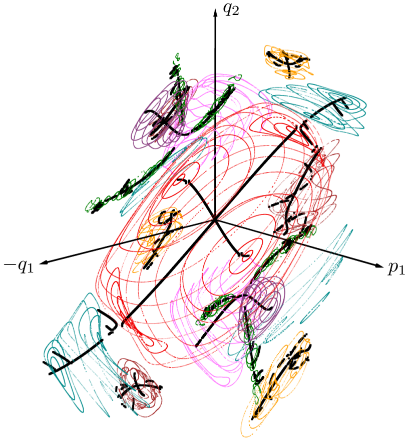


Invariant objects

- 2D tori

Organization of phase space

[Lange, Richter, Onken, AB, Ketzmerick: Chaos 2014]

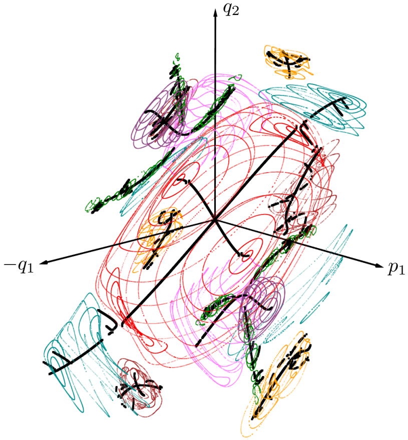


Invariant objects

- 2D tori
- 1D tori

Organization of phase space

[Lange, Richter, Onken, AB, Ketzmerick: Chaos 2014]

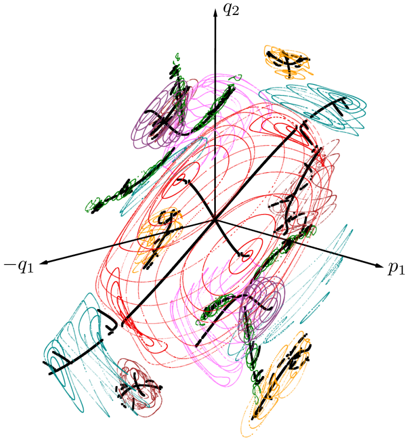


Invariant objects

- 2D tori
- 1D tori
- 0D: Elliptic-Elliptic fixed points/p.o.

Organization of phase space

[Lange, Richter, Onken, AB, Ketzmerick: Chaos 2014]



Invariant objects

- 2D tori
- 1D tori
- 0D: Elliptic-Elliptic fixed points/p.o.

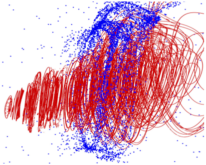
Further aspects

- Hierarchy
- Bifurcations of 1D tori
- Hyperbolic objects

Applications

- Power law trapping

  [Ding, Bountis, Ott 1990;
  Altmann, Kantz 2007;
  Shepelyansky 2010; ...]



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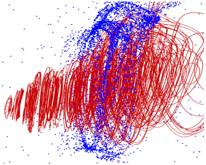
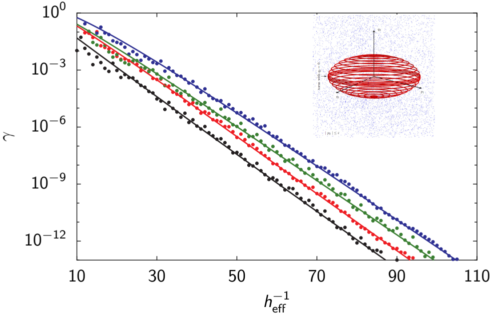


- Tunneling



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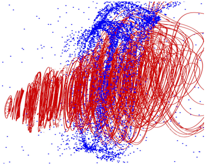
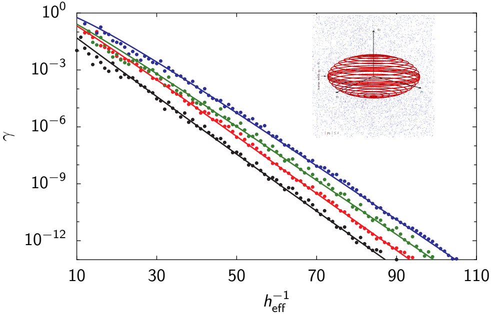


- Tunneling



  - *Experimental observation of resonance-assisted tunneling*

    Gehler, Löck, Shinohara, AB, Ketzmerick, Kuhl, Stöckmann, arXiv:1502.04263
  - *Complex Paths for Resonance-Assisted Tunneling*

    Fritzsch, Mertig, Löbner, AB, Ketzmerick (in prep. 2015)

Applications

- Power law trapping

  [Ding, Bountis, Ott 1990;
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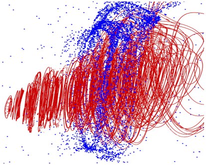

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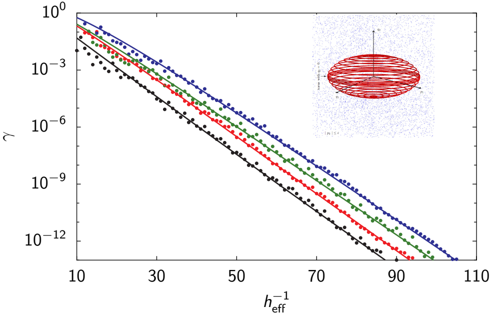
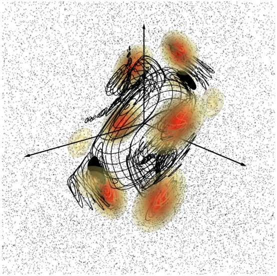
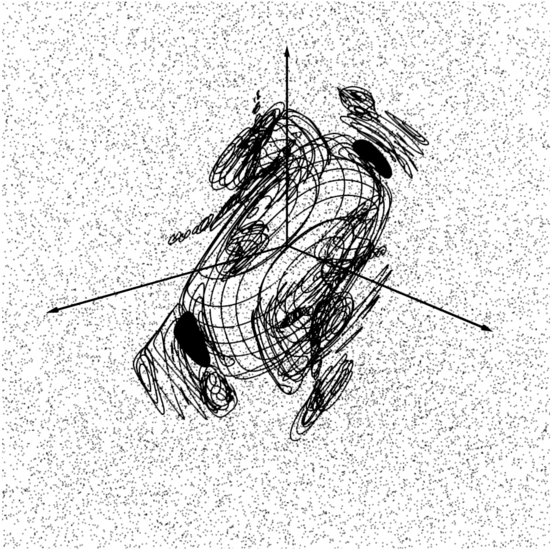




- Visualization of eigenstates

Quantum eigenstates on the phase-space slice



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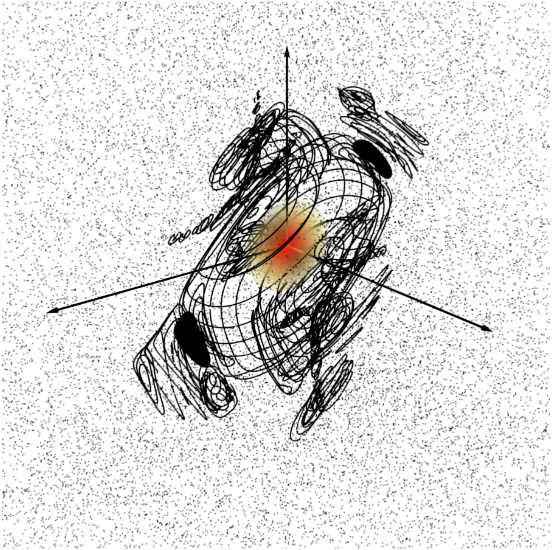

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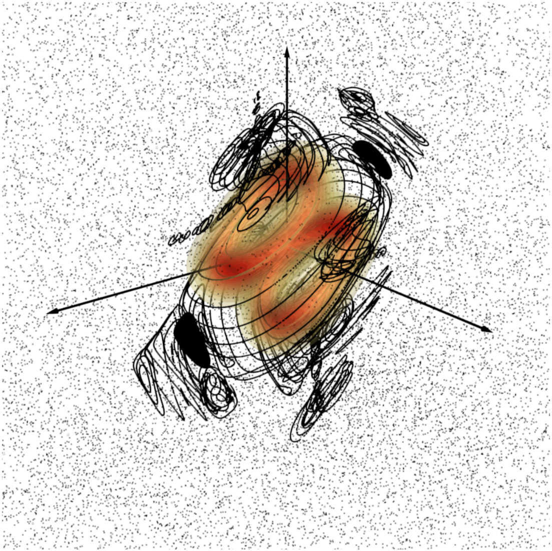

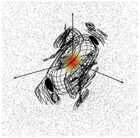

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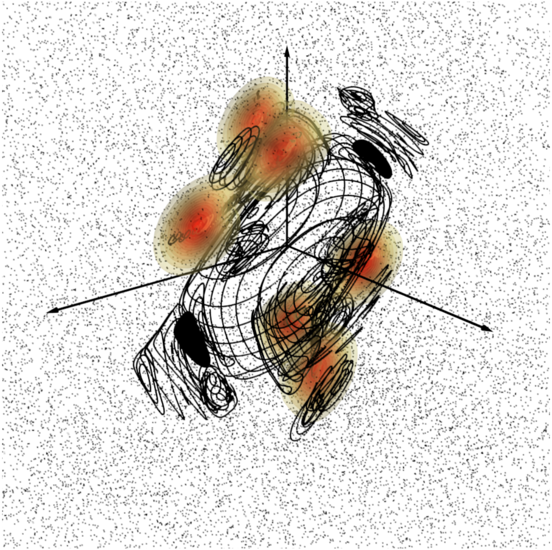
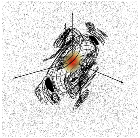
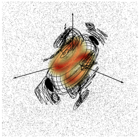
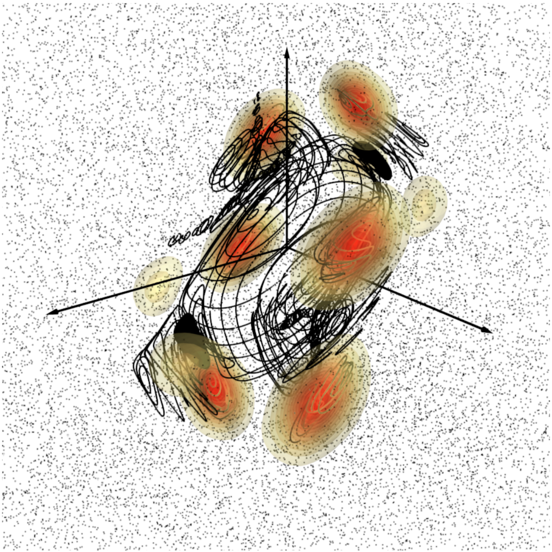
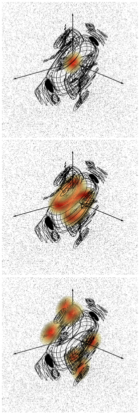






Quantum eigenstates on the phase-space slice





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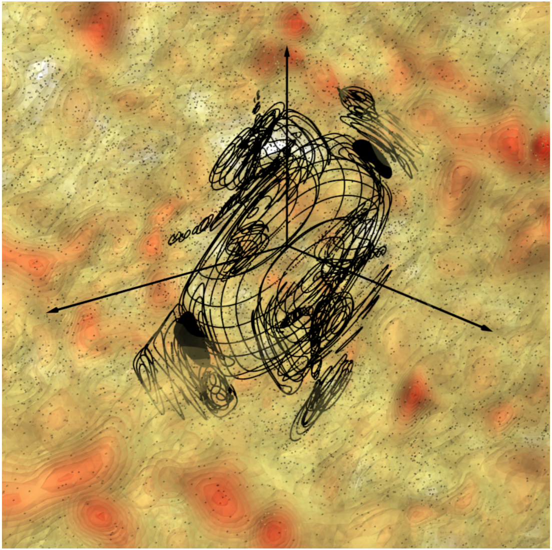

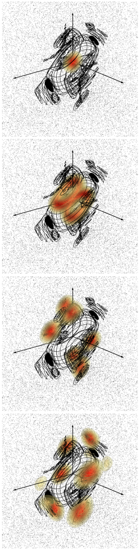

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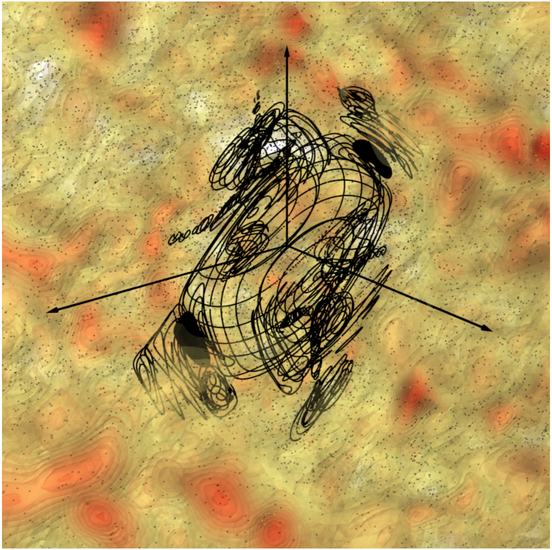
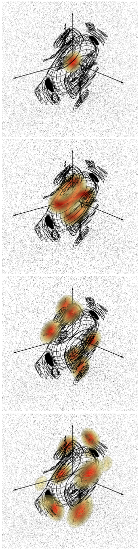


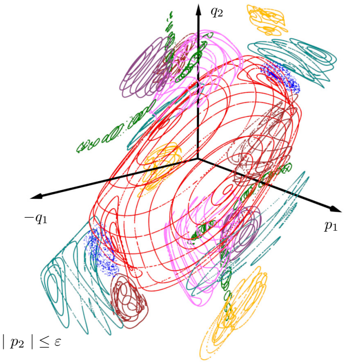


Confirms localization of eigenstates on invariant objects in a higher dimensional system

Summary and Outlook

Phase space of 4D maps

- 3D phase space slices:
  global view of regular dynamics



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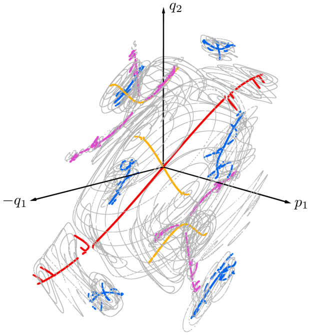
- Organization:
  Skeleton of 1D–tori



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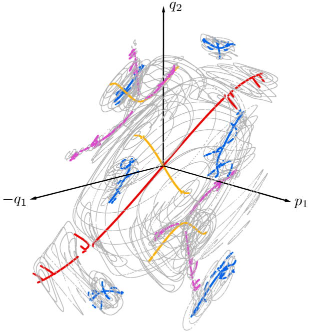
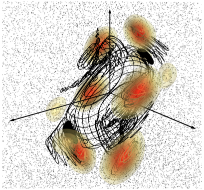


Applications

- Power-law trapping
- Regular to chaotic tunneling
- Spectral and eigenvector statistics
- Husimi representation of eigenstates



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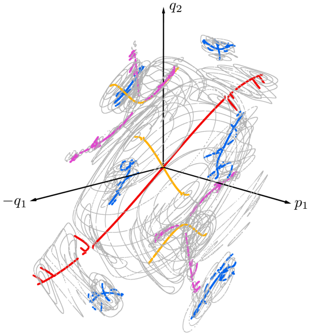

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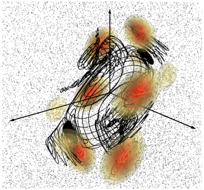

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Outlook

- Hyperbolic structures
- Chaotic transport

References

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  Chaos **24**, 024409 (2014)
  http://dx.doi.org/10.1063/1.4882163

- All movies can be found at
  http://www.comp-phys.tu-dresden.de/supp/