Dynamics of cold atoms in chaotic/disordered potentials

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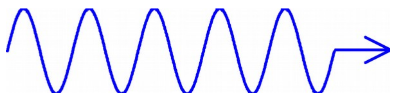
Outline

- Anderson localization with cold atoms in a disordered optical potential
- Mobility edge in 3D
- Semiclassical spectral function
  - Classical spectral function
  - Smooth semiclassical correction
  - Singular semiclassical correction

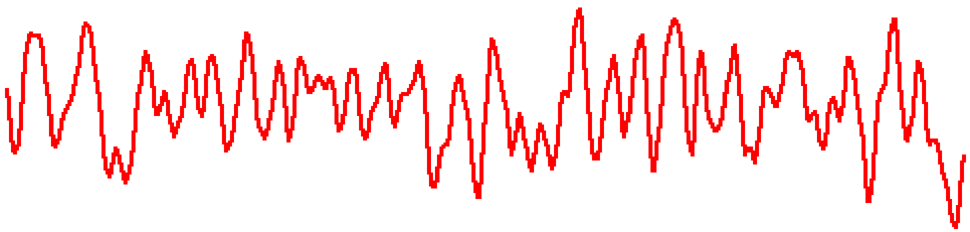
Anderson (a.k.a. Strong) localization

- Particle in a disordered (random) potential:

One-dimensional system

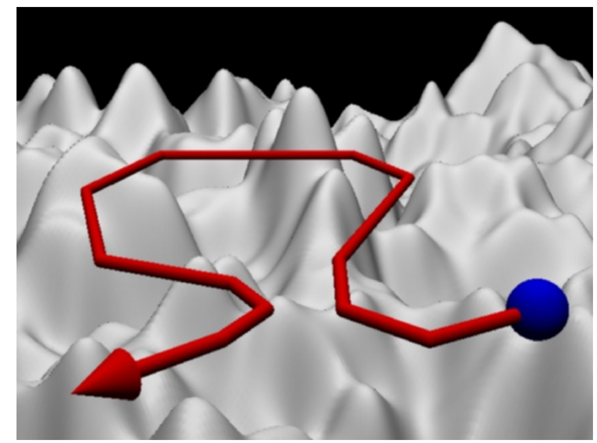


Particle with energy *E*



Disordered potential $V(z)$ (typical value V_0)

Two-dimensional system

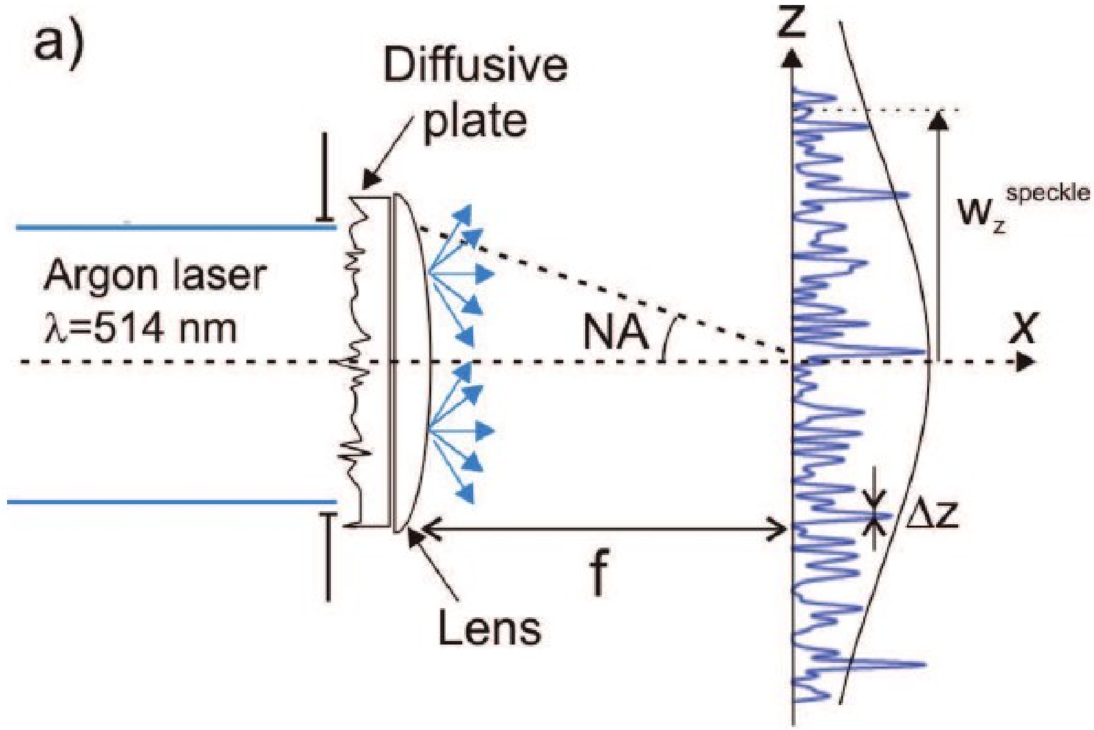


- When $E \ll V_0$, the particle is classically trapped in the potential wells.
- When $E \gg V_0$, the classical motion is ballistic in 1d, typically diffusive in dimension 2 and higher.
- Quantum interference may inhibit diffusion at long times =>

Anderson localization

Speckle optical potential (2D version)

- Speckle created by shining a laser on a diffusive plate:



V. Josse et al, Institut d'Optique (Palaiseau)

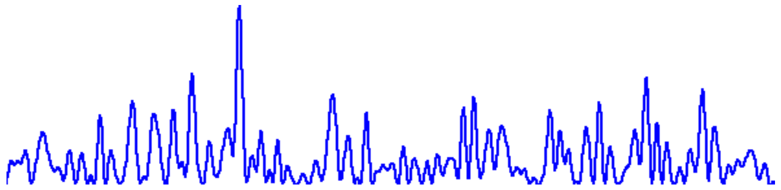
Speckle spot size

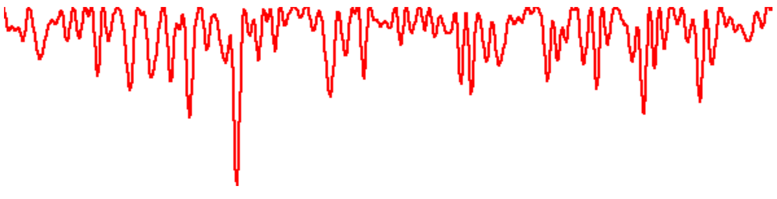
$$\sigma \approx \frac{\lambda}{\mathrm{NA}}$$

λ : laser wavelength

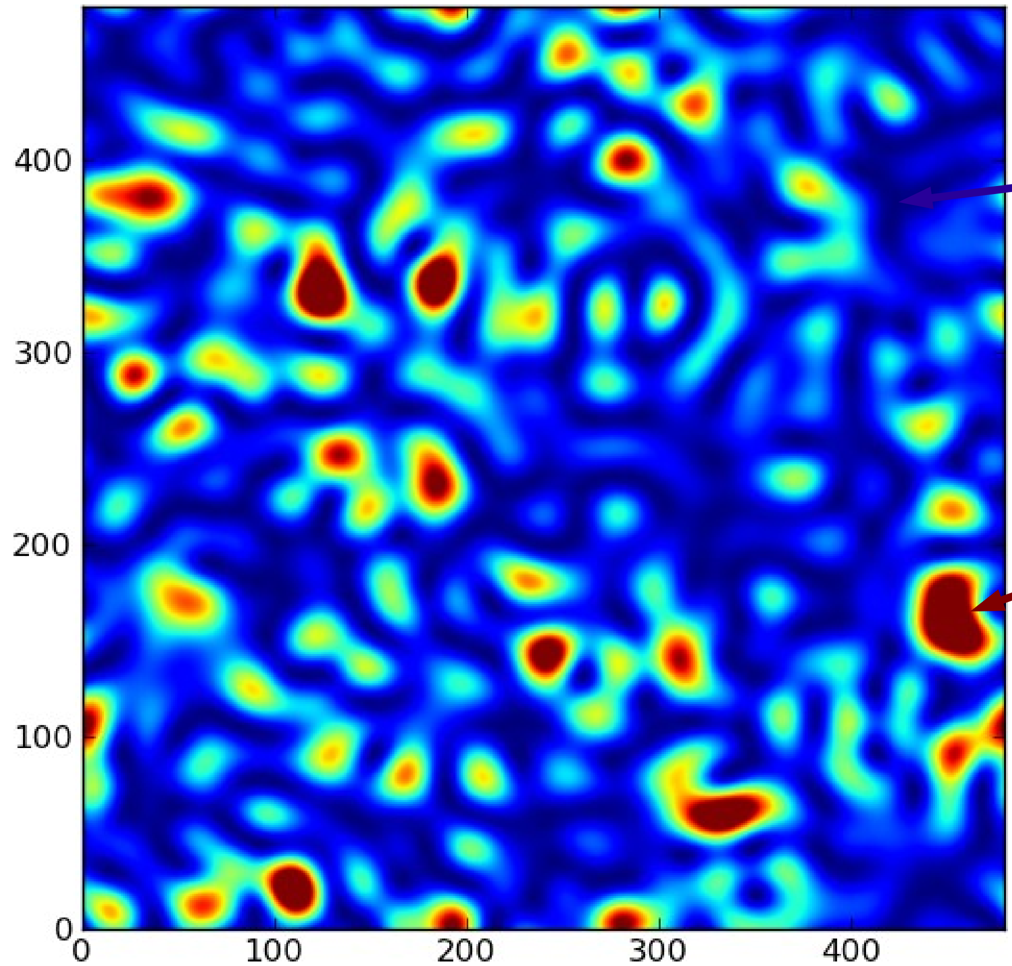
NA : Numerical Aperture

- The speckle electric field is a (complex) random variable with Gaussian statistics. All correlation functions can be computed.
- Depending on the sign of the detuning, the optical potential is bounded either from above or from below





A typical realization of a 2D blue-detuned speckle potential



Dark region (low potential, ocean floor, zero energy)

Bright spot (high potential)

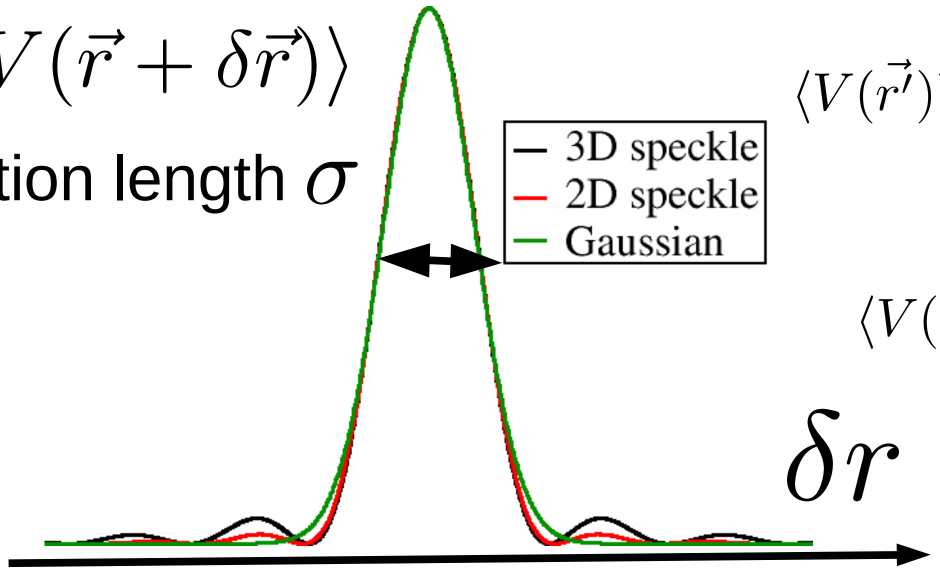
Distribution of potential value $\boxed{P(V) = \frac{\exp(-V/V_0)}{V_0}\Theta(V)}$

Rigorous low energy bound, no high energy bound

Spatial correlation function for speckle potential

$$\langle V(\vec{r})V(\vec{r}+\delta\vec{r})\rangle$$

correlation length σ



$$\langle V(\vec{r'})V(\vec{r'}+\vec{r})\rangle = V_0^2\left(1+4\frac{J_1^2(\alpha k_{\mathrm{L}} r)}{\alpha^2 k_{\mathrm{L}}^2 r^2}\right)$$

2D

$$\langle V(\vec{r'})V(\vec{r'}+\vec{r})\rangle = V_0^2\left(1+\frac{\sin^2 k_{\mathrm{L}} r}{k_{\mathrm{L}}^2 r^2}\right)$$

3D

Important energy scales: potential strength V_0

correlation energy $E_\sigma = \dfrac{\hbar^2}{m\sigma^2}$

When $E = E_\sigma$ the de Broglie wavelength is equal to σ

$V_0 \ll E_\sigma$

“quantum” regime

$V_0 \gg E_\sigma$

“classical” regime

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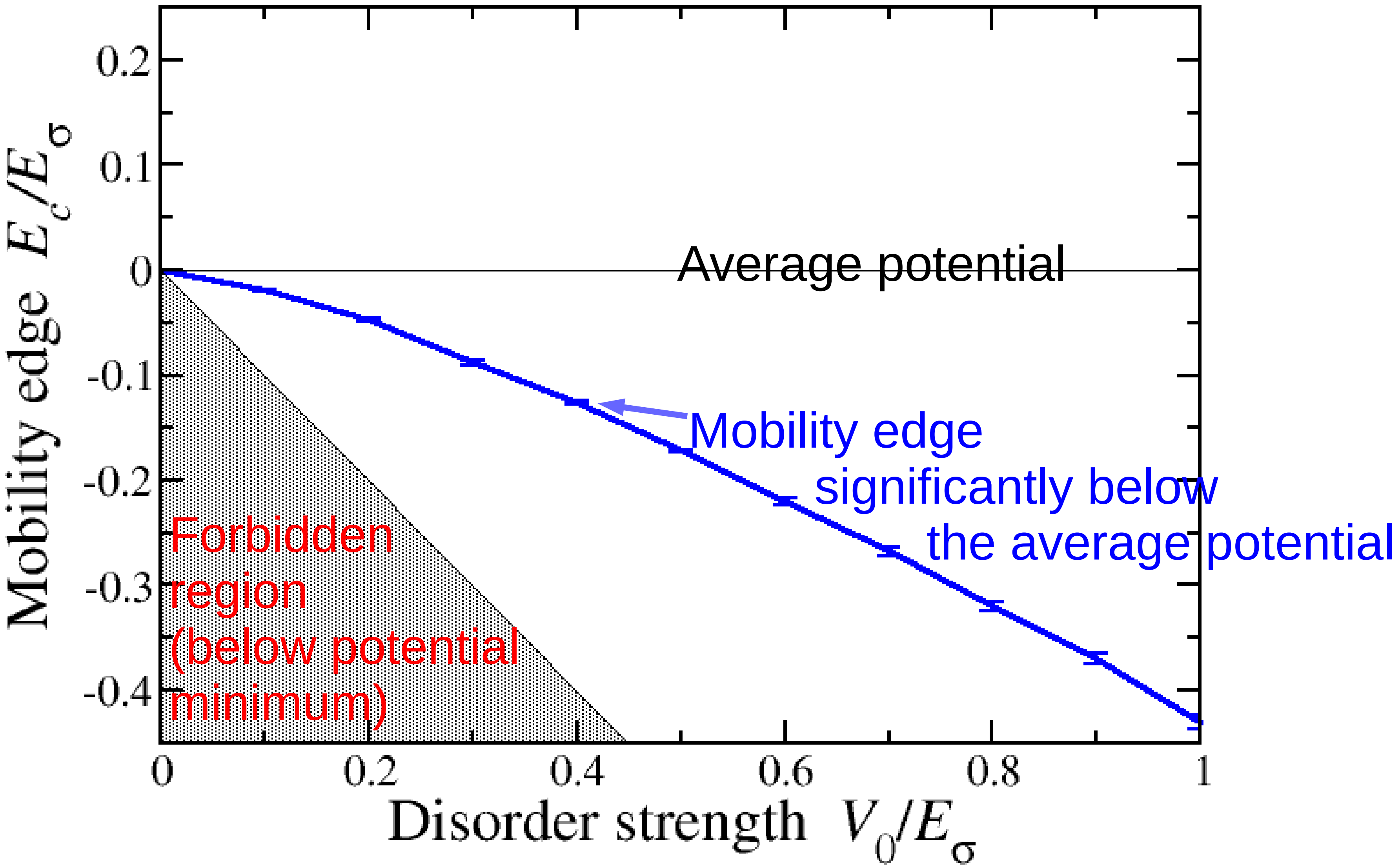
Numerical results for the mobility edge

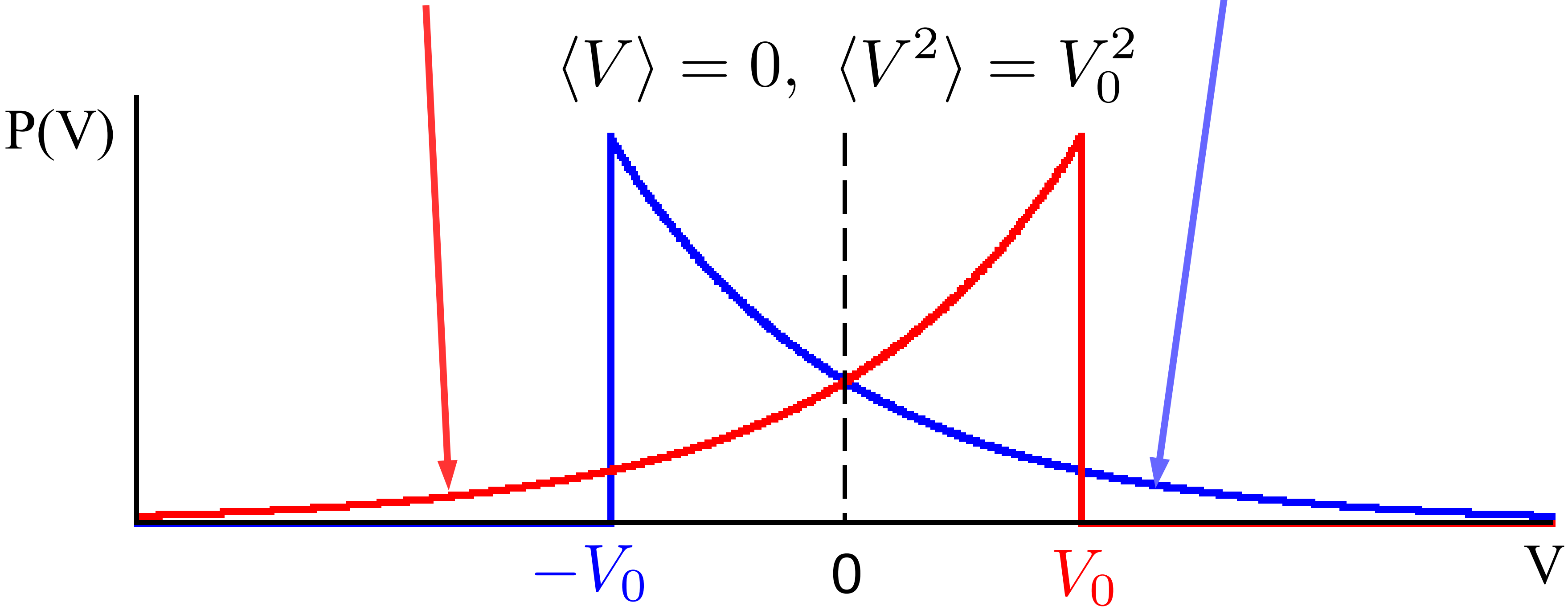


Blue-detuned 3D spherical speckle

Delande and Orso,
PRL 113, 060601 (2014)

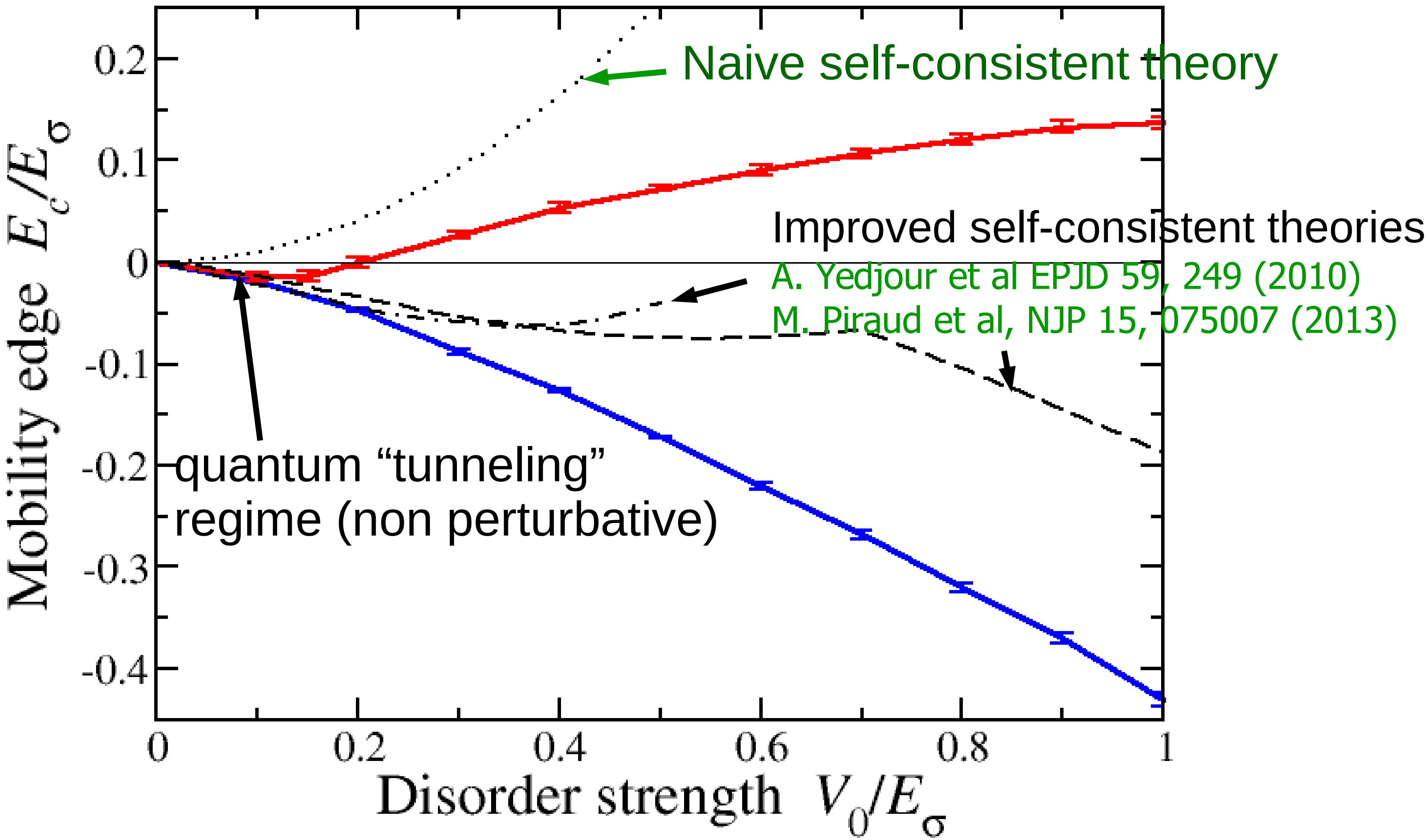
Effect of on-site potential distribution

- Use red-detuned speckle instead of blue-detuned speckle



- Very asymmetric distributions
- Blue speckle has a strict lower energy bound, red does not
- Even order (in V_0) contributions are identical for blue and red
- Odd order contributions have opposite signs
- Naive and improved self-consistent theories **predict the same mobility edge**.

Huge blue-red asymmetry



What is the spectral function?

- Makes the connection between momentum *k* and energy *E*

Averaged over disorder realizations

$$A_{\boldsymbol{k}}(E) = -\frac{1}{\pi}\mathrm{Im}\overline{\langle \boldsymbol{k}|G(E+i\epsilon)|\boldsymbol{k}\rangle} = \langle \boldsymbol{k}|\overline{\delta(E-\hat{H})}|\boldsymbol{k}\rangle$$

Spectral function Green function Hamiltonian $\hat{H} = \hat{T} + \hat{V}$

kinetic+potential

- Probability density that a plane wave $|\boldsymbol{k}\rangle$ has energy *E*.
- Normalization: $\int \mathrm{d}E\ A_{\boldsymbol{k}}(E) = 1$
- Link with density of states: $\rho(E) = \sum_{\boldsymbol{k}} A_{\boldsymbol{k}}(E)$
- In the absence of disorder:

$A_{\boldsymbol{k}}(E) = \delta(E - T_{\boldsymbol{k}})$ with $T_{\boldsymbol{k}} = \langle \boldsymbol{k}|\hat{T}|\boldsymbol{k}\rangle$(usually $k^2/2m$)

- “Sum rules”: $\int \mathrm{d}E\ E\ A_{\boldsymbol{k}}(E) = T_{\boldsymbol{k}} + \overline{V}$ ← potential average

$\int \mathrm{d}E\ E^2\ A_{\boldsymbol{k}}(E) = (T_{\boldsymbol{k}} + \overline{V})^2 + \overline{\delta V^2}$ ← potential variance

Spectral function in weak disorder

- The self-energy $\Sigma_{\boldsymbol{k}}(E)$ is defined by the Dyson equation:

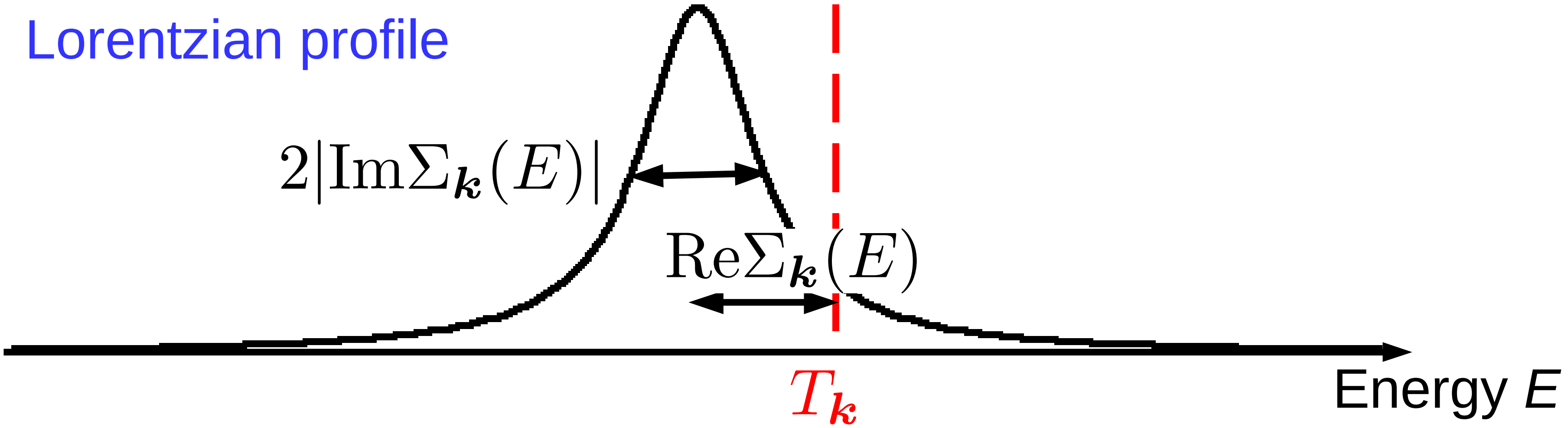
$$\overline{\langle \boldsymbol{k}|G(E)|\boldsymbol{k}\rangle} = \frac{1}{E - T_{\boldsymbol{k}} - \Sigma_{\boldsymbol{k}}(E)}$$

  It is a smooth function of *k* and *E*.

- Then:

$$A_{\boldsymbol{k}}(E) = -\frac{1}{\pi}\ \frac{\mathrm{Im}\Sigma_{\boldsymbol{k}}(E)}{[E - T_{\boldsymbol{k}} - \mathrm{Re}\Sigma_{\boldsymbol{k}}(E)]^2 + \mathrm{Im}\Sigma_{\boldsymbol{k}}(E)^2}$$

Lorentzian profile

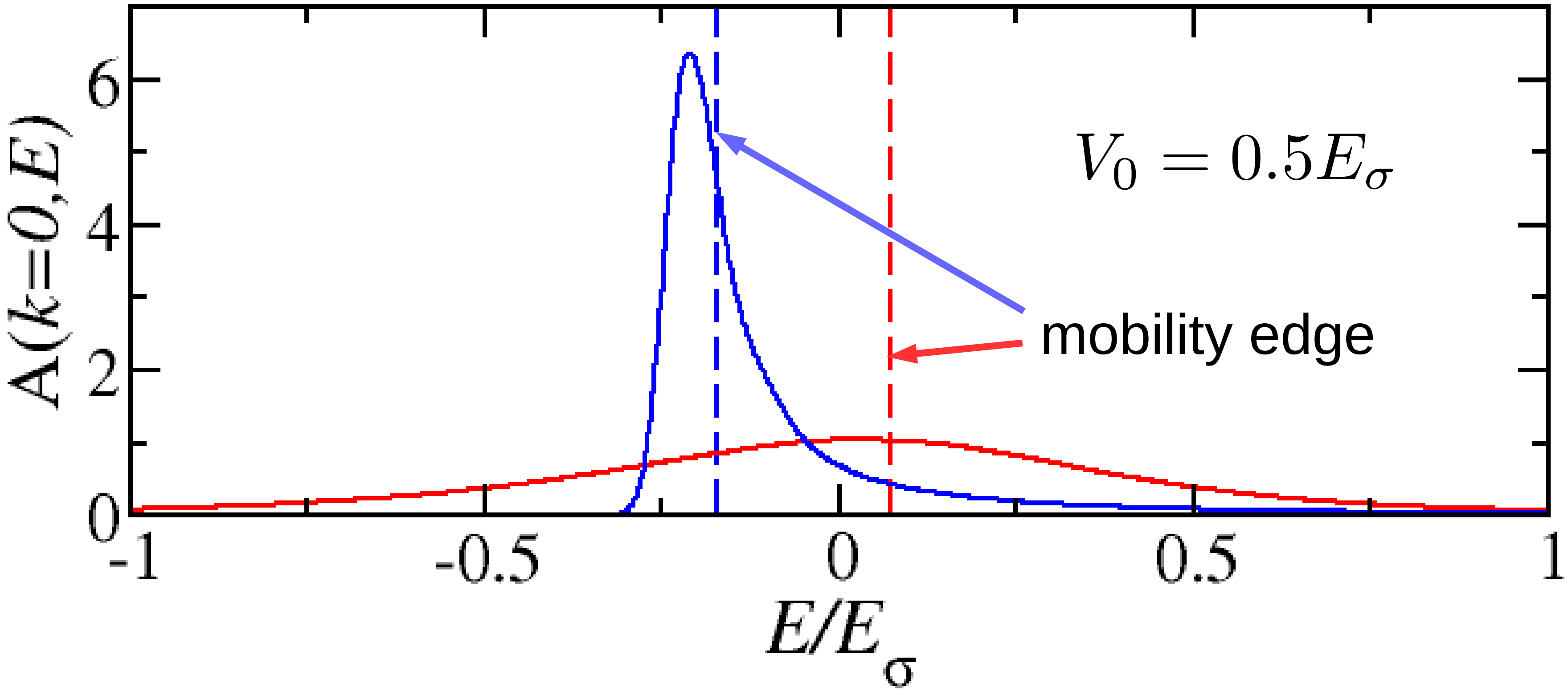


Blue-red asymmetry

- We compute numerically the spectral function:

$$A_{\boldsymbol{k}}(E) = \overline{\langle \boldsymbol{k} | \delta(E - \hat{H}) | \boldsymbol{k} \rangle}$$

Average probability that a plane wave with wave-vector k has energy E



- On-shell approximation: $A_0(E) = \delta(E)$
- "Better" approximation: shifted δ-function, Lorentzian
- Needs a better approximation for the spectral function

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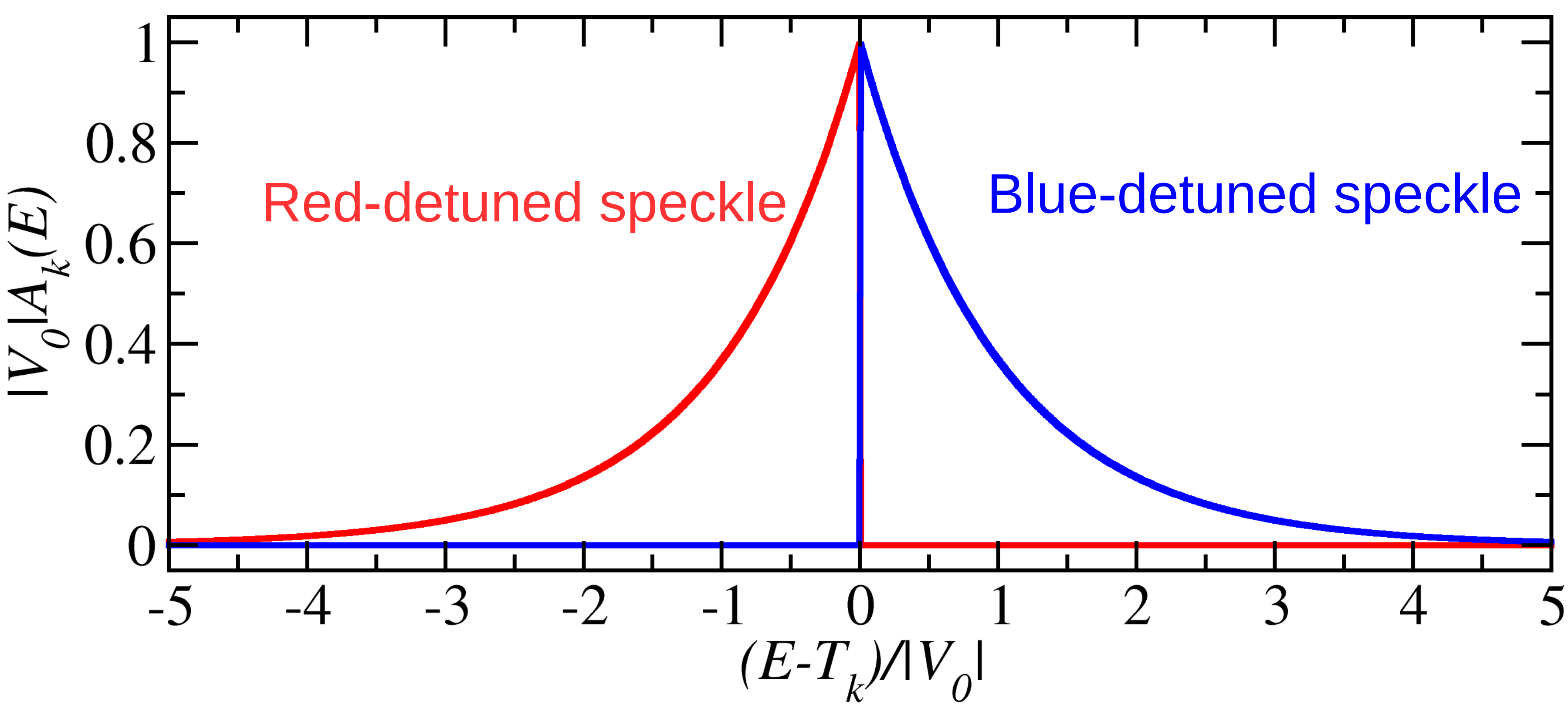
Classical spectral function

- Neglect entirely non-commutativity of *r* and *p:*

$$A_{\boldsymbol{k}}(E) = \overline{\langle \boldsymbol{k}|\delta(E - \hat{H})|\boldsymbol{k}\rangle} \approx \overline{\delta(E - T_{\boldsymbol{k}} - \hat{V})} = P(E - T_{\boldsymbol{k}})$$

where *P(V)* is the distribution of potential strength



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Semiclassical spectral function

- Use the Weyl symbol (Wigner transform) of the spectral function:

$$A_{\boldsymbol{k}}(E) = \overline{\delta(E-\hat{H})_W(\boldsymbol{k})}$$

where:

$$X_W(\boldsymbol{r},\boldsymbol{p}) = \int \mathrm{d}\boldsymbol{\rho}\; e^{i\boldsymbol{p}.\boldsymbol{\rho}/\hbar}\; \langle \boldsymbol{r}-\boldsymbol{\rho}/2|X(\hat{\boldsymbol{r}},\hat{\boldsymbol{p}})|\boldsymbol{r}+\boldsymbol{\rho}/2\rangle$$

- Expand the Wigner transform in powers of $\hbar$:

$$\delta(E-\hat{H})_W \approx \delta(E-H) - \frac{\hbar^2}{16}\left\{H \overleftrightarrow{\Lambda}^2 H\right\}\delta''(E-H)$$
$$-\frac{\hbar^2}{24}\left\{H \overleftrightarrow{\Lambda} H \overleftrightarrow{\Lambda} H\right\}\delta'''(E-H)$$

where $\overleftrightarrow{\Lambda} = \overleftarrow{\partial_{\boldsymbol{r}}} \cdot \overrightarrow{\partial_{\boldsymbol{k}}} - \overleftarrow{\partial_{\boldsymbol{k}}} \cdot \overrightarrow{\partial_{\boldsymbol{r}}}$ is the Poisson bracket.

- The leading order is the classical spectral function

Semiclassical spectral function (continued)

- Leading order quantum correction:

$$\Delta A_{\boldsymbol{k}}(E) \approx -\frac{\hbar^2}{12} \sum_{i,j=1}^{d} \left[m_{ij}^{-1} C_{ij}^{(2)}(E - T_{\boldsymbol{k}}) - \frac{v_i v_j}{2} C_{ij}^{(3)}(E - T_{\boldsymbol{k}}) \right]$$

Effective mass: $m_{ij}^{-1} = \partial_{k_i} \partial_{k_j} T_{\boldsymbol{k}}$

Group velocity: $v_i = \partial_{k_i} T_{\boldsymbol{k}}$

$$C_{ij}^{(n)}(x) := \overline{(\partial_{r_i} \partial_{r_j} V(\boldsymbol{r}))\delta^{(n)}(x - V(\boldsymbol{r}))}$$

- What is left is to compute the $C_{ij}^{(n)}$ correlation functions

Detour: Gaussian potential

- Gaussian distribution of potential

$$P(V) = \frac{1}{\sqrt{2\pi}V_0} \exp\left(-\frac{V^2}{2V_0^2}\right)$$

- Gaussian correlation function:

$$\overline{V(\boldsymbol{\rho})V(\boldsymbol{\rho}+\boldsymbol{r})} = V_0^2 \exp\left(-\frac{\boldsymbol{r}^2}{2\sigma^2}\right)$$

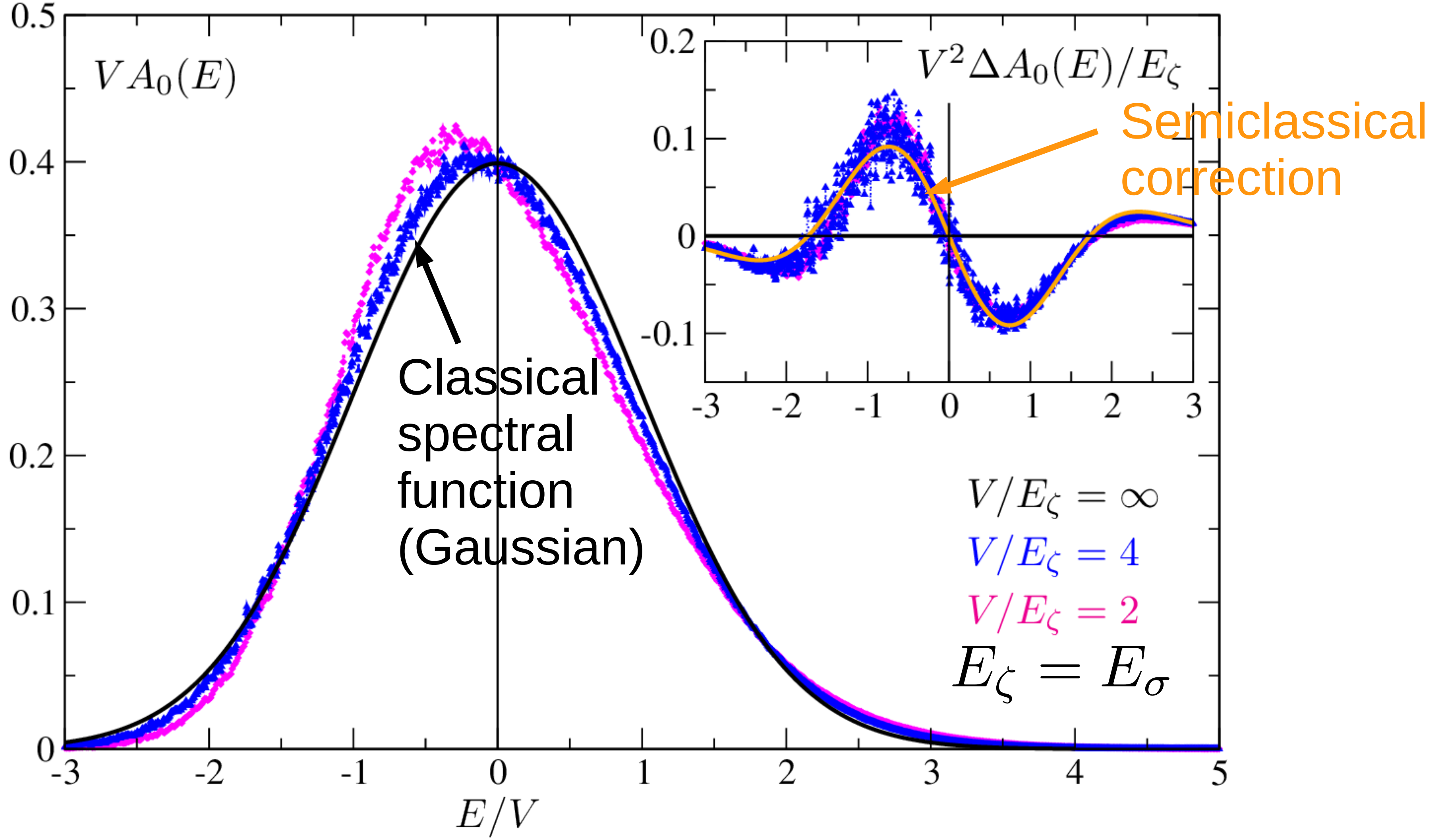
- Then: $\boxed{\Delta A_{\boldsymbol{k}}(E) \approx -\frac{V_0^2 E_\sigma}{12}\left[d\partial_E^3 - T_{\boldsymbol{k}}\partial_E^4\right] P(E - T_{\boldsymbol{k}})}$

$$E_\sigma = \frac{\hbar^2}{m\sigma^2}$$ correlation energy

- Sum rules of order 0, 1 and 2 automatically satisfied
- The two terms in the correction have relative strengths

$\frac{E_\sigma}{V_0}$ and $\frac{E_\sigma T_{\boldsymbol{k}}}{V_0^2}$ Semiclassical regime: $E_\sigma \ll V_0$

Numerics for the 2D Gaussian potential



$$\Delta A_0(E) \approx -\frac{E E_\sigma}{6\sqrt{2\pi}V_0^3}\left(3-\frac{E^2}{V_0^2}\right)\exp\left(-\frac{E^2}{2V_0^2}\right)$$ Works very well!

M.I. Trappe et al, arxiv:1411.2412

Back to the speckle potential

- A similar calculation for a speckle potential with Gaussian correlation function gives:

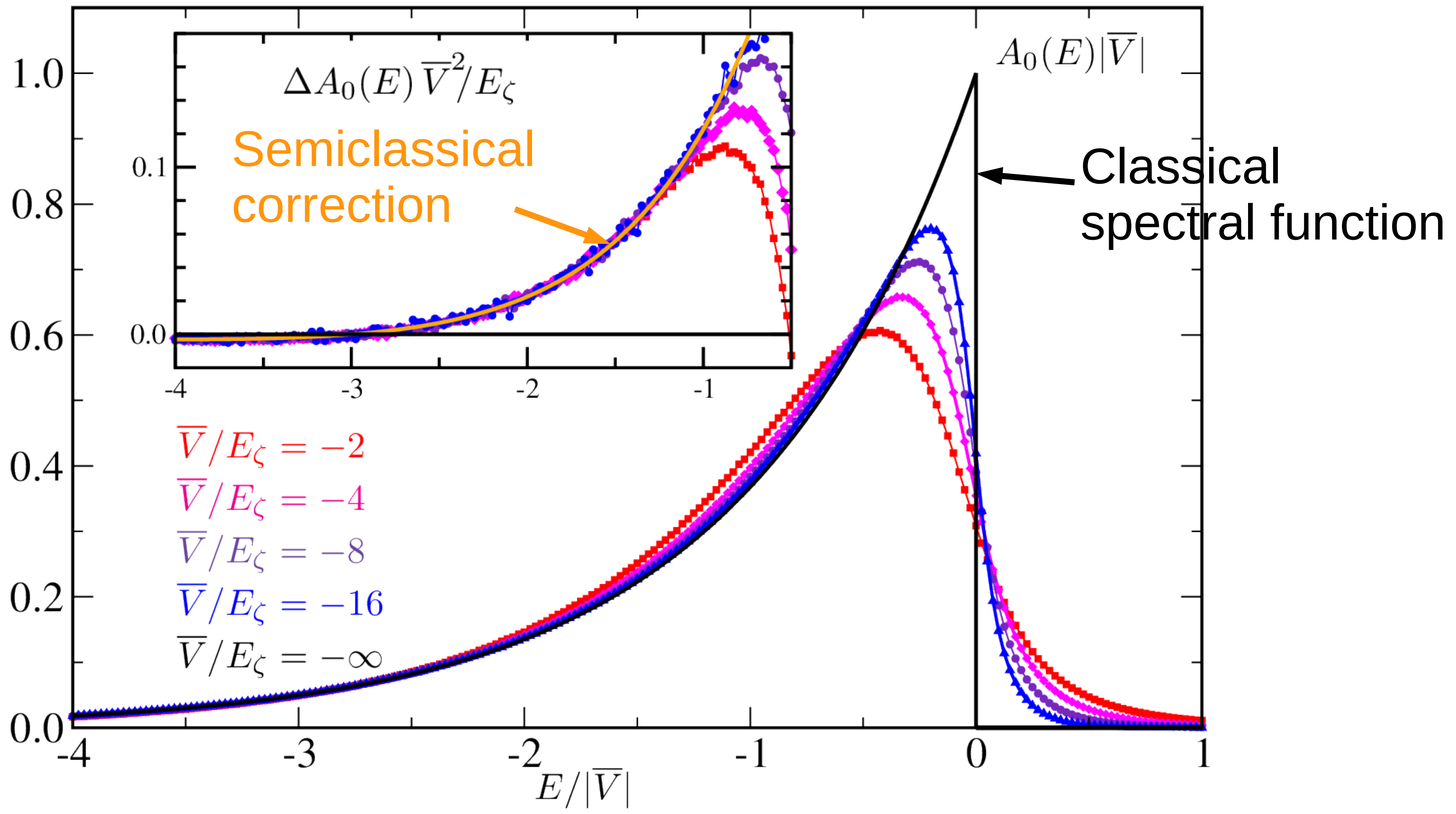
$$\Delta A_{\boldsymbol{k}}(E) \approx -\frac{|V_0|E_\sigma}{12}\left[d\partial_E^3 - T_{\boldsymbol{k}}\partial_E^4\right](E-T_{\boldsymbol{k}})P(E-T_{\boldsymbol{k}})$$

- Sum rules are again automatically satisfied.
- Especially simple for *k*=0:

$$\Delta A_0(E) \approx -\frac{dE_\sigma}{12V_0^2}f'''(E/V_0) \quad \text{with} \quad f(x) = \Theta(x)xe^{-x}$$

$$f'''(x) = \delta'(x) - 2\delta(x) + \Theta(x)(3-x)e^{-x}$$

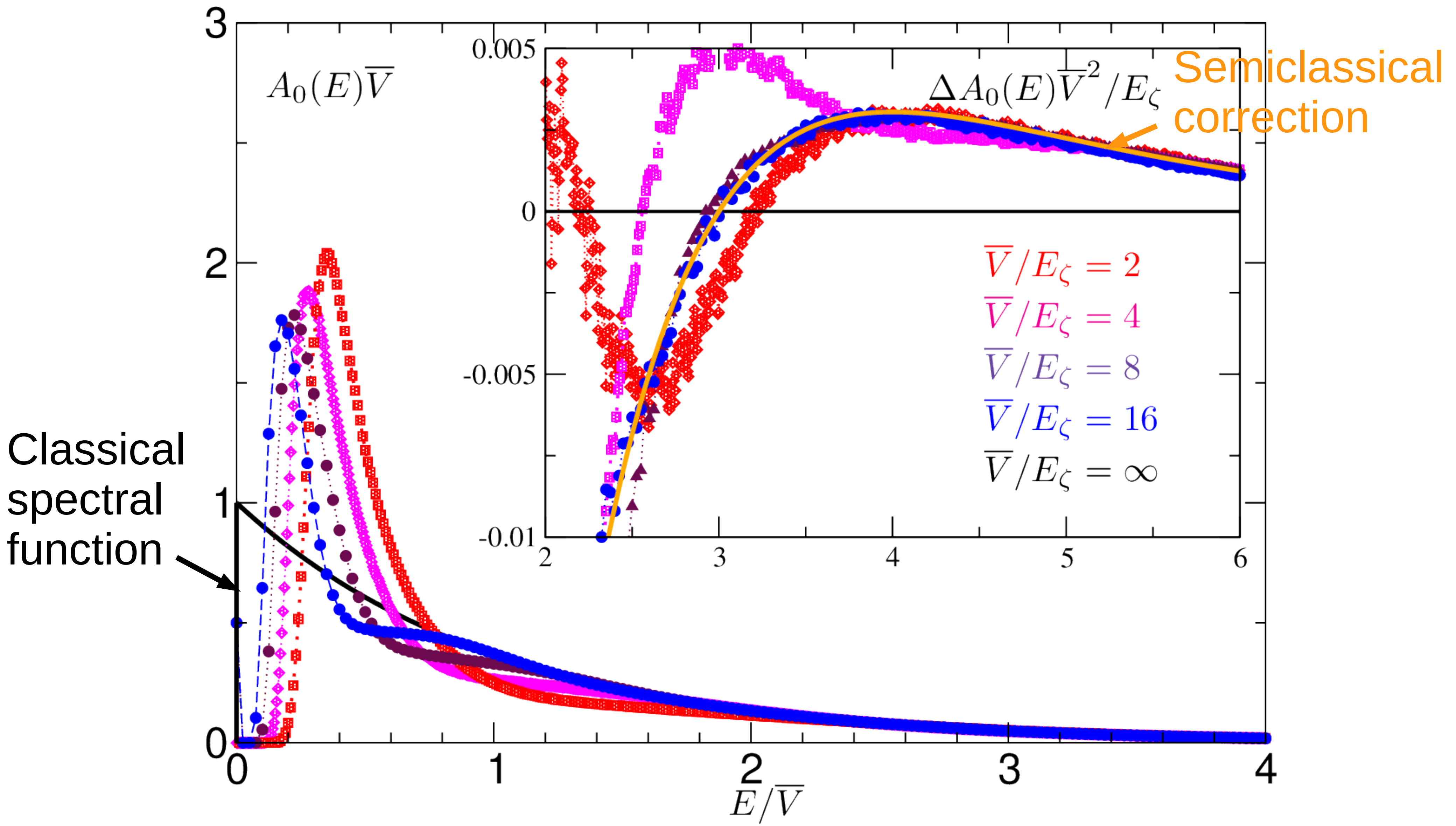
2D red-detuned speckle potential



Excellent agreement in the tail, but large deviation near E=0!

M.I. Trappe et al, arxiv:1411.2412

2D blue-detuned speckle



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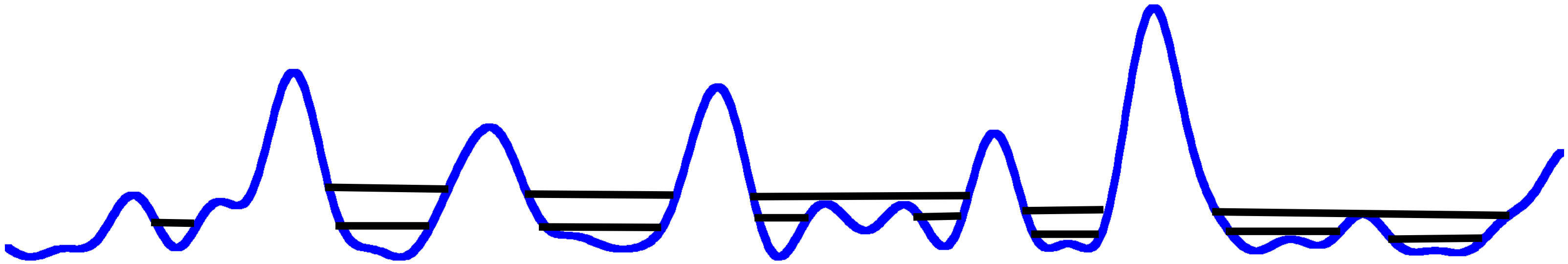
The role of periodic orbits

- Density of states (and spectral function) at low energy is dominated by states trapped in potential minima



- Use semiclassical Green function and average over statistical properties of potential minima.
- Simple model: approximate each potential minimum by an harmonic potential filled by a series of equally spaced energy levels.
- Requires to know the probability distribution of energy minima and local curvature.
- Can be completely computed in 1D.

Statistical properties of energy minima (1D speckle)

- Joint distribution for the potential V, its derivative V' and V”

$$P(V,V',V") = \frac{\sigma^4}{4\sqrt{2\pi}V_0^3 V} e^{-\frac{24V+16V"\sigma^2+\frac{(V'^2-2VV")^2\sigma^4}{V^3}}{16V_0}}$$

$$\sqrt{\frac{(-V'^2+2VV")V_0}{V}}\left(I_{-\frac{1}{4}}\left(\frac{(V'^2-2VV")^2\sigma^4}{16V^3V_0}\right)+I_{\frac{1}{4}}\left(\frac{(V'^2-2VV")^2\sigma^4}{16V^3V_0}\right)\right)$$

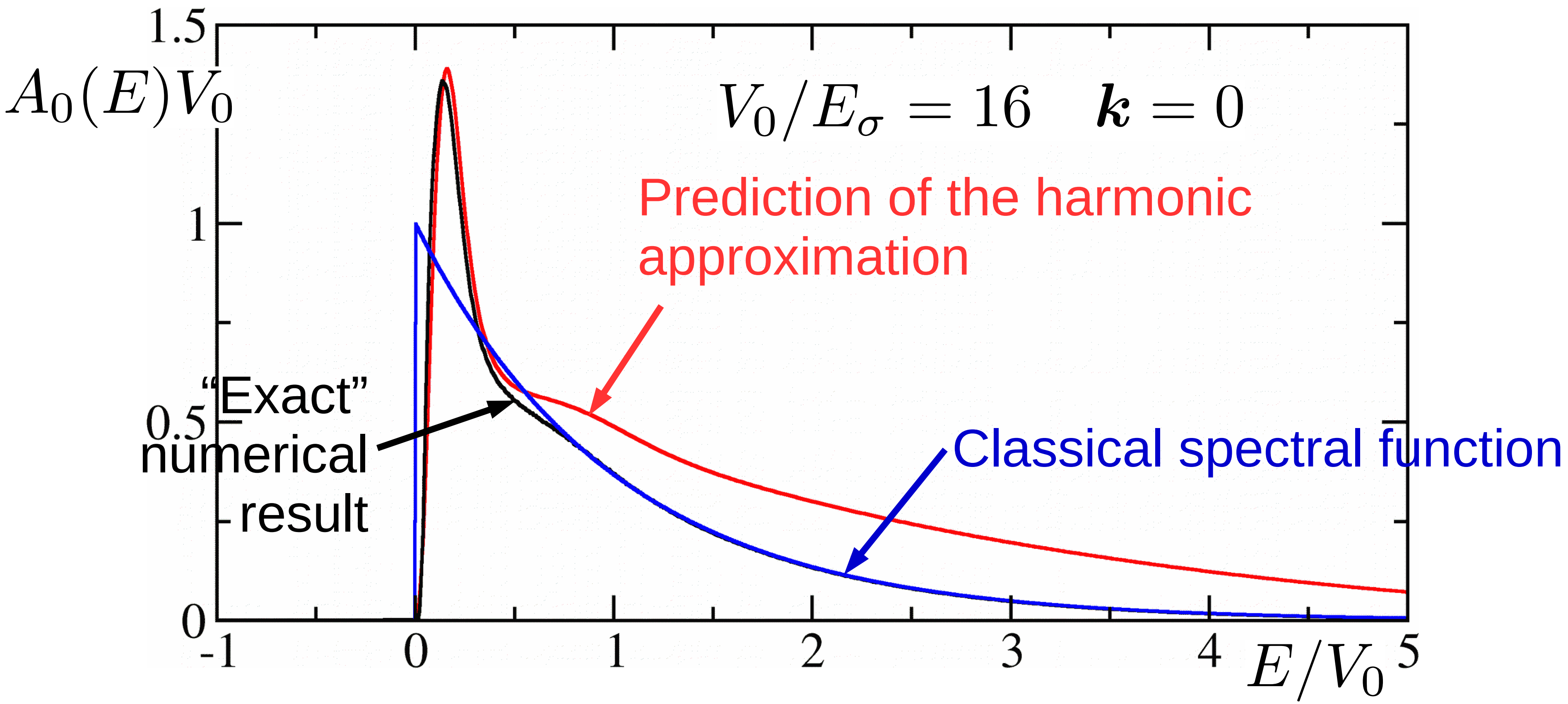
- At the potential minima, the joint distribution for V and V” is approximately:

$$P(V,V") \propto \boxed{\frac{e^{-V/V_0}}{\sqrt{V}}}\boxed{\sqrt{V"}\exp\left(-\frac{\sigma^2 V"}{V_0}\right)}$$

Porter-Thomas distribution
Mostly minima close to 0

Typical curvature V_0/σ^2
Almost no shallow potential well

Approximate spectral function for blue-detuned speckle

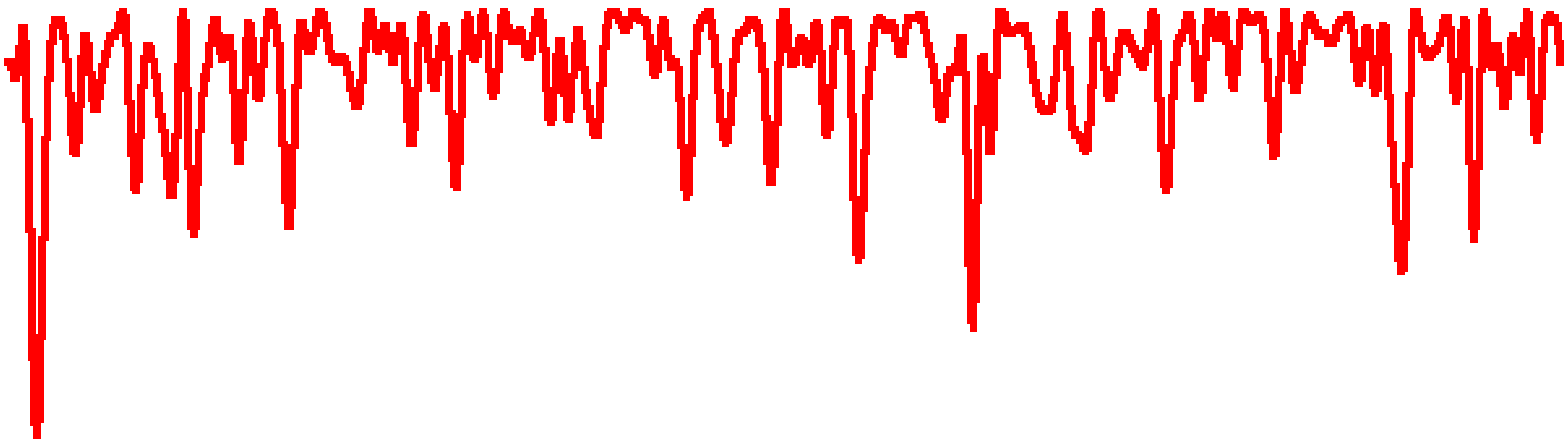


* Rather good agreement near *E*=0 where the peak is well reproduced
* The small energy structure has a characteristic energy:

$$\hbar\sqrt{\frac{V_0}{m\sigma^2}} = \sqrt{V_0 E_\sigma}$$ => convergence to the classical limit is slow

What about red-detuned potential?

- Obtained by turning a blue-detuned potential upside down => same statistical properties of potential extrema, modulo a change of sign of the curvature.



- Periodic orbits are now very complicated!
- Use a different method => go to the time domain:

$$A_{\boldsymbol{k}}(E) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty \overline{\langle \boldsymbol{k}|e^{-i\hat{H}t/\hbar}|\boldsymbol{k}\rangle}\; e^{iEt/\hbar}\;\mathrm{d}t$$

Evolution operator: can use semiclassics

Semiclassical approximations for the propagator

- Very short time: use the Baker-Campbell-Haussdorf formula:

$$e^{-i\hat{H}t/\hbar} = e^{-i\hat{T}t/\hbar}\; e^{-i\hat{V}t/\hbar}\; e^{-t^2[\hat{V},\hat{T}]/\hbar^2}\ldots$$

- At lowest order, generates the classical spectral function:

$$\overline{\langle \boldsymbol{k}|e^{-i\hat{H}t/\hbar}|\boldsymbol{k}\rangle} = \frac{e^{-iT_{\boldsymbol{k}}/\hbar}}{1+iV_0t/\hbar}$$

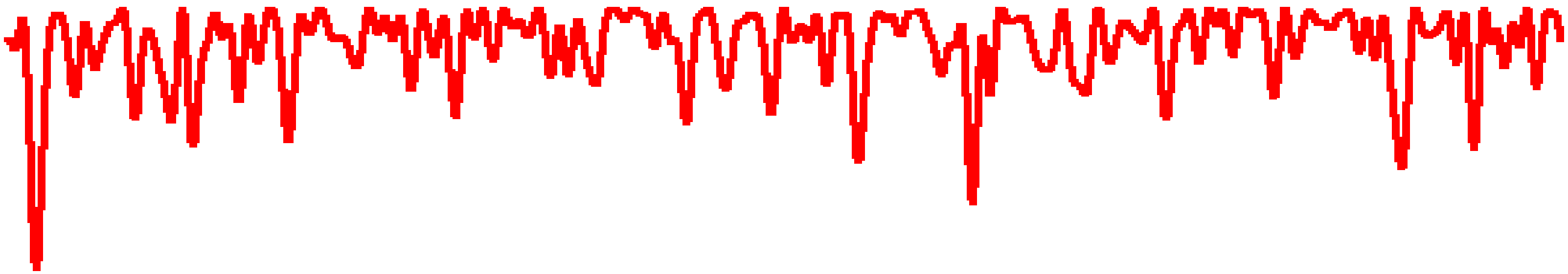
- Next orders generate exactly the same corrections than the Wigner expansion in powers of $\hbar$.
- At longer time, use the semiclassical Van Vleck propagator:

$$\langle \boldsymbol{k}|e^{-i\hat{H}t/\hbar}|\boldsymbol{k}\rangle = \sum_{\text{classical orbits } j} [\text{prefactor}]\; e^{iS_j(\boldsymbol{k},\boldsymbol{k},t)/\hbar}$$

classical action

- For blue-detuned speckle, only short orbits trapped in the low-energy potential minima survive the disorder averaging => equivalent to the harmonic oscillator approximation in the energy domain.

Semiclassical propagator for the red-detuned case

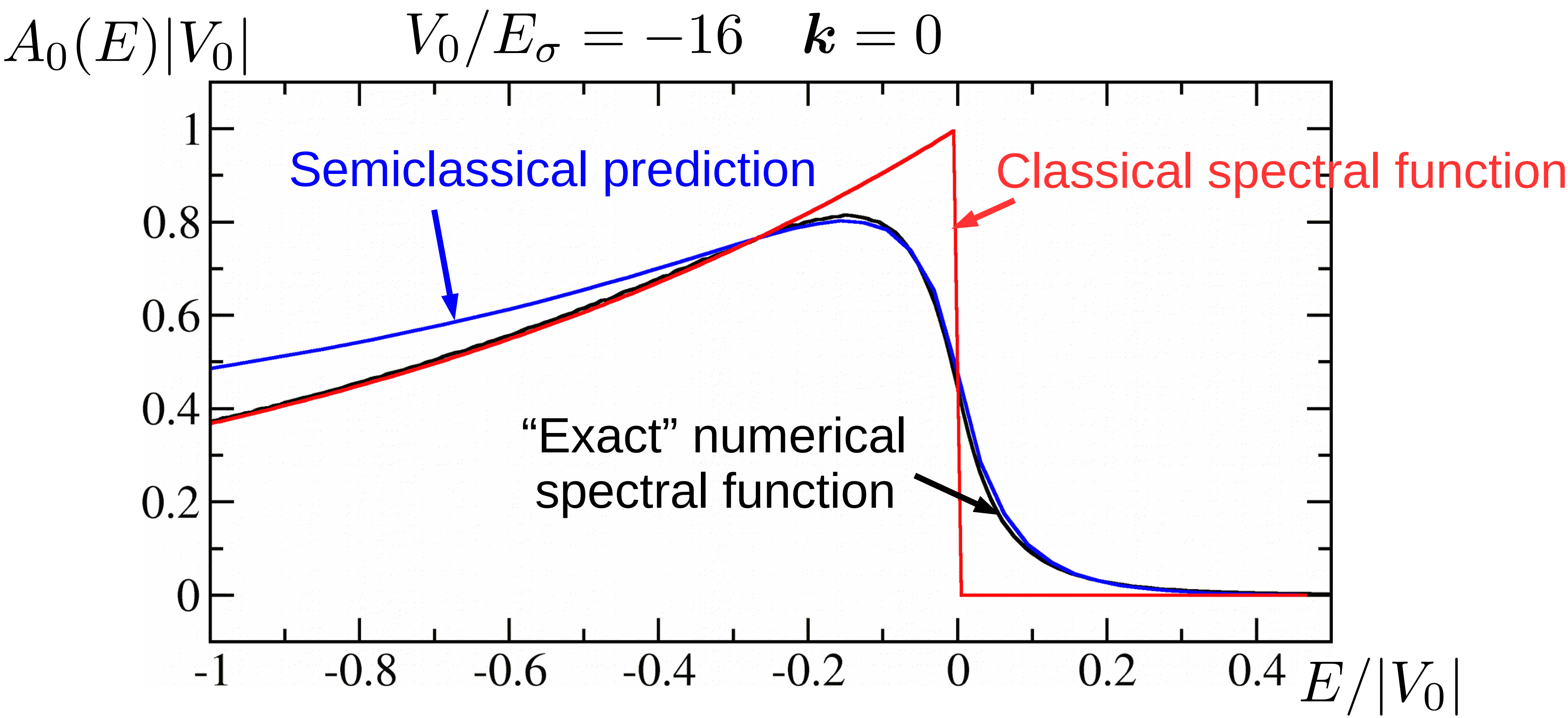


- For small momentum *k*, the only relevant classical trajectories are in the vicinity of the potential maxima near *E*=0.
- Potential maxima are hyperbolic fixed point => exponential stretching along the unstable direction => contribution to $\langle \boldsymbol{k}|e^{-i\hat{H}t/\hbar}|\boldsymbol{k}\rangle$ decays like $\exp(-\lambda t)$

Lyapounov exponent related to the (negative) potential curvature

- All the statistical properties of the speckle potential (derived in the blue-detuned case) can be readily reused.

Approximate spectral function for the red-detuned case

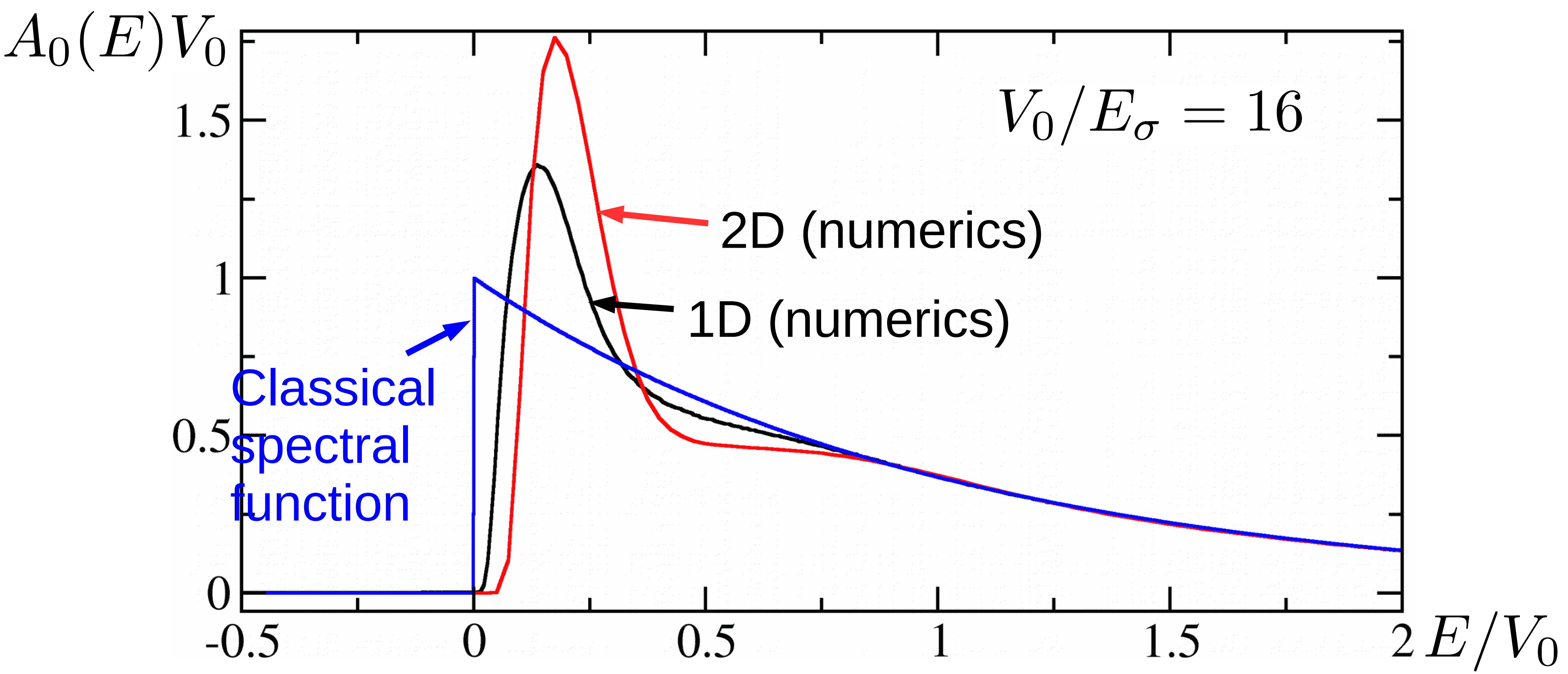


- The semiclassical prediction is excellent around *E*=0.
- The spectral function is less singular than for the blue speckle.
- But the typical energy scale is the same $\sqrt{|V_0|E_\sigma}$
- Deviations at low energy...

Summary and perspectives

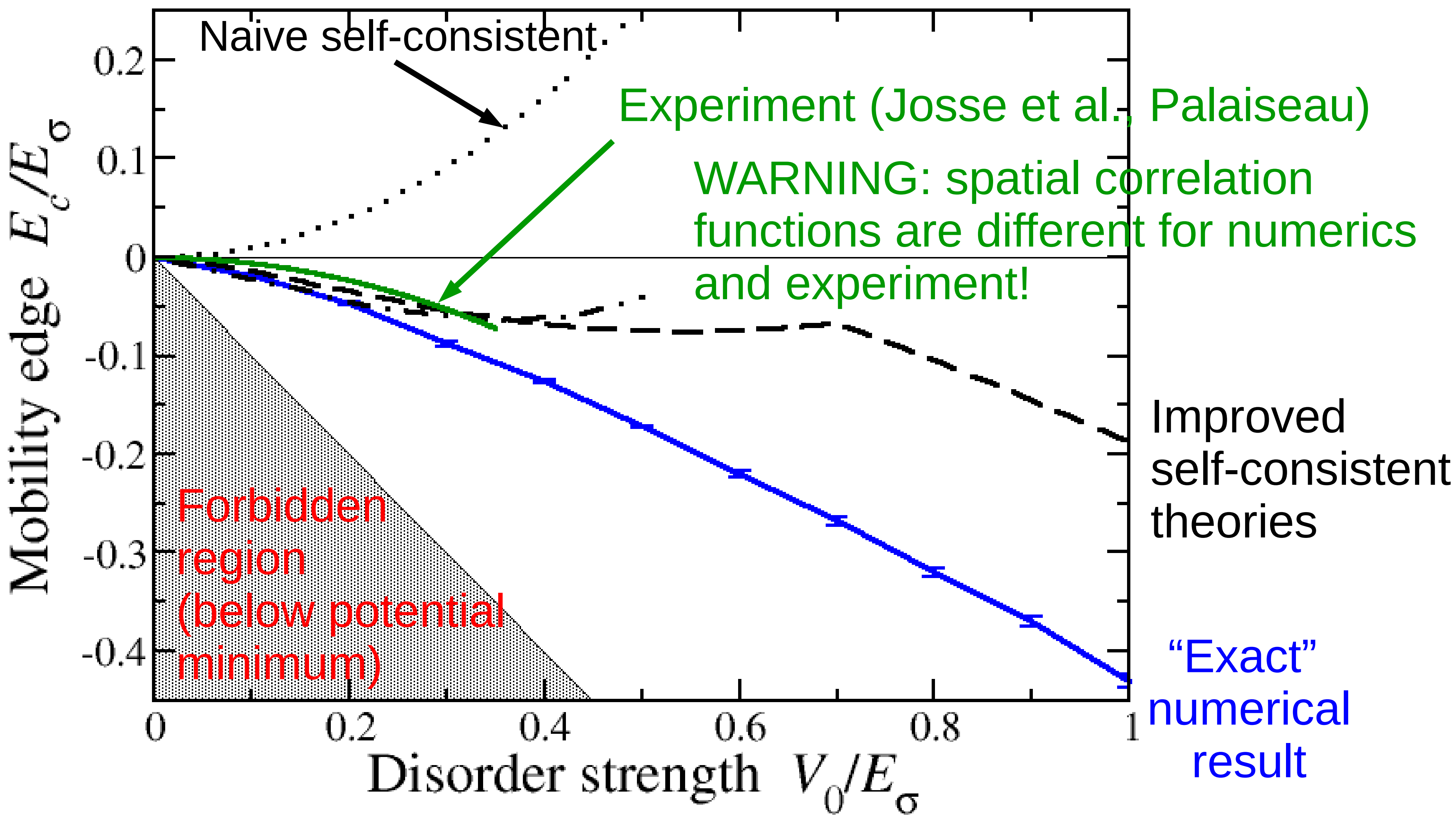
- It is possible to compute a semiclassical prediction for the spectral function of a disordered potential.
- Systematic expansion for “smooth” contributions.
- Ad-hoc methods can be developed for “singular” contributions.
- Opens the way to a semiclassical+self-consistent calculation of the mobility edge for Anderson localization.
- Presently done in 1D. Work in progress for 2D and 3D.

Spectral function in 1D and 2D (blue detuned)



- The 2D spectral function looks more singular.
- Could be related to the more singular distribution of potential minima (finite fraction at exactly V=0, see Weinrib and Halperin, PRB 26,1362 (1982)).

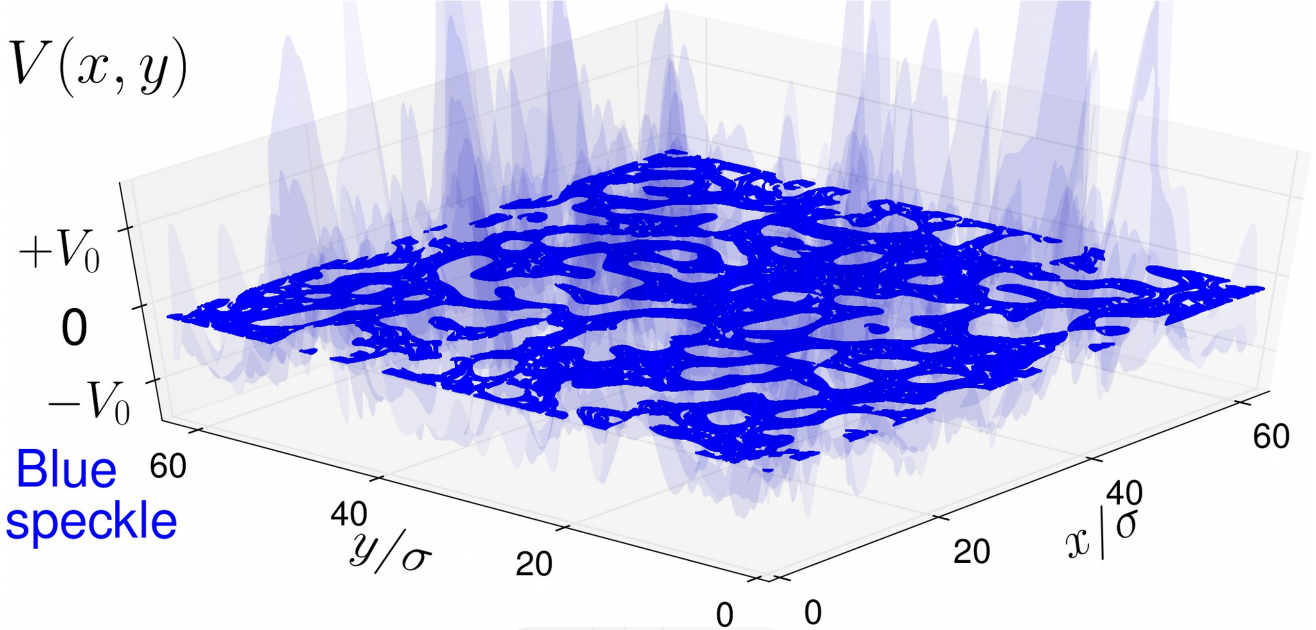
Position of the mobility edge



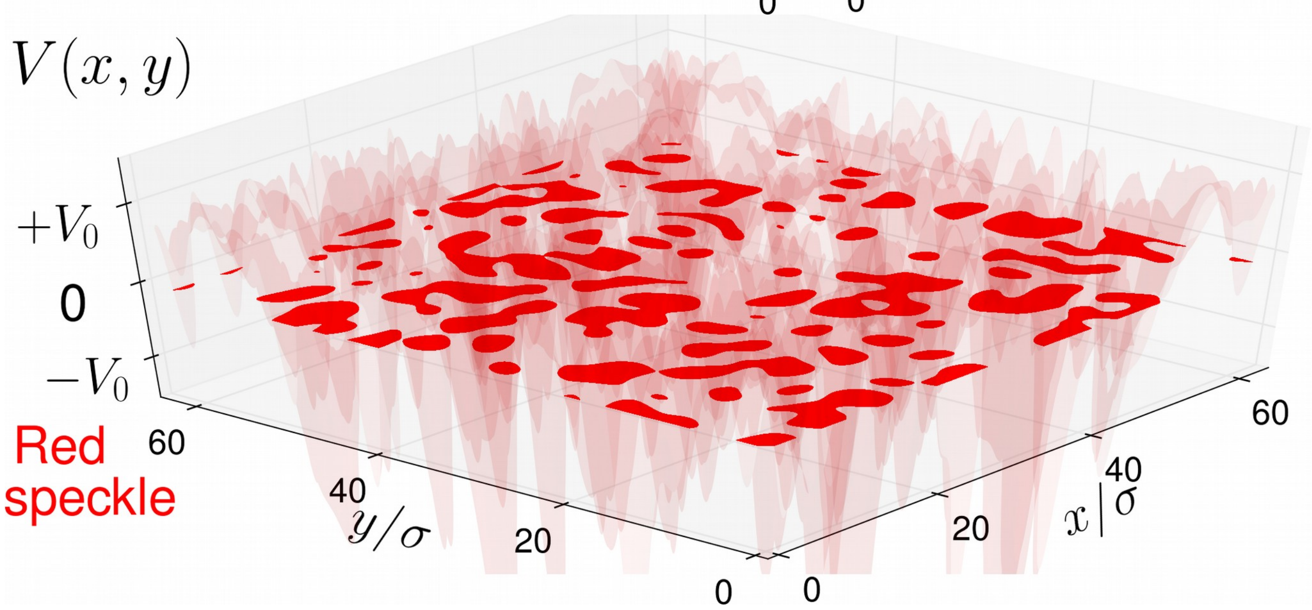
Blue-detuned 3D spherical speckle

Classical percolation

- Classical allowed region at an energy half-way between the red and blue mobility edges (pictures in 2D!).



Connected



Not connected