Multifractality and extreme value statistics

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Luchon, March 18, 2015

Outline

- Multifractality
- Logarithmically correlated random fields
- Disorder-generated multifractals
- Critical random matrix ensembles

[Y. V. Fyodorov and O. Giraud, Chaos, Solitons and Fractals **74**, 15 (2015)]

Multifractals

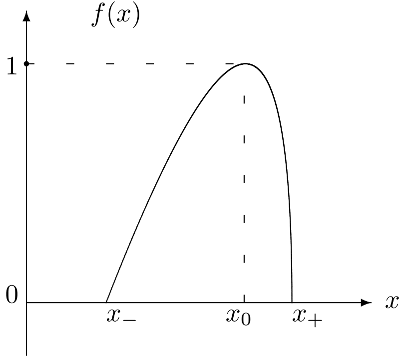
- d-dimensional lattice
- linear size L, lattice spacing a
- $M = (L/a)^d \gg 1$ lattice sites with intensities $h_i > 0$

$$h_i \sim M^{x_i}$$

Multifractal Ansatz :

$$\begin{aligned} \rho_M(x) &= \sum_{i=1}^{M} \delta\left(\frac{\ln h_i}{\ln M} - x\right) \\ &\approx c_M(x)\sqrt{\ln M}\, M^{f(x)}, \\ &\quad (M \gg 1) \end{aligned}$$

$f(x)$ singularity spectrum



Multifractality is characterized by :

- Power-law correlation of intensities

$$\mathbb{E}\left\{h^q(\mathbf{r}_1)h^s(\mathbf{r}_2)\right\} \propto \left(\frac{L}{a}\right)^{y_{q,s}} \left(\frac{|\mathbf{r}_1-\mathbf{r}_2|}{a}\right)^{-z_{q,s}}, \quad a \ll |\mathbf{r}_1-\mathbf{r}_2| \ll L$$

- Spatial homogeneity

$$\mathbb{E}\left\{h^q(\mathbf{r})\right\} = \mathbb{E}\left\{\frac{1}{M}\sum_{\mathbf{r}} h^q(\mathbf{r})\right\} \propto \left(\frac{L}{a}\right)^{d(\zeta_q-1)}$$

If

- intensities do not vary much over the scale a

$$\mathbb{E}\left\{h^q(\mathbf{r}_1)h^s(\mathbf{r}_2)\right\} \sim \mathbb{E}\left\{h^{q+s}(\mathbf{r}_1)\right\} \qquad |\mathbf{r}_1-\mathbf{r}_2| \sim a$$

- intensities are uncorrelated at scale L

$$\mathbb{E}\left\{h^q(\mathbf{r}_1)h^s(\mathbf{r}_2)\right\} \sim \mathbb{E}\left\{h^q(\mathbf{r}_1)\right\}\mathbb{E}\left\{h^s(\mathbf{r}_2)\right\} \qquad |\mathbf{r}_1-\mathbf{r}_2| \sim L$$

then

$$y_{q,s} = d(\zeta_{q+s}-1), \qquad z_{q,s} = d(\zeta_{q+s}-\zeta_q-\zeta_s+1)$$

$\Rightarrow$ multifractal pattern characterized by ζ_q

Large deviations

Saddle-point approximation for partition function :

$$Z_q = \sum_{i=1}^{M} h_i^q = \int_{-\infty}^{\infty} M^{qy} \rho_M(y)\, dy \approx \frac{c_M(y_*)}{\sqrt{|f''(y_*)|}}\, M^{\zeta_q}, \quad M \gg 1$$

with $f'(y_*) = -q$ and $\zeta_q = f(y_*) + q\, y_*$

(Recall multifractal Ansatz :

$$\rho_M(x) = \sum_{i=1}^{M} \delta\left(\frac{\ln h_i}{\ln M} - x\right) \approx c_M(x)\sqrt{\ln M}\, M^{f(x)}, \quad M \gg 1)$$

Counting function

$$\mathcal{N}_M(x) = \int_x^{\infty} \rho_M(y)\, dy \approx \frac{c_M(x)}{|f'(x)|\sqrt{\ln M}}\, M^{f(x)}$$

Statistics of extreme values of $h = M^x \quad \Leftrightarrow \quad \mathcal{N}_M(x) \sim 1$

Log-correlated fields

Logarithm of the multifractal field :

$$V(\mathbf{r}) = \ln h(\mathbf{r}) - \mathbb{E}\left\{\ln h(\mathbf{r})\right\}$$

With

$$\frac{d}{ds} h^s|_{s=0} = \ln h$$

one gets

$$\mathbb{E}\left\{V(\mathbf{r_1})V(\mathbf{r_2})\right\} = -d\,\zeta_0'' \ln \frac{|\mathbf{r}_1 - \mathbf{r}_2|}{L}$$

(ζ_0'' : second derivative of ζ_q taken at $q = 0$).

i.e. multifractal pattern $\Leftrightarrow$ log-correlated random field

Gaussian $1/f$ noises

$$V(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[v_n e^{int} + \overline{v}_n e^{-int} \right] \ , \quad t \in [0, 2\pi)$$

$v_n, \overline{v}_n$ complex normal i.i.d. variables with mean zero and variance 1

Then

$$\mathbb{E}\left\{V(t_1)V(t_2)\right\} = -2\ln|2\sin\frac{t_1 - t_2}{2}|, \quad t_1 \neq t_2$$

Discrete version : $M \gg 1$, $V_k \equiv V\left(t = \frac{2\pi}{M}k\right)$ random variables with covariance matrix $C_{km} = \mathbb{E}\left\{V_k V_m\right\}$ given by

$$\mathbb{E}\left\{V_k V_m\right\} = -2\ln\left|2\sin\frac{\pi(k-m)}{M}\right|, \quad C_{kk} = \mathbb{E}\left\{V_k^2\right\} > 2\ln M$$

$$h_i = e^{V_i}$$

$Z_q = \sum h_i^q$ and $\mathcal{N}_M(x) = \int_x^{\infty} \rho_M(y)dy$ can be obtained analytically

Moment distribution

Discrete periodic Gaussian $1/f$ noise

$$\mathcal{P}(Z_q) = \frac{1}{q^2\, Z_e}\left(\frac{Z_e}{Z_q}\right)^{1+\frac{1}{q^2}} e^{-\left(\frac{Z_e}{Z_q}\right)^{\frac{1}{q^2}}}, \quad Z_e = \frac{M^{1+q^2}}{\Gamma(1-q^2)}$$

for $Z_q < M^2$ and $|q| < 1$ [Fyodorov Bouchaud (2008)]

$$Z_q \approx \frac{c_M(y_*)}{\sqrt{|f''(y_*)|}}\, M^{\zeta_q}, \qquad \mathcal{N}_M(x) \approx \frac{c_M(x)}{|f'(x)|\sqrt{\ln M}}\, M^{f(x)}$$

$\Rightarrow$ distribution of $\mathcal{N}_M(x)$ via c_M

Power-law tail

$$\mathcal{P}(z) \sim z^{-1-\frac{1}{q^2}}$$

for the scaled variable $z = Z_q/Z_e$

Typical extreme value

Typical counting function $\mathcal{N}_t(x)$:

$$e^{\mathbb{E}\{\ln \mathcal{N}_M(x)\}} \sim \mathcal{N}_t(x)$$

Scaled counting function $n = \mathcal{N}_M(x)/\mathcal{N}_t(x)$ characterized by

$$\mathcal{P}_x(n) = \frac{4}{x^2}\, e^{-n^{-\frac{4}{x^2}}}\, n^{-\left(1+\frac{4}{x^2}\right)}, \qquad 0 < x < 2$$

and

$$\mathcal{N}_t(x) = \frac{M^{f(x)}}{x\sqrt{\ln M}} \frac{1}{\Gamma(1 - x^2/4)}, \qquad f(x) = 1 - x^2/4$$

Threshold for typical value $\mathcal{N}_t(x) \sim 1$

$$x_m = 2 - \frac{3}{2}\, \frac{\ln \ln M}{\ln M} + O(1/\ln M)$$

Average extreme value

$$\mathcal{N}_M(x) = n\,\mathcal{N}_t(x) \qquad \text{and} \quad \mathbb{E}(n) = \Gamma(1 - x^2/4)$$
$$\Rightarrow \qquad \mathbb{E}\left\{\mathcal{N}_M(x)\right\} = \Gamma(1 - x^2/4)\mathcal{N}_t(x)$$

$$\Gamma\left(1 - x^2/4\right) \sim_{x \to 2} \frac{1}{2 - x}$$

Threshold for average value $\mathbb{E}\left\{\mathcal{N}_M(x)\right\} \sim 1$

$$x_m = 2 - \frac{1}{2}\frac{\ln\ln M}{\ln M} + O(1/\ln M)$$

Threshold for typical value $\mathcal{N}_t(x) \sim 1$

$$x_m = 2 - \frac{3}{2}\frac{\ln\ln M}{\ln M} + O(1/\ln M)$$

Disorder-generated multifractals

$|\psi\rangle = \sum_{i=1}^{M} \psi_i |i\rangle$ normalized vector :

$$h_i = |\psi_i|^2 \sim M^{-\alpha_i},\; i = 1, \dots M,$$

multifractal Ansatz $\rho_M(\alpha) \propto M^{f(\alpha)}$,
($\rho_M(\alpha)$ the density of exponents α_i)

or

$$Z_q = \sum_{i=1}^{M} |\psi_i|^{2q} \propto M^{-\tau_q}, \qquad \tau_q = D_q(q-1)$$

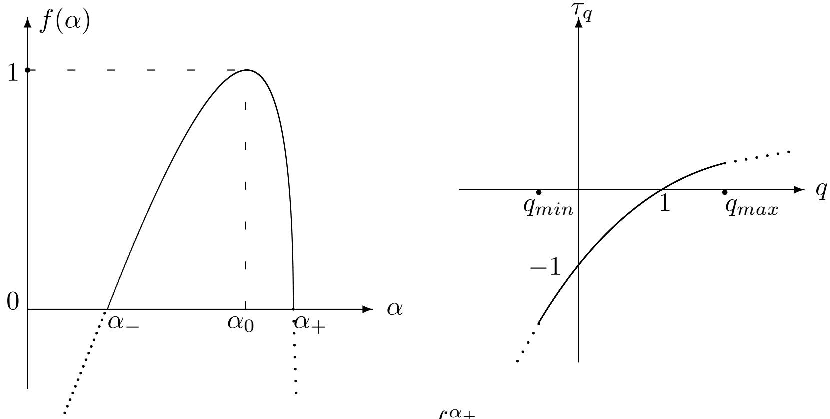
Inverse participation ratios

$$\mathbb{E}\{Z_q\} \sim M^{-\tau_q} \qquad Z_q^{\mathrm{typ}} = \exp \mathbb{E}\{\ln Z_q\} \sim M^{-\tau_q^{\mathrm{typ}}}$$

Scaling

$$Z_q = \sum_{i=1}^{M} h_i^q = \int_0^\infty M^{-q\alpha} \rho_M(\alpha)\, d\alpha$$

Singularity spectrum



$$f(\alpha) = \min_q (q\alpha - \tau_q), \qquad Z_q^{\mathrm{typ}} = \int_{\alpha_-}^{\alpha_+} M^{-q\alpha + f^{\mathrm{typ}}(\alpha)} d\alpha \sim M^{-\tau_q^{\mathrm{typ}}}$$

Counting function

$$\mathcal{N}_M(\alpha) = \int_{-\infty}^{\alpha} \rho_M(\alpha)\, d\alpha$$

and scaled variable

$$\mathcal{N}_M(\alpha) \simeq n \mathcal{N}_t(\alpha)$$

Extreme value statistics

Distribution of the scaled variable $z = Z_q/Z_q^{\text{typ}}$

$$P(z) \sim z^{-1-\omega_q}$$

$\Rightarrow$ Power-law tail [Mirlin-Evers 2000]

Tail with $\omega_q \to 1$ for $q \to q_c = q_{\max}$

$$\Rightarrow \text{Divergence of } \mathbb{E}(z) \sim_{q \to q_c} |q - q_c|^{-1}$$
$$\Rightarrow \text{Divergence of } \mathbb{E}(n) \sim_{\alpha \to \alpha_-} |\alpha - \alpha_-|^{-1}$$

$$\mathcal{N}_t(x) = \frac{1}{x\sqrt{\ln M}} \frac{M^{f(x)}}{\Gamma(1 - x^2/4)} \qquad \longrightarrow \qquad \mathcal{N}_t(\alpha) \propto \frac{1}{\sqrt{\ln M}} \frac{M^{f(\alpha)}}{\mathbb{E}(n)}$$

Threshold for typical value $\mathcal{N}_t(\alpha) \sim 1$

$$\alpha_m \approx \alpha_- + \frac{3}{2} \frac{1}{f'(\alpha_-)} \frac{\ln \ln M}{\ln M}$$

Extreme value $|\psi_{\max}|^2 = M^{-\alpha_m}$

Random matrix ensembles with multifractal eigenvalues

One-dimensional N-body models with Hamiltonian $H(\mathbf{p},\mathbf{q})$

- equations of motion are equivalent to

$$\dot{L} = K\,L - L\,K$$

  L, K pair of Lax matrices of size $M \times M$

- explicit canonical transformation to action-angle variables

We choose $L(\mathbf{p},\mathbf{q})$ as a random matrix with some measure

$$\mathrm{d}L = P(\mathbf{p}\,,\mathbf{q}\,)\,\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}$$

Canonical transformation

$$\mathrm{d}L = \mathcal{P}(\boldsymbol{\lambda}\,,\boldsymbol{\phi}\,)\,\mathrm{d}\boldsymbol{\lambda}\,\mathrm{d}\boldsymbol{\phi}$$

with $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_M)$ eigenvalues of L.

Integration over $\boldsymbol{\phi}$ yields $P(\boldsymbol{\lambda})$

Ruijsenaars-Schneider model

Hamiltonian $H(\mathbf{p},\mathbf{q}) = \sum_j \cos(p_j) \prod_{k\neq j} \left(1 - \frac{\sin^2\tau}{\sin^2[\frac{q_j-q_k}{2}]}\right)^{\frac{1}{2}}$

Lax matrix :

$$L_{jk} = \prod_{s\neq j} \frac{\sin[\frac{q_j-q_s}{2}+\tau]^{\frac{1}{2}}}{\sin[\frac{q_j-q_s}{2}]^{\frac{1}{2}}} \frac{\mathrm{e}^{\mathrm{i}[\tau(N-1)+p_j+\frac{q_k-q_j}{2}]}\sin\tau}{\sin[\frac{q_j-q_k}{2}+\tau]} \prod_{s\neq k} \frac{\sin[\frac{q_k-q_s}{2}-\tau]}{\sin[\frac{q_k-q_s}{2}]}$$

For $q_k = 2\pi k/M$ and $\tau = \pi a/M$,

$$L_{jk} = \frac{\mathrm{e}^{\mathrm{i}p_j}}{M} \frac{1-\mathrm{e}^{2\pi\mathrm{i}a}}{1-\mathrm{e}^{2\pi\mathrm{i}(j-k+a)/M}}$$

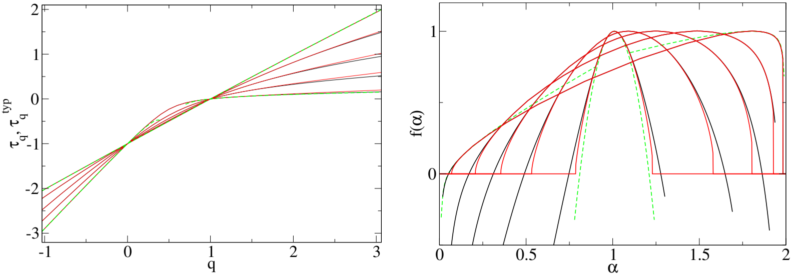
p_j = independent random variables uniformly distributed in $[0, 2\pi]$

[Phys. Rev. Lett. **103**, 054103 (2009)]

Multifractality of eigenvectors

Eigenvectors of

$$L_{jk} = \frac{e^{ip_j}}{M} \frac{1 - e^{2\pi i a}}{1 - e^{2\pi i (j-k+a)/M}},$$



$a = 0.1, 0.3, 0.5, 0.7, 0.9$

Perturbation expansion for RS

Fractal dimensions are accessible via perturbation series

$$L_{mn} = \frac{\mathrm{e}^{\mathrm{i}\Phi_m}}{M} \frac{1 - \mathrm{e}^{2\pi\mathrm{i}\mathbf{a}}}{1 - \mathrm{e}^{2\pi\mathrm{i}(m-n+\mathbf{a})/M}}$$

Perturbation series are possible around all *integer* points $\mathbf{a} = \kappa$, $\mathbf{a} = \kappa + \epsilon$

$$L_{mn} = L_{mn}^{(0)} \left(1 + \frac{\pi\mathrm{i}(M-1)}{M}\epsilon \right) + \epsilon L_{mn}^{(1)} + \mathcal{O}(\epsilon^2)$$

where

$$\begin{aligned} L_{mn}^{(0)} &= \mathrm{e}^{\mathrm{i}\Phi_m} \delta_{n,\,m+\kappa} \\ L_{mn}^{(1)} &= \mathrm{e}^{\mathrm{i}\Phi_m} (1 - \delta_{n,\,m+\kappa}) \frac{\pi \mathrm{e}^{-\pi\mathrm{i}(m-n+\kappa)/M}}{M \sin(\pi(m-n+\kappa)/M)} \end{aligned}$$

($\delta_{n,\,m+\kappa} = 1$ when $n \equiv m + \kappa \mod M$ and 0 otherwise)

Fractal dimensions for RS

- **Strong multifractality** (almost localized) :
  $\mathbf{a} \ll \mathbf{1}$, $L^{(0)}_{mn}$ diagonal
  - Unperturbed eigenfunctions $\Psi^{(0)}_j(\alpha) = \delta_{j\alpha}$
  - Unperturbed eigenvalues $\lambda_\alpha = \mathrm{e}^{\mathrm{i}\Phi_\alpha}$

  At first order in $\mathbf{a}$

$$\tau_q = 2a \frac{\Gamma\left(q - \frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(q-1)}$$

- **Weak multifractality** (almost extended) :
  $a = \kappa + \epsilon$ and $\kappa \neq 0$. The unperturbed matrix

$$L^{(0)}_{mn} = \mathrm{e}^{\mathrm{i}\Phi_m} \delta_{n,\, m+\kappa}$$

  is the shift matrix and its eigenfunctions are extended.

$$\tau_q = q - 1 - q(q-1)\frac{(a-\kappa)^2}{\kappa^2}, \qquad |a - \kappa| \ll 1$$

[Phys. Rev. Lett. **106**, 044101 (2011)]

Correlations in the Ruijsenaars-Schneider model

$$V_i = \ln|\Psi_i|^2 - \mathbb{E}\left\{\ln|\Psi_i|^2\right\}$$

Weak multifractality limit $a = \kappa + \epsilon$ with $\kappa \neq 0$
Expansion to order 2 in ϵ, $\kappa = 1$:

$$\mathbb{E}\left\{V_k(\alpha)V_{k+r}(\alpha)\right\} = \frac{\pi^2\epsilon^2}{M^3}\left(\sum_{x<r}\frac{x(2r-x-M))}{\sin^2\frac{\pi x}{M}} + \sum_{x\geq r}\frac{(x-2r)(M-x)}{\sin^2\frac{\pi x}{M}}\right)$$

($\mathbb{E}$ = average over eigenvectors α, phases Φ and position k)
For $r = cM$, $M \to \infty$, c fixed,

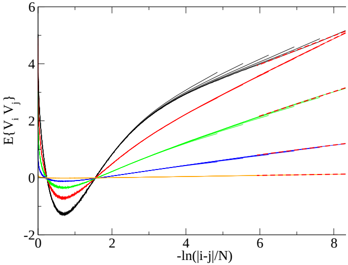
$$\mathbb{E}\left\{V_k(\alpha)V_{k+r}(\alpha)\right\} \sim -2\epsilon^2 \ln\frac{r}{M}, \qquad r \ll M$$

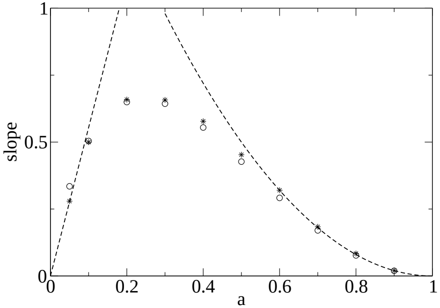
$\Rightarrow$ hidden logarithmic structure of the RS model. Compare with

$$\mathbb{E}\left\{V(\mathbf{r_1})V(\mathbf{r_2})\right\} = -d\,\zeta_0'' \ln\frac{|\mathbf{r}_1 - \mathbf{r}_2|}{L}$$

Here $\tau_q = q - 1 - q(q-1)\frac{(a-k)^2}{k^2}$ $\qquad \Rightarrow \tau_0'' = -2\epsilon^2$

Correlations in the Ruijsenaars-Schneider model



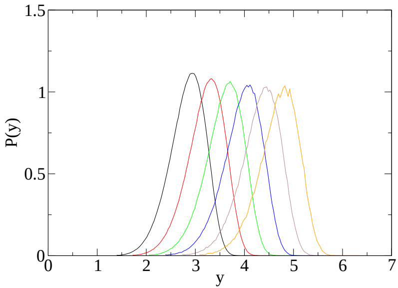


$a = 0.1$ (black), 0.3 (red), 0.5 (green), 0.7 (blue), 0.9 (orange)

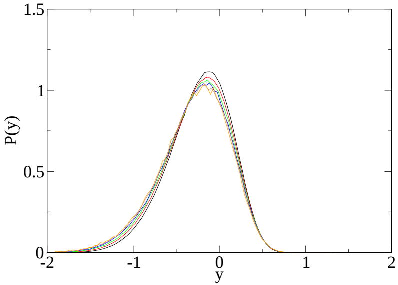
stars : τ_q'' at $q = 0$
circles : slope of the correlator

$$-\tau_q''|_{q=0} = 4a \ln 4, \quad a \simeq 0, \qquad -\tau_q''|_{q=0} = 2(1-a)^2, \quad a \simeq 1$$

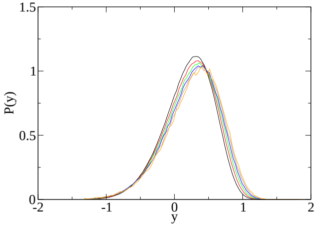
Extreme values in RS



$$y = -\ln h_m$$



$$y \to y - \alpha_- \ln M - \frac{3}{2f'(\alpha_-)} \ln \ln M$$

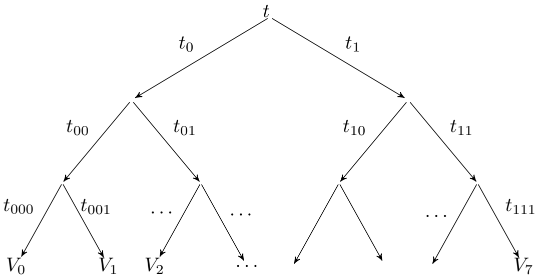


REM and SDM

$$h_i = e^{\beta V_i}/Z(\beta), \qquad Z(\beta) = \sum_{i=1}^{M} e^{\beta V_i}$$

V_i Gaussian random variables $\langle V_i \rangle = 0$, $\langle V_i^2 \rangle = 2 \ln M$

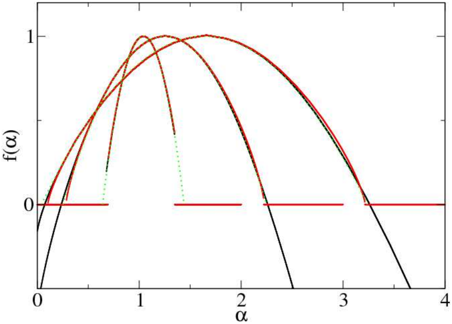
- Random Energy Model [Derrida 1981] : V_i are i.i.d.
- Derrida-Spohn Model [Derrida-Spohn 1988] : $V_i = \sum t_{i_1 i_2 \ldots}$ along a path, $t_{i_1 i_2 \ldots}$ i.i.d Gaussian with variance $\frac{2n}{n+1} \ln 2$, $M = 2^n$



REM and Spohn Derrida models

Typical singularity spectrum is the same for both :

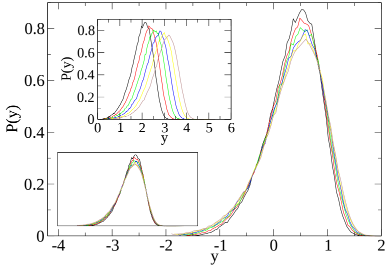
$$f^{\mathrm{typ}}(\alpha) = 1 - \frac{1}{4\beta^2}(1+\beta^2-\alpha)^2, \qquad \alpha \in [(1-\beta)^2, (1+\beta)^2]$$



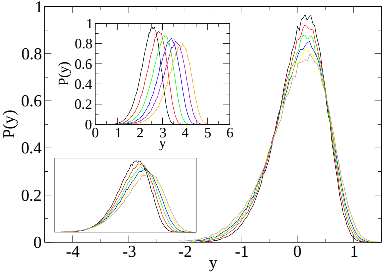
But

- REM : $-\ln h_m \simeq (1-\beta)^2 \ln M + \frac{1}{2}\beta \ln\ln M$
- DSM : $-\ln h_m \simeq (1-\beta)^2 \ln M + \frac{3}{2}\beta \ln\ln M$

Extreme value distribution in REM and Spohn



shift by $\frac{1}{2}\beta \ln \ln M$
(3/2 in the inset)



shift by $\frac{3}{2}\beta \ln \ln M$
(1/2 in the inset)

Conclusions

Extreme values in multifractal patterns

- logarithm of a disorder-generated multifractal = log-correlated random field
- relationship between logarithmically correlated random processes and disorder-generated multifractals
- parallel between features of their extreme values
- Ruijsenaars-Schneider ensemble and models with (DSM) or without (REM) logarithmic correlations