String-like theory of many-particle Quantum Chaos

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Basic Question

Quantum Chaos Theory:

Standard semiclassical limit:
fixed N (number of particles), $\hbar_{eff} \to 0$

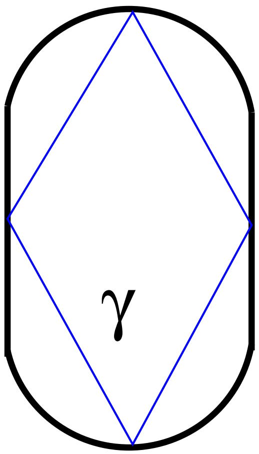
Non-standard: fixed $\hbar_{eff}$, $N \to \infty$ (Bosons)

But what if $\hbar_{eff} \to 0$ and $N \to \infty$?

Single-particle Quantum Chaos

Gutzwiller's trace formula:

$$\rho(E) = \sum_n \delta(E - E_n) \sim \underbrace{\bar{\rho}(E)}_{Smooth} + \underbrace{\Re \sum_{\gamma \in \mathrm{PO}} \mathcal{A}_\gamma \exp\left(\frac{i}{\hbar} S_\gamma(E)\right)}_{Oscillating}$$



$\mathcal{A}_\gamma$ stability factor,
$S_\gamma(E)$ action of a **periodic orbit** γ

Number of periodic orbits grows exponentially with length

– No prediction on E_n from an individual γ
– **All** $\{\gamma\}$ **together** $\Longleftrightarrow$ **spectrum**

Two-point correlation function

$$R(\varepsilon) = \frac{1}{\bar{\rho}^2} \left\langle \rho(E + \varepsilon/\bar{\rho}) \rho(E) \right\rangle_E - 1$$

$$K(\tau) = \int_{-\infty}^{+\infty} R(\varepsilon) e^{-2\pi i \tau \varepsilon} d\varepsilon \approx \text{ (Semiclassically)}$$

$$\approx \frac{1}{T_H^2} \left\langle \sum_{\gamma,\gamma'} \mathcal{A}_\gamma \mathcal{A}^*_{\gamma'} e^{\frac{i}{\hbar}(S_\gamma - S_{\gamma'})} \delta\left(\tau - \frac{(T_\gamma + T_{\gamma'})}{2T_H} \right) \right\rangle_E ,$$

$T_\gamma, T_{\gamma'}$ are periods of γ, γ', $\quad T_H = 2\pi\hbar\bar{\rho}$ (Heisenberg time)

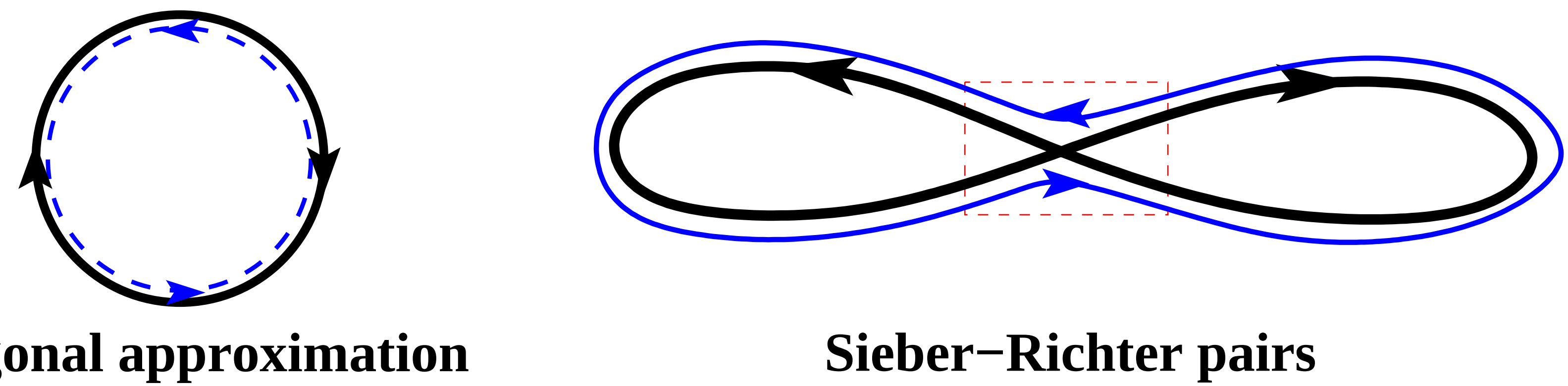
Spectral correlations $\Longleftrightarrow$

Correlations between actions of periodic orbits

Semiclassical origins of universality

$$K(\tau) = c_1\tau + c_2\tau^2 \dots$$

Diagonal approximation $\gamma = \gamma' \implies$
Leading order: c_1 [M. Berry 1985]



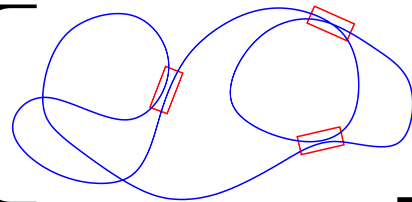
Sieber-Richter pairs (Non-trivial correlations) $\implies$
Second order: c_2 [M. Sieber K. Richter 2001]

$S_\gamma - S_{\gamma'} \sim \hbar \implies$ Duration of encounter $\sim \underbrace{\tau_E = \lambda^{-1}|\log\hbar|}_{\text{Ehrenfest time}}$

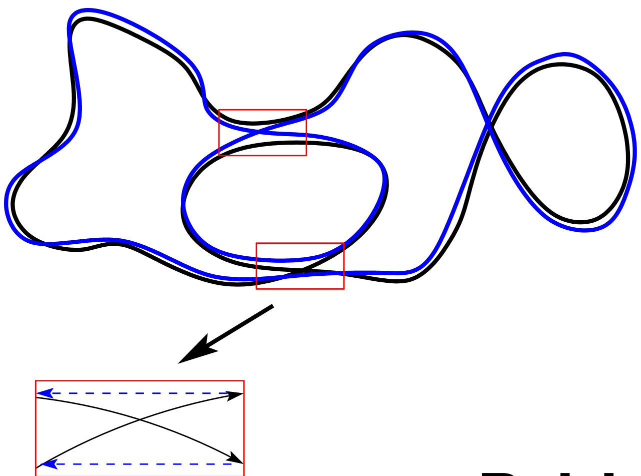
Full theory – all orders in τ

S. Müller, S. Heusler, P. Braun, F. Haake, A. Altland 2004

$$K(\tau)=\sum_{\substack{\text{Structures of}\\ \text{Periodic Orbits}}} \Big\{$$



$$\Big\} \, \tau^{n}$$



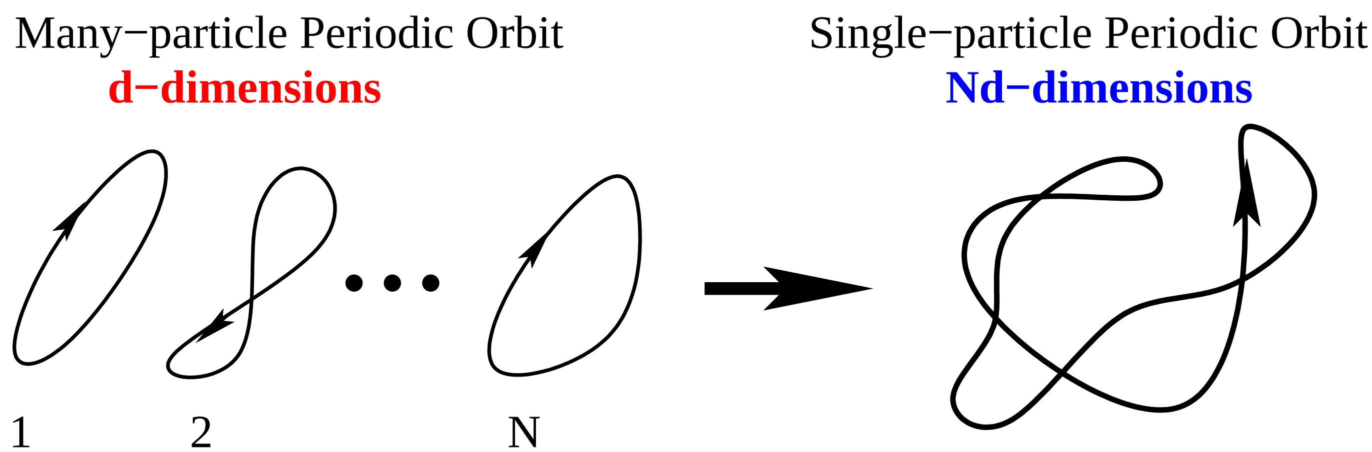
Pairing is robust under perturbation
Other correlations are washed out!

Pairing mechanism $\Longrightarrow$
Universality of spectral correlations

Many-particle systems

$$\mathcal{H} = \sum_{n=1}^{N} \frac{p_n^2}{2m} + V(x_n) + V_{\text{int}}(x_n - x_{n+1})$$

Chaos; Local, Homogeneous interactions
i.e, invariance under $n \to n+1$



Q: Is the single-particle theory of Quantum Chaos applicable?

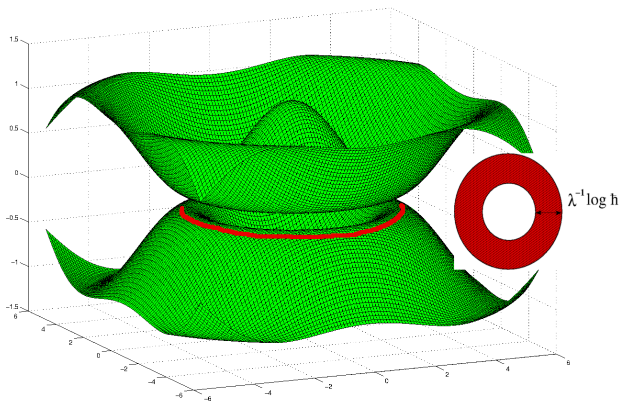
A: Depends how large N is

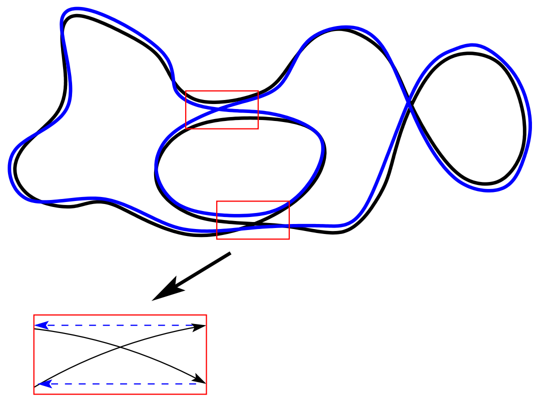
Caricature: Semiclassical Field Theory

Continues limit $n \to \eta \in [0, \ell]$**,** $\quad x_n(t) \to x(\eta, t)$

$$\mathcal{L} = \sum_{n=1}^{N} \frac{\dot{x}_n^2}{2m} - V(x_n) - V_{\text{int}}(x_n - x_{n+1}) \longrightarrow$$

$$\mathcal{L} = \int_0^\ell d\eta (\partial_t x(\eta, t))^2 + (\partial_\eta x(\eta, t))^2 - V(x(\eta, t))$$

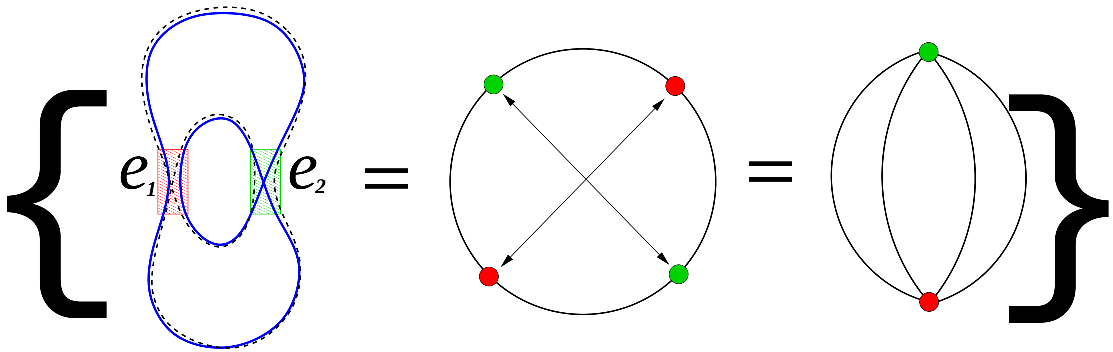




1) **PO** -are **2D toric surfaces** in d-dim space (Rather than 1D lines in $N \cdot d$-dim)
2) **Encounters** are **"rings"** (Rather than 1D stretches) of "width" $\sim \lambda^{-1} |\log \hbar_{eff}|$

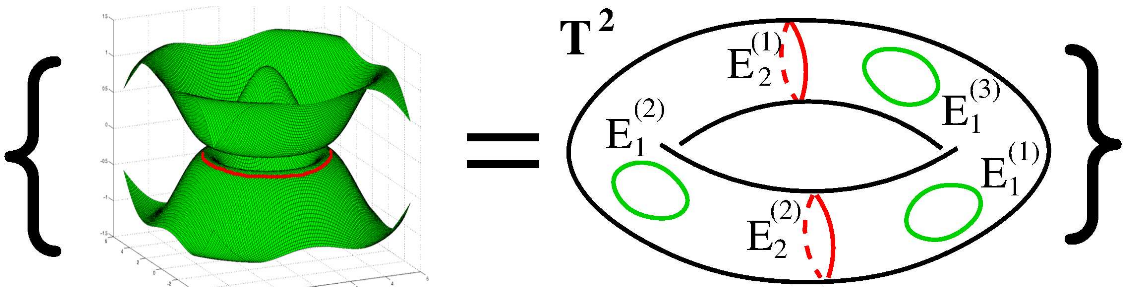
Caricature: Semiclassical Field Theory

Single-particle structure diagrams:



Distinguished by order of encounters

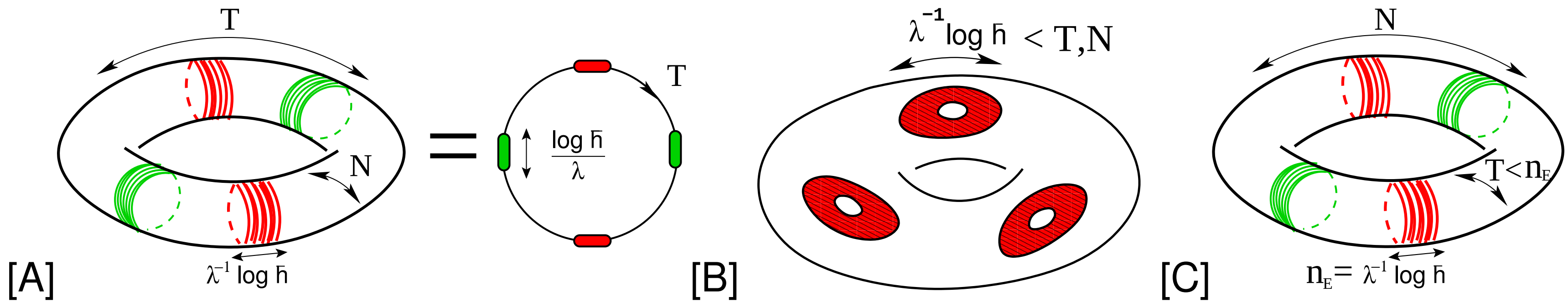
Many-particle structure diagrams:



Distinguished by order and winding numbers ω of encounters!

Many-particle Quantum Chaos

$\lambda^{-1}|\log \hbar_{eff}| =: n_E$- **Ehrenfest "number"**



A. If $N \lesssim n_E$, $T \gtrsim n_E \Longrightarrow$ only $\omega = (0,1)$ "single-particle" encounters

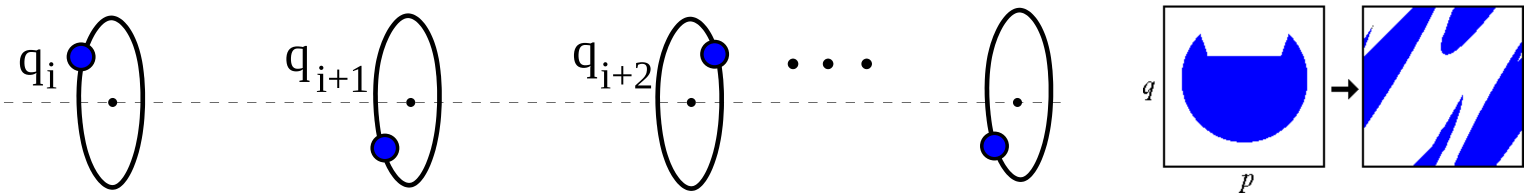
Effectively single-particle Quantum Chaos

B. $N, T \gtrsim n_E$, $\Longrightarrow \omega = (0,0)$ encounters dominate!

C. $N \gtrsim n_E$, $T \lesssim n_E \Longrightarrow$ only $\omega = (1,0)$ encounters

B, C - Genuine many-particle Quantum Chaos!

Coupled-Cat Maps



$S(\mathsf{q}_t, \mathsf{q}_{t+1}) = S_0(\mathsf{q}_t, \mathsf{q}_{t+1}) + S_{\rm int}(\mathsf{q}_t)$, $\mathsf{q}_t = (q_{1,t}, q_{2,t} \ldots q_{N,t})$

N uncoupled cat maps, $q_{n,t}, p_{n,t} \in [0, 1]$:

$$S_0 = \sum_{n=1}^{N} S_0^{(n)}(q_{n,t}, q_{n,t+1}) + V(q_{n,t})$$

Interactions:

$$S_{\rm int} = -\sum_{n=1}^{N} q_{n,t} q_{1+(n \bmod N),t}.$$

Coupled-Cat Maps $V = 0$

$$Z_{t+1} = \mathcal{B}_N Z_t \text{mod} 1, \qquad Z_t = (q_{1,t}, p_{1,t}, \ldots q_{N,t}, p_{N,t})^\mathsf{T},$$

with $2N \times 2N$ matrix $\mathcal{B}_N$ given by:

$$\mathcal{B}_N = \begin{pmatrix} A & B & \mathbf{0} & \ldots & \mathbf{0} & B \\ B & A & B & \ldots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B & A & \ldots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & A & B \\ B & \mathbf{0} & \mathbf{0} & \ldots & B & A \end{pmatrix}, A = \begin{pmatrix} a & 1 \\ ab-1 & b \end{pmatrix}, B = -\begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}$$

Lyapunov exponents:

$$\cosh \lambda_k = (a+b)/2 - \cos(2\pi k/N), \qquad k = 1, \ldots N$$

Full chaos: $|\text{Re}\lambda_k| > 0$, i.e. $|a+b| > 4$

Particle-time Duality

Newtonian form:

$$\Delta_t^2 q_{n,t} + \Delta_n^2 q_{n,t} = (a+b-4)q_{n,t} + V'(q_{n,t}) \bmod 1$$

Δ_α^2- discrete Laplacian: $\Delta_\alpha^2 f_\alpha \equiv f_{\alpha+1} - 2f_\alpha + f_{\alpha-1}$

Particle-time duality:

If $\{q_{n,t}\}$ solution then $\{q'_{n,t} = q_{t,n}\}$ also solution! $\Longrightarrow$ N-part. PO Γ of period T $\Longleftrightarrow$ T-part. PO Γ' of period N

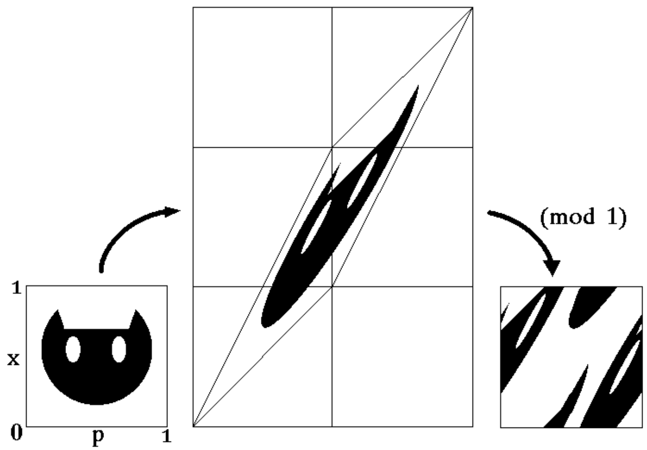
1) $S(\Gamma) = S(\Gamma')$

2) [$\#$ of N-particle PO of period T] = [$\#$ of T-particle PO of period N] $\Longleftrightarrow$ $|\det(I - \mathcal{B}_N^T)| = |\det(I - \mathcal{B}_T^N)|$

Corollary: $$T^2 \prod_{k=1}^{N-1} 4\sin^2\left(\frac{\pi k T}{N}\right) = N^2 \prod_{m=1}^{T-1} 4\sin^2\left(\frac{\pi m N}{T}\right)$$

2D Symbolic Dynamics

$$Z_{t+1} + \mathcal{M}_t = \mathcal{B}_N Z_t, \mathcal{M}_t = (\mathrm{m}_{1,t}, \ldots \mathrm{m}_{N,t})^\intercal, \mathrm{m}_{n,t} = (m^q_{n,t}, m^p_{n,t})$$



T

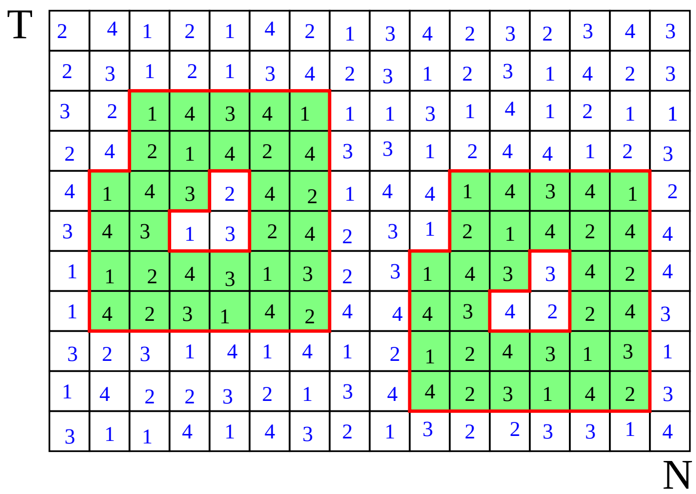
2	4	1	2	1	4	2	1	3	4	2	3	2	3	4	3
2	3	1	2	1	3	4	2	3	1	2	3	1	4	2	3
3	2	1	4	3	4	1	1	1	3	1	4	1	2	1	1
2	4	2	1	4	2	4	3	3	1	2	4	4	1	2	3
4	1	4	3	2	4	2	1	4	4	1	4	3	4	1	2
3	4	3	1	3	2	4	2	3	1	2	1	4	2	4	4
1	1	2	4	3	1	3	2	3	1	4	3	3	4	2	4
1	4	2	3	1	4	2	4	4	4	3	4	2	2	4	3
3	2	3	1	4	1	4	1	2	1	2	4	3	1	3	1
1	4	2	2	3	2	1	3	4	4	2	3	1	4	2	3
3	1	1	4	1	4	3	2	1	3	2	2	3	3	1	4

N

$$\mathrm{M}_\Gamma = \begin{pmatrix} \mathrm{m}_{1,1} & \mathrm{m}_{2,1} & \ldots & \mathrm{m}_{N,1} \\ \mathrm{m}_{1,2} & \mathrm{m}_{2,2} & \ldots & \mathrm{m}_{N,2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{m}_{1,T} & \mathrm{m}_{2,T} & \ldots & \mathrm{m}_{N,T} \end{pmatrix}$$

2D Symbolic Dynamics

$$Z_{t+1} + \mathcal{M}_t = \mathcal{B}_N Z_t, \mathcal{M}_t = (\mathrm{m}_{1,t}, \dots \mathrm{m}_{N,t})^\intercal, \mathrm{m}_{n,t} = (m^q_{n,t}, m^p_{n,t})$$



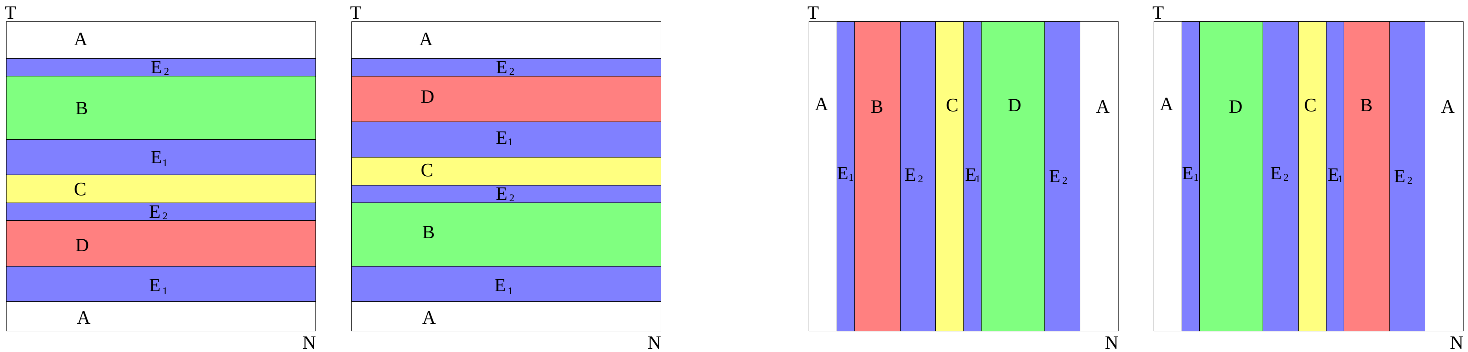
$$\mathbb{M}_\Gamma = \begin{pmatrix} \mathrm{m}_{1,1} & \mathrm{m}_{2,1} & \dots & \mathrm{m}_{N,1} \\ \mathrm{m}_{1,2} & \mathrm{m}_{2,2} & \dots & \mathrm{m}_{N,2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{m}_{1,T} & \mathrm{m}_{2,T} & \dots & \mathrm{m}_{N,T} \end{pmatrix}$$

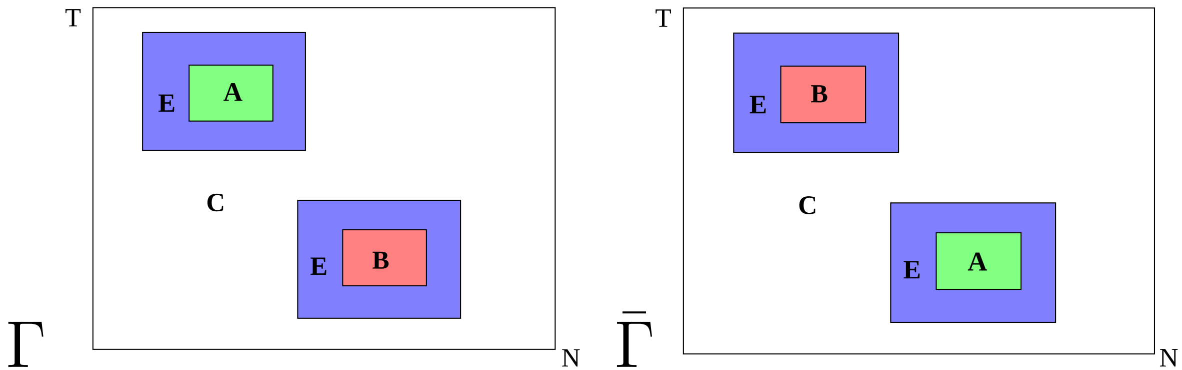
1) **Small alphabet** (does not grow with N)

2) **Uniqueness:** Each PO Γ is uniquely encoded by $\mathbb{M}_\Gamma$
Γ can be easily restored from $\mathbb{M}_\Gamma$

3) **Locality:** $r \times r$ square of symbols around (n, t) defines position of the n'th particle at the time t up to error $\sim \Lambda^{-r}$

Partner Orbits



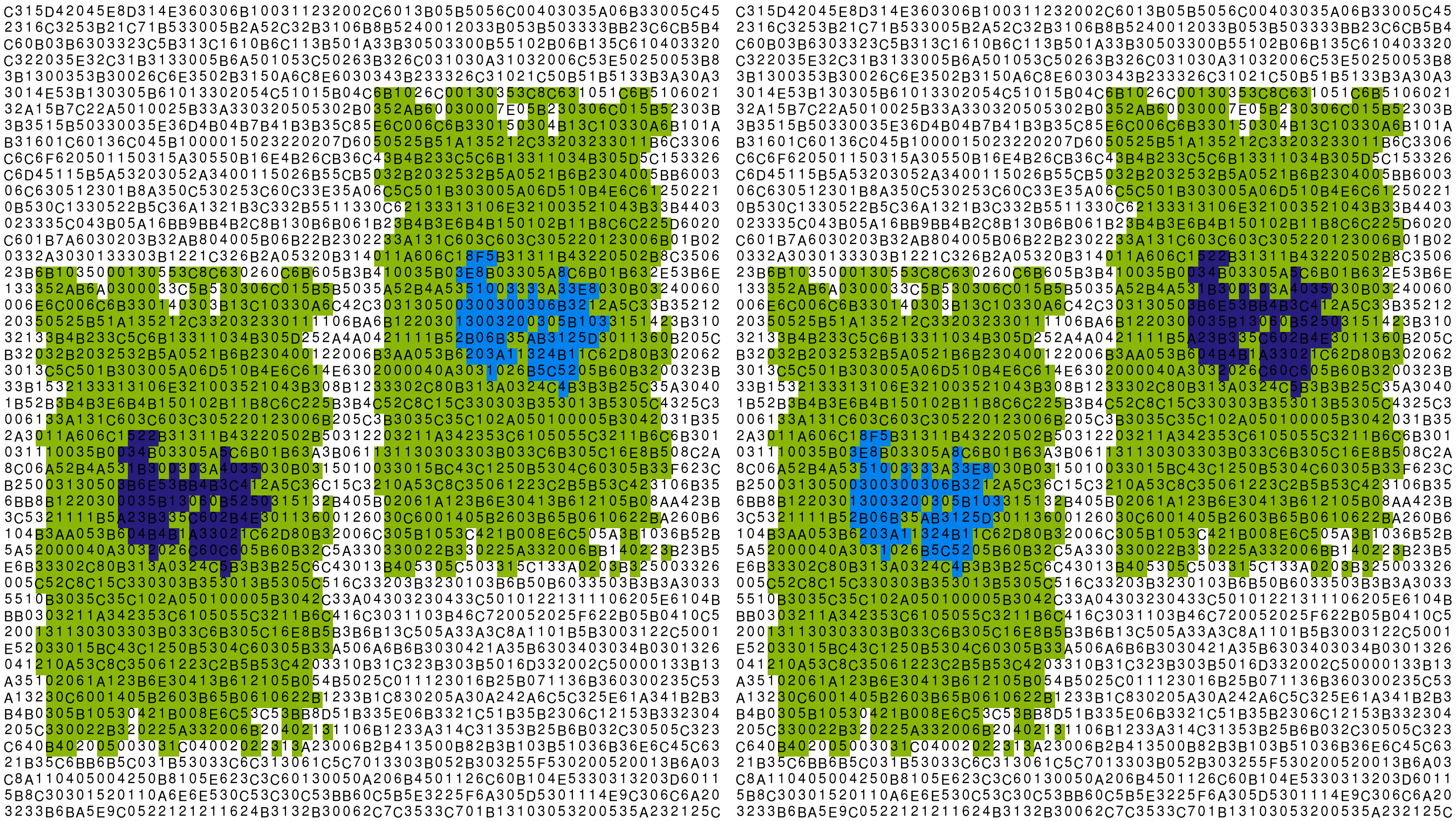


$\mathbb{M}_{\bar{\Gamma}}$ is obtained by reshuffling $\mathbb{M}_{\Gamma}$

Note: One encounter is enough, even if time reversal symmetry is broken

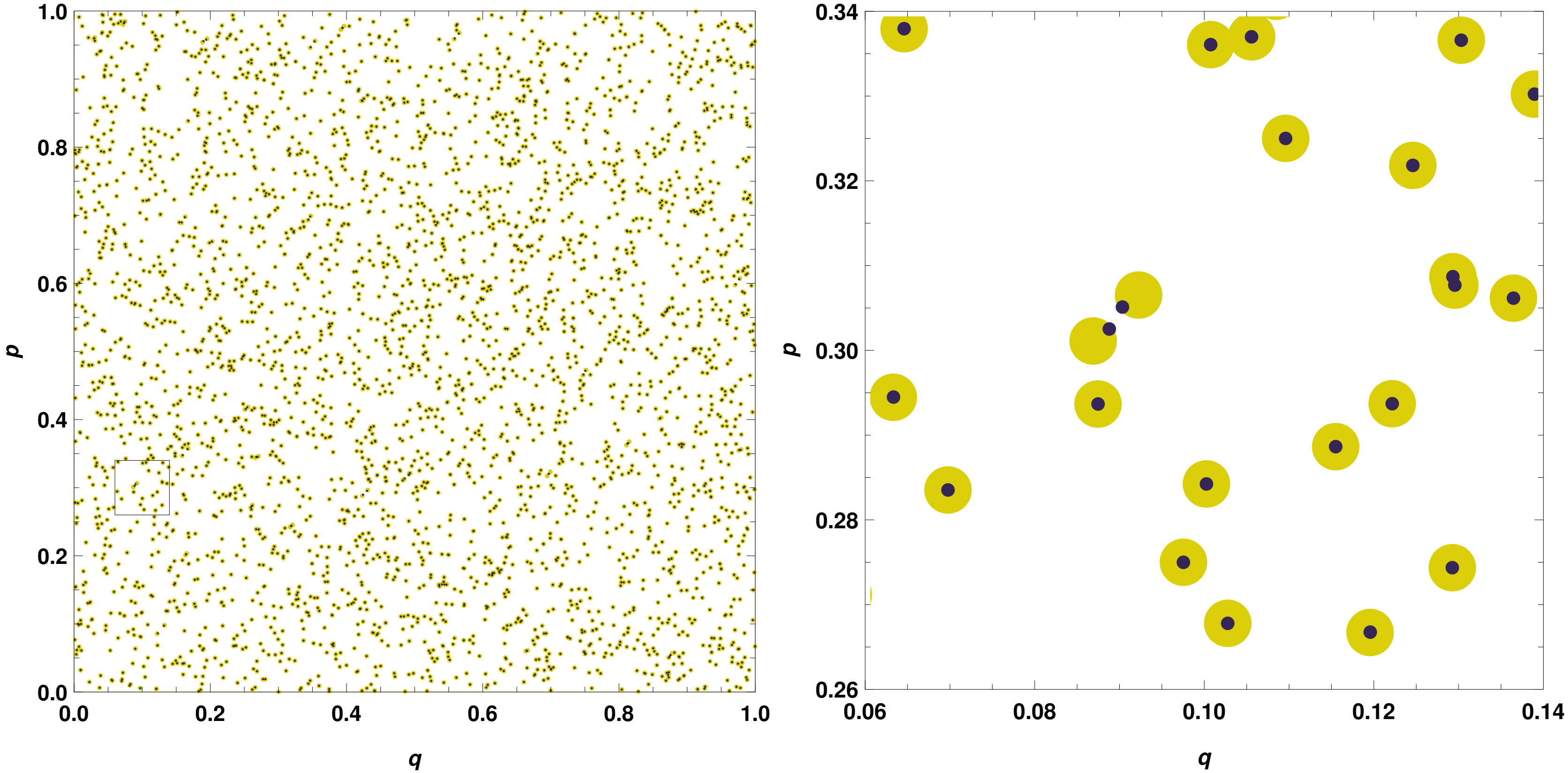
Example of Partner Orbits

$$T = 50,\ N = 70,\ a = 3,\ b = 2$$



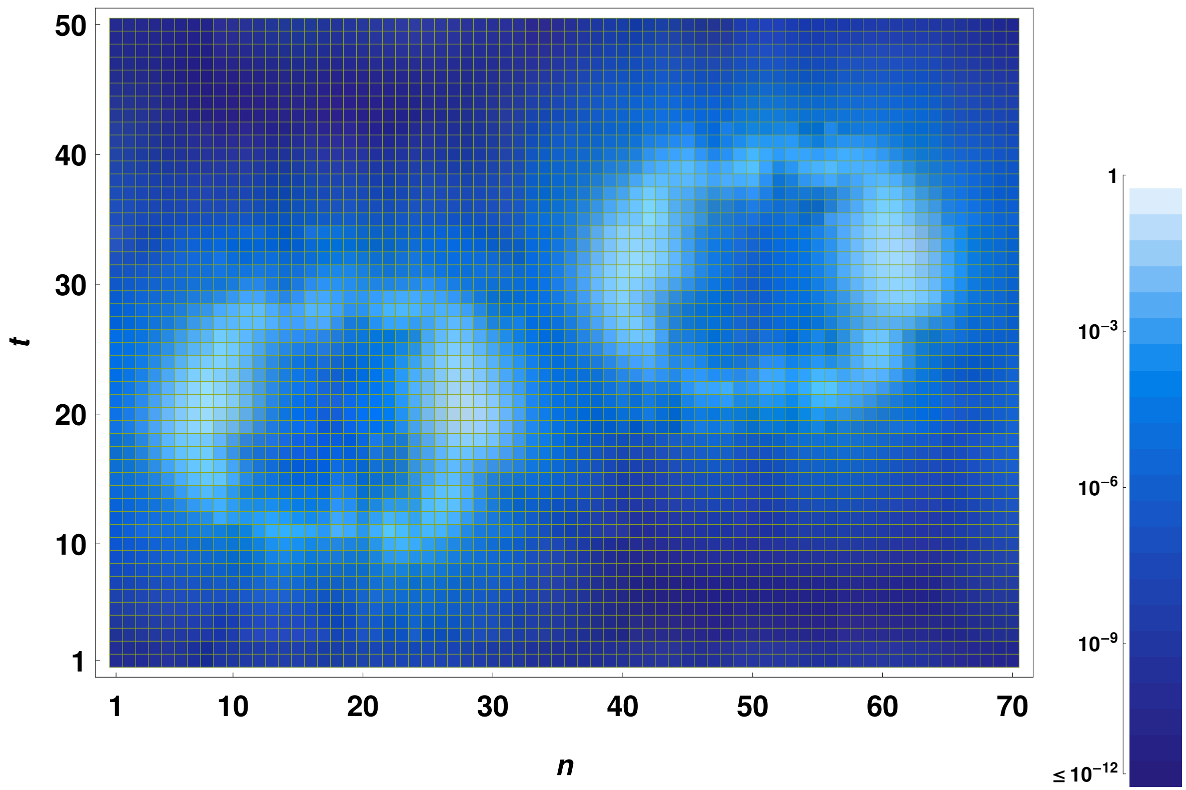
m	$\binom{0}{0}$	$\binom{0}{1}$	$\binom{1}{1}$	$\binom{1}{2}$	$\binom{1}{3}$	$\binom{2}{3}$	$\binom{2}{4}$	$\binom{2}{5}$	$\binom{3}{5}$	$\binom{3}{6}$	$\binom{0}{-1}$	$\binom{-1}{-1}$	$\binom{-1}{-2}$	$\binom{-1}{-3}$	$\binom{-2}{-3}$	$\binom{-2}{-4}$
a	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

Example of Partner Orbits



All the points of $\Gamma = \{(q_{n,t}, p_{n,t})\}$ **and** $\bar{\Gamma} = \{(\bar{q}_{n,t}, \bar{p}_{n,t})\}$ **are paired**

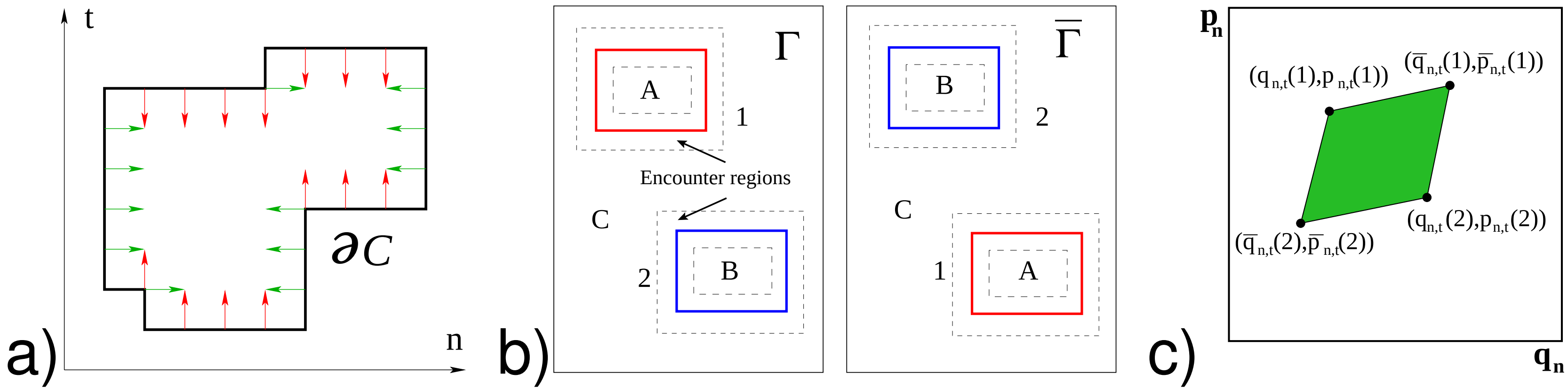
Distances between paired points



$$d_{n,t} = \sqrt{(q_{n,t} - \bar{q}_{n',t'})^2 + (p_{n,t} - \bar{p}_{n',t'})^2},$$

Lardgest distances $\sim 2 \cdot 10^{-3}$ are between points in encounters

Action differences



$$S(\Gamma) - S(\bar{\Gamma}) = \sum_{(n,t)\in\partial C_{\|}} \Delta S^{\|}_{n,t} + \sum_{(n,t)\in\partial C_{\perp}} \Delta S^{\perp}_{n,t},$$

$\Delta S^{a}_{n,t}$ $a \in \{\|, \perp\}$ - symplectic area of the region formed by the points $(q_{n,t}(k), p^{a}_{n,t}(k))$, $(\bar{q}_{n,t}(k), \bar{p}^{a}_{n,t}(k))$, $k = 1, 2$

$S(\Gamma) - S(\bar{\Gamma})$ **independent of** ∂C **choice as long as it is inside of the encounter**

Quantisation

Hannay, Berry (1980); Keating (1991)

U_N is $L^N \times L^N$ unitary matrix, $L = \hbar_{eff}^{-1}$

Translational symmetries: $\Longrightarrow$ N subspectra approximately of the same size $= L^N/N$. Almost all are paired i.e., mostly doubly degenerate levels

Gutzwiller trace formula

Rivas, Saraceno, A. de Almeida (2000)

$$\mathrm{Tr}\,(U_N)^T = \left|\det(\mathcal{B}_N^T - 1)\right|^{-\frac{1}{2}} \sum_{\Gamma \in \mathrm{PO}} \exp(-i2\pi L S_\Gamma).$$

All entries are symmetric under exchange $N \leftrightarrow T$

Quantum Duality

Particle-time duality (Quantum):

$$\mathrm{Tr}\,(U_N)^T = \mathrm{Tr}\,(U_T)^N$$

Form Factor: $K_N(T) = \frac{1}{2L^N}\left\langle \left|\mathrm{Tr}\,(U_N)^T\right|^2 \right\rangle$

For short times $T < n_E = \lambda^{-1}\log L$, $\quad N \sim L^T$

Regime dual to universal:

$$K_N(T) = L^{T-N} K_\beta(TN/L^T)$$

In particular for very short times $L^T/T < N$, $\quad K_\beta \approx 1$

$$K_N(T) \approx L^T/L^N$$

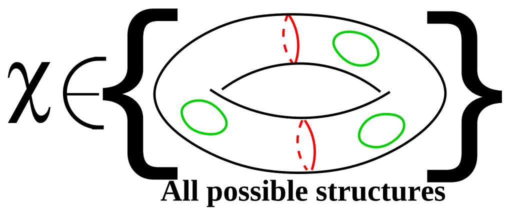
Short time exponential growth instead of linear TN/L^N

Many-particle Semiclassics

$$\left(\frac{L^N}{NT}\right) K_N(T) = \mathcal{K}_{\rm diag}(N,T) + \mathcal{K}_{\rm off}(N,T).$$

Diagonal: $\mathcal{K}_{\rm diag}(N,T) = 2/\beta$

Off-diagonal:



$$\mathcal{K}_{\rm off} = \sum_{\chi} \sum_{\Gamma,\bar{\Gamma}\in\chi} |A_\Gamma|^2 e^{i\frac{S_\Gamma - S_{\bar{\Gamma}}}{\hbar_{eff}}}$$

For a given encounter type ω:

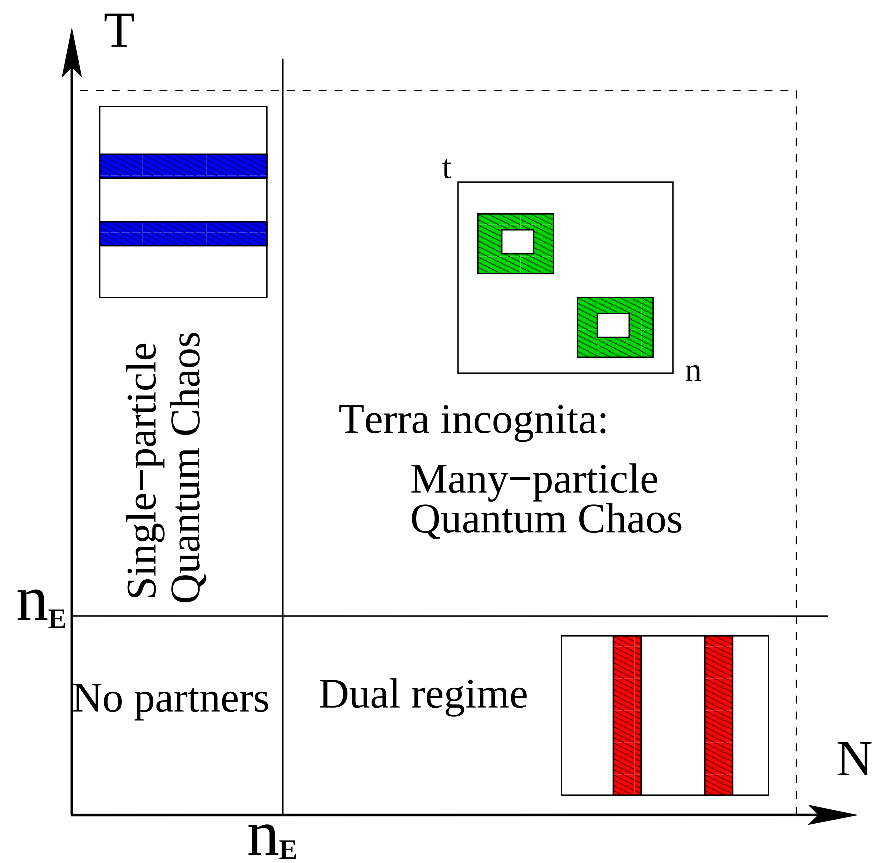
$$\mathcal{K}_{\rm off}^{(\omega)}(N,T) = \sum_{k=1}^{\infty} \alpha_\omega^{(k)} \left(\frac{NT}{L^{d_\omega}}\right)^k$$

The scale L^{d_ω} is defined by length of encounter:

$d_{(0,1)} = N$, $d_{(1,0)} = T$, $d_{(0,0)} = n_E$

For non-interacting particles $d = 1$.

Summary 1



$$\mathcal{K} = \frac{1}{TN} \left\langle \left| \mathrm{Tr}\,(U_N)^T \right|^2 \right\rangle$$

Duality:

$$\mathcal{K}(N,T) = \mathcal{K}(T,N)$$

Challenge: Contributions from new partners. Applications beyond spectral correlations. Extension to:

- **Hamiltonian flows (continues T)**
- **Quantum Field Theory (continues T, N)**

Summary 2

> “Sadly, searching for periodic orbits will never become as popular as a week on Côte d’Azur, or publishing yet another log-log plot in Phys. Rev. Letters.”

— P. Cvitanović, et al., Chaos: Classical and Quantum

Preprint: arXiv:1503.02676