

# String-like theory of many-particle Quantum Chaos

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March 2015

# Basic Question

## Quantum Chaos Theory:

Standard semiclassical limit:

fixed  $N$  (number of particles),  $\hbar_{eff} \rightarrow 0$

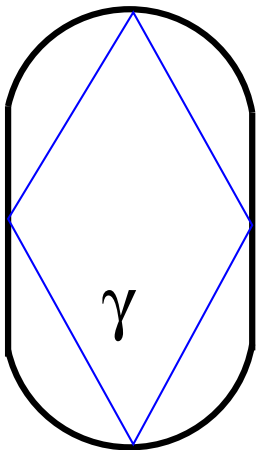
Non-standard: fixed  $\hbar_{eff}$ ,  $N \rightarrow \infty$  (Bosons)

**But what if  $\hbar_{eff} \rightarrow 0$  and  $N \rightarrow \infty$ ?**

# Single-particle Quantum Chaos

Gutzwiller's trace formula:

$$\rho(E) = \sum_n \delta(E - E_n) \sim \underbrace{\bar{\rho}(E)}_{\text{Smooth}} + \underbrace{\Re \sum_{\gamma \in \text{PO}} \mathcal{A}_\gamma \exp\left(\frac{i}{\hbar} S_\gamma(E)\right)}_{\text{Oscillating}}$$



$\mathcal{A}_\gamma$  stability factor,  
 $S_\gamma(E)$  action of a **periodic orbit**  $\gamma$

**Number of periodic orbits grows exponentially** with length

- No prediction on  $E_n$  from an individual  $\gamma$
- **All  $\{\gamma\}$  together  $\iff$  spectrum**

# Two-point correlation function

$$R(\varepsilon) = \frac{1}{\bar{\rho}^2} \langle \rho(E + \varepsilon/\bar{\rho}) \rho(E) \rangle_E - 1$$

$$K(\tau) = \int_{-\infty}^{+\infty} R(\varepsilon) e^{-2\pi i \tau \varepsilon} d\varepsilon \approx \text{(Semiclassically)}$$

$$\approx \frac{1}{T_H^2} \left\langle \sum_{\gamma, \gamma'} \mathcal{A}_\gamma \mathcal{A}_{\gamma'}^* e^{\frac{i}{\hbar}(S_\gamma - S_{\gamma'})} \delta \left( \tau - \frac{(T_\gamma + T_{\gamma'})}{2T_H} \right) \right\rangle_E,$$

$T_\gamma, T_{\gamma'}$  are periods of  $\gamma, \gamma'$ ,  $T_H = 2\pi\hbar\bar{\rho}$  (Heisenberg time)

**Spectral correlations**  $\iff$

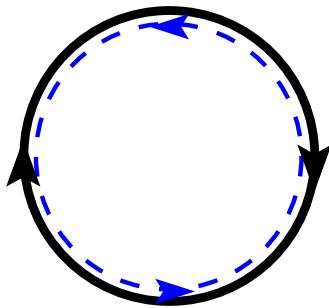
**Correlations between actions of periodic orbits**

# Semiclassical origins of universality

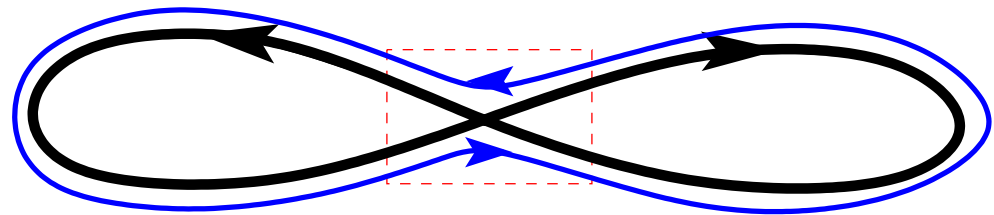
$$K(\tau) = c_1\tau + c_2\tau^2 \dots$$

**Diagonal approximation**  $\gamma = \gamma' \implies$

Leading order:  $c_1$  [M. Berry 1985]



**Diagonal approximation**



**Sieber-Richter pairs**

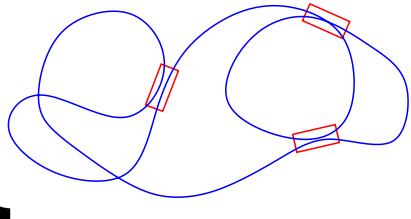
**Sieber-Richter pairs** (Non-trivial correlations)  $\implies$

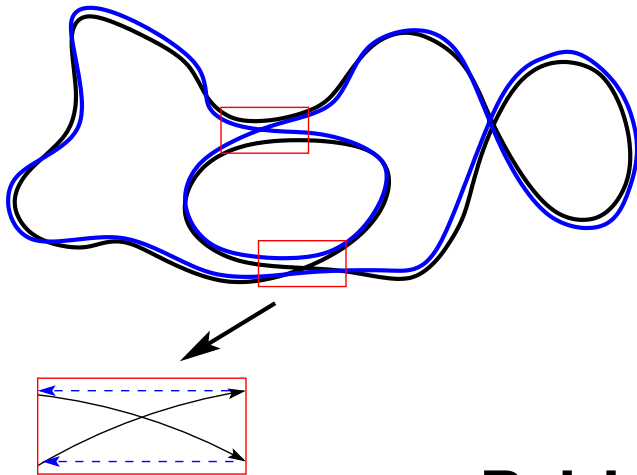
Second order:  $c_2$  [M. Sieber K. Richter 2001]

$S_\gamma - S_{\gamma'} \sim \hbar \implies$  Duration of encounter  $\sim \underbrace{\tau_E = \lambda^{-1} |\log \hbar|}_{\text{Ehrenfest time}}$

# Full theory – all orders in $\tau$

S. Müller, S. Heusler, P. Braun, F. Haake, A. Altland 2004

$$K(\tau) = \sum_{\text{Structures of Periodic Orbits}} \left\{ \text{Diagram} \right\} \tau^n$$




Pairing is robust under perturbation  
Other correlations are washed out!

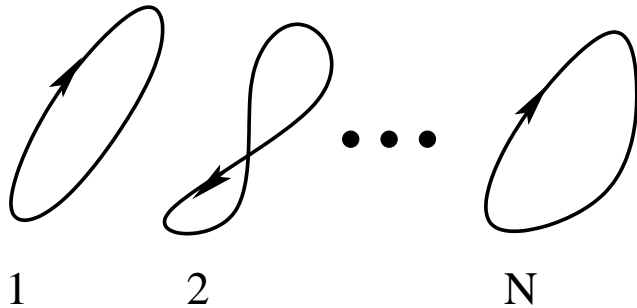
**Pairing mechanism  $\implies$   
Universality of spectral correlations**

# Many-particle systems

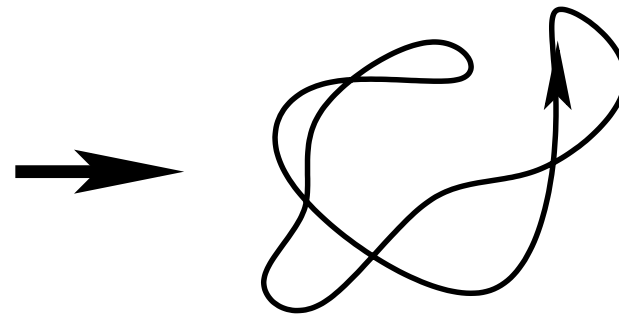
$$\mathcal{H} = \sum_{n=1}^N \frac{p_n^2}{2m} + V(x_n) + V_{\text{int}}(x_n - x_{n+1})$$

Chaos; Local, Homogeneous interactions  
i.e, invariance under  $n \rightarrow n + 1$

Many-particle Periodic Orbit  
**d-dimensions**



Single-particle Periodic Orbit  
**Nd-dimensions**



**Q: Is the single-particle theory of Quantum Chaos applicable?**

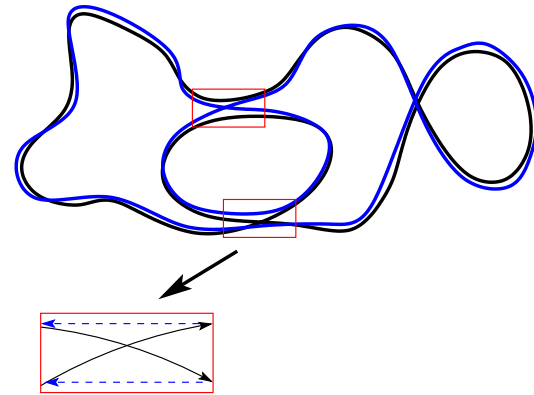
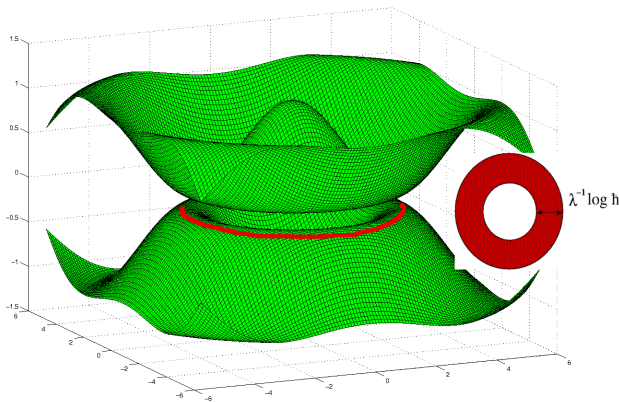
**A: Depends how large  $N$  is**

# Caricature: Semiclassical Field Theory

**Continues limit**  $n \rightarrow \eta \in [0, \ell]$ ,  $x_n(t) \rightarrow x(\eta, t)$

$$\mathcal{L} = \sum_{n=1}^N \frac{\dot{x}_n^2}{2m} - V(x_n) - V_{\text{int}}(x_n - x_{n+1}) \longrightarrow$$

$$\mathcal{L} = \int_0^\ell d\eta (\partial_t x(\eta, t))^2 + (\partial_\eta x(\eta, t))^2 - V(x(\eta, t))$$

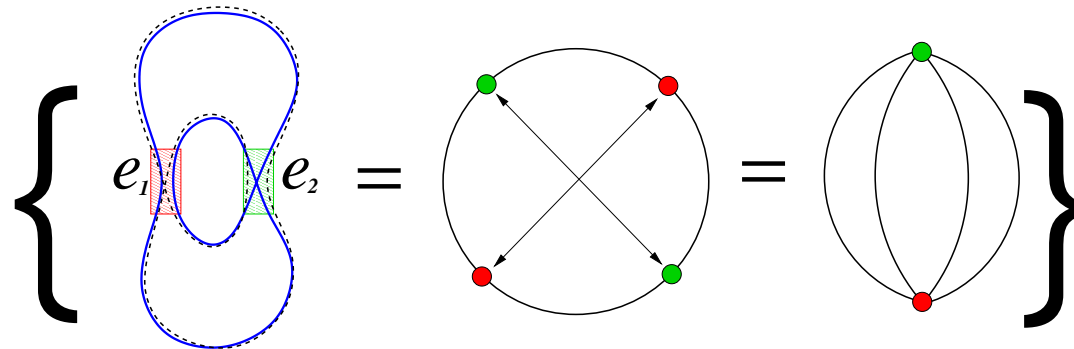


- 1) **PO** -are **2D toric surfaces** in  $d$ -dim space (Rather than 1D lines in  $N \cdot d$ -dim)
- 2) **Encounters** are “**rings**” (Rather than 1D stretches) of “width”  $\sim \lambda^{-1} |\log \hbar_{eff}|$



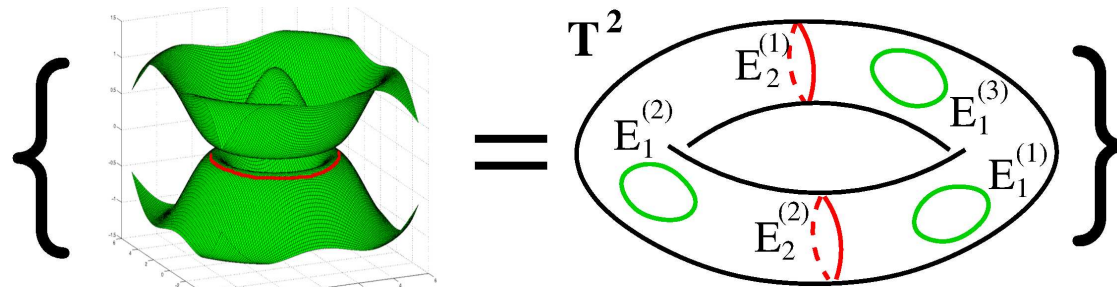
# Caricature: Semiclassical Field Theory

Single-particle structure diagrams:



Distinguished by order of encounters

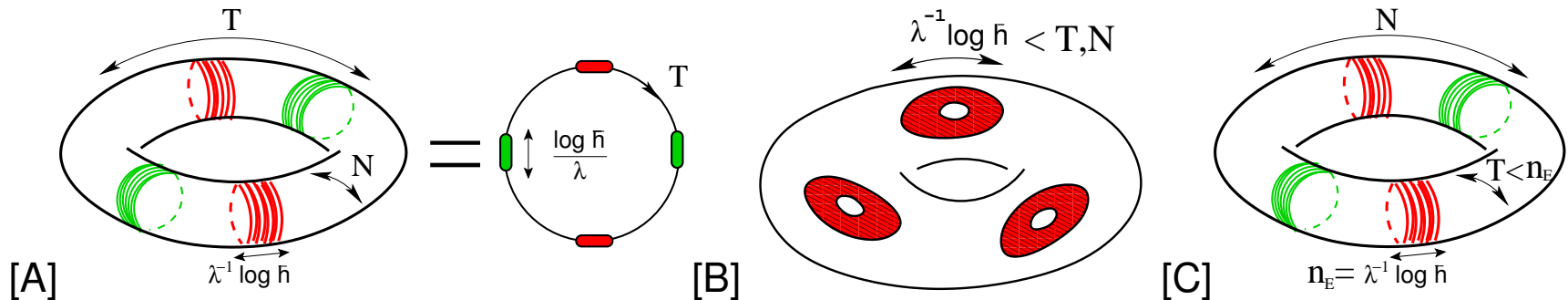
Many-particle structure diagrams:



Distinguished by order and winding numbers  $\omega$  of encounters!

# Many-particle Quantum Chaos

$\lambda^{-1} |\log \hbar_{eff}| =: n_E$  - Ehrenfest “number”



**A.** If  $N \lesssim n_E, T \gtrsim n_E \implies$  only  $\omega = (0, 1)$  “single-particle” encounters

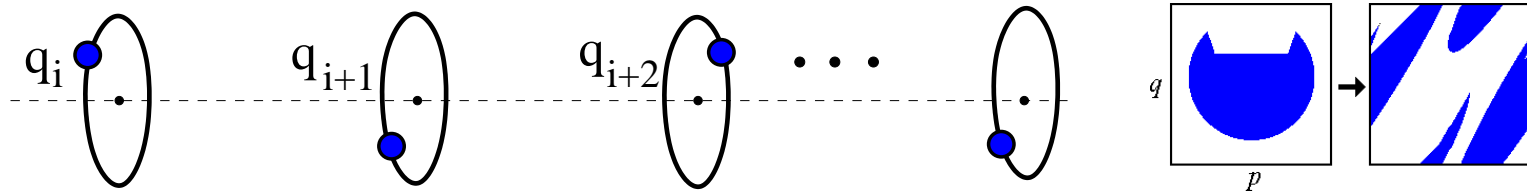
**Effectively single-particle Quantum Chaos**

**B.**  $N, T \gtrsim n_E, \implies \omega = (0, 0)$  encounters dominate!

**C.**  $N \gtrsim n_E, T \lesssim n_E \implies$  only  $\omega = (1, 0)$  encounters

**B, C - Genuine many-particle Quantum Chaos!**

# Coupled-Cat Maps



$$S(\mathbf{q}_t, \mathbf{q}_{t+1}) = S_0(\mathbf{q}_t, \mathbf{q}_{t+1}) + S_{\text{int}}(\mathbf{q}_t), \quad \mathbf{q}_t = (q_{1,t}, q_{2,t} \dots q_{N,t})$$

$N$  uncoupled cat maps,  $q_{n,t}, p_{n,t} \in [0, 1]$ :

$$S_0 = \sum_{n=1}^N S_0^{(n)}(q_{n,t}, q_{n,t+1}) + V(q_{n,t})$$

Interactions:

$$S_{\text{int}} = - \sum_{n=1}^N q_{n,t} q_{1+(n \bmod N), t}$$

# Coupled-Cat Maps $V = 0$

$$Z_{t+1} = \mathcal{B}_N Z_t \bmod 1, \quad Z_t = (q_{1,t}, p_{1,t}, \dots, q_{N,t}, p_{N,t})^\top,$$

with  $2N \times 2N$  matrix  $\mathcal{B}_N$  given by:

$$\mathcal{B}_N = \begin{pmatrix} A & B & \mathbf{0} & \dots & \mathbf{0} & B \\ B & A & B & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B & A & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & A & B \\ B & \mathbf{0} & \mathbf{0} & \dots & B & A \end{pmatrix}, \quad A = \begin{pmatrix} a & 1 \\ ab - 1 & b \end{pmatrix}, \quad B = - \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}$$

Lyapunov exponents:

$$\cosh \lambda_k = (a + b)/2 - \cos(2\pi k/N), \quad k = 1, \dots, N$$

**Full chaos:**  $|\operatorname{Re} \lambda_k| > 0$ , i.e.  $|a + b| > 4$

# Particle-time Duality

Newtonian form:

$$\Delta_t^2 q_{n,t} + \Delta_n^2 q_{n,t} = (a + b - 4)q_{n,t} + V'(q_{n,t}) \pmod{1}$$

$$\Delta_\alpha^2 \text{- discrete Laplacian: } \Delta_\alpha^2 f_\alpha \equiv f_{\alpha+1} - 2f_\alpha + f_{\alpha-1}$$

**Particle-time duality:**

**If  $\{q_{n,t}\}$  solution then  $\{q'_{n,t} = q_{t,n}\}$  also solution!  $\implies$   
 $N$ -part. PO  $\Gamma$  of period  $T \iff T$ -part. PO  $\Gamma'$  of period  $N$**

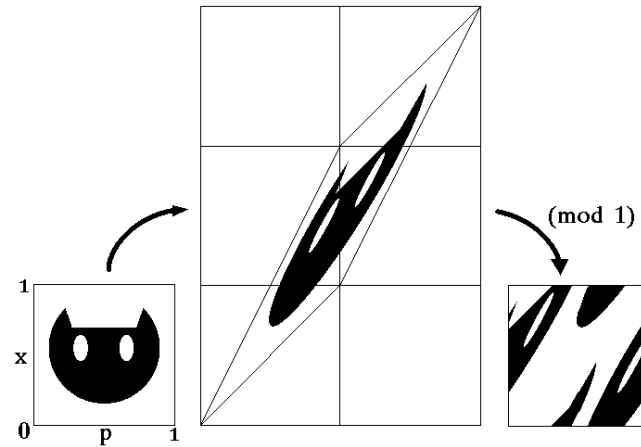
1)  $S(\Gamma) = S(\Gamma')$

2)  $[\# \text{ of } N\text{-particle PO of period } T] = [\# \text{ of } T\text{-particle PO of period } N] \iff |\det(I - \mathcal{B}_N^T)| = |\det(I - \mathcal{B}_T^N)|$

Corollary: 
$$T^2 \prod_{k=1}^{N-1} 4 \sin^2 \left( \frac{\pi k T}{N} \right) = N^2 \prod_{m=1}^{T-1} 4 \sin^2 \left( \frac{\pi m N}{T} \right)$$

# 2D Symbolic Dynamics

$$Z_{t+1} + \mathcal{M}_t = \mathcal{B}_N Z_t, \mathcal{M}_t = (m_{1,t}, \dots, m_{N,t})^\top, m_{n,t} = (m_{n,t}^q, m_{n,t}^p)$$



2	4	1	2	1	4	2	1	3	4	2	3	2	3	4	3
2	3	1	2	1	3	4	2	3	1	2	3	1	4	2	3
3	2	1	4	3	4	1	1	1	3	1	4	1	2	1	1
2	4	2	1	4	2	4	3	3	1	2	4	4	1	2	3
4	1	4	3	2	4	2	1	4	4	1	4	3	4	1	2
3	4	3	1	3	2	4	2	3	1	2	1	4	2	4	4
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3	2	3	1	4	1	4	1	2	1	2	4	3	1	3	1
1	4	2	2	3	2	1	3	4	4	2	3	1	4	2	3
3	1	1	4	1	4	3	2	1	3	2	2	3	3	1	4

$$\mathbb{M}_\Gamma = \begin{pmatrix} m_{1,1} & m_{2,1} & \dots & m_{N,1} \\ m_{1,2} & m_{2,2} & \dots & m_{N,2} \\ \vdots & \vdots & \ddots & \vdots \\ m_{1,T} & m_{2,T} & \dots & m_{N,T} \end{pmatrix}$$

# 2D Symbolic Dynamics

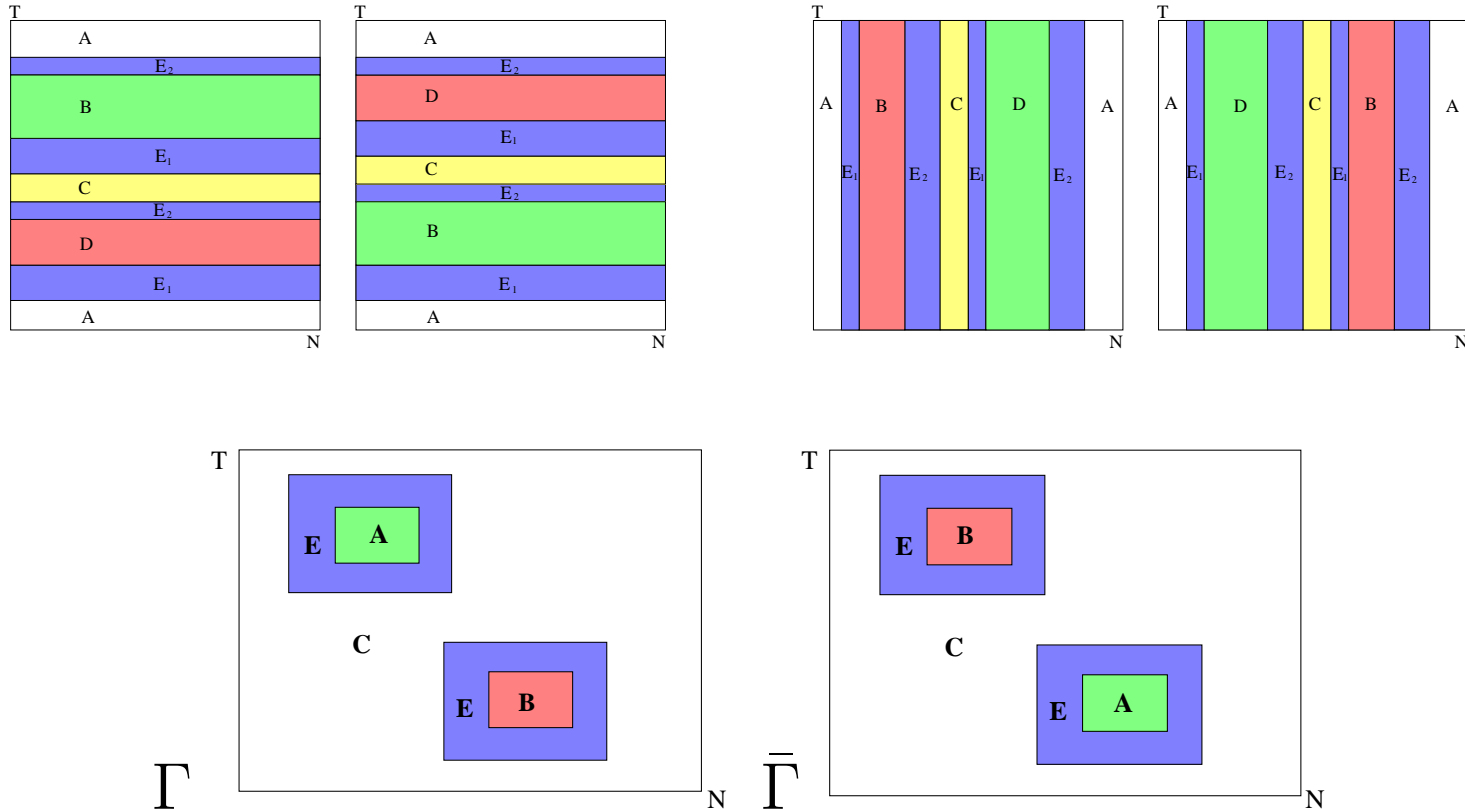
$$Z_{t+1} + \mathcal{M}_t = \mathcal{B}_N Z_t, \mathcal{M}_t = (m_{1,t}, \dots, m_{N,t})^\top, m_{n,t} = (m_{n,t}^q, m_{n,t}^p)$$

T	2	4	1	2	1	4	2	1	3	4	2	3	2	3	4	3
	2	3	1	2	1	3	4	2	3	1	2	3	1	4	2	3
	3	2	1	4	3	4	1	1	1	3	1	4	1	2	1	1
	2	4	2	1	4	2	4	3	3	1	2	4	4	1	2	3
	4	1	4	3	2	4	2	1	4	4	1	4	3	4	1	2
	3	4	3	1	3	2	4	2	3	1	2	1	4	2	4	4
	1	1	2	4	3	1	3	2	3	1	4	3	3	4	2	4
	1	4	2	3	1	4	2	4	4	4	3	4	2	2	4	3
	3	2	3	1	4	1	4	1	2	1	2	4	3	1	3	1
	1	4	2	2	3	2	1	3	4	4	2	3	1	4	2	3
	3	1	1	4	1	4	3	2	1	3	2	2	3	3	1	4
																N

$$\mathbb{M}_\Gamma = \begin{pmatrix} m_{1,1} & m_{2,1} & \dots & m_{N,1} \\ m_{1,2} & m_{2,2} & \dots & m_{N,2} \\ \vdots & \vdots & \ddots & \vdots \\ m_{1,T} & m_{2,T} & \dots & m_{N,T} \end{pmatrix}$$

- 1) **Small alphabet** (does not grow with  $N$ )
- 2) **Uniqueness:** Each PO  $\Gamma$  is uniquely encoded by  $\mathbb{M}_\Gamma$   
 $\Gamma$  can be easily restored from  $\mathbb{M}_\Gamma$
- 3) **Locality:**  $r \times r$  square of symbols around  $(n, t)$  defines position of the  $n$ 'th particle at the time  $t$  up to error  $\sim \Lambda^{-r}$

# Partner Orbits



$M_{\bar{\Gamma}}$  is obtained by reshuffling  $M_{\Gamma}$

Note: One encounter is enough, even if time reversal symmetry is broken



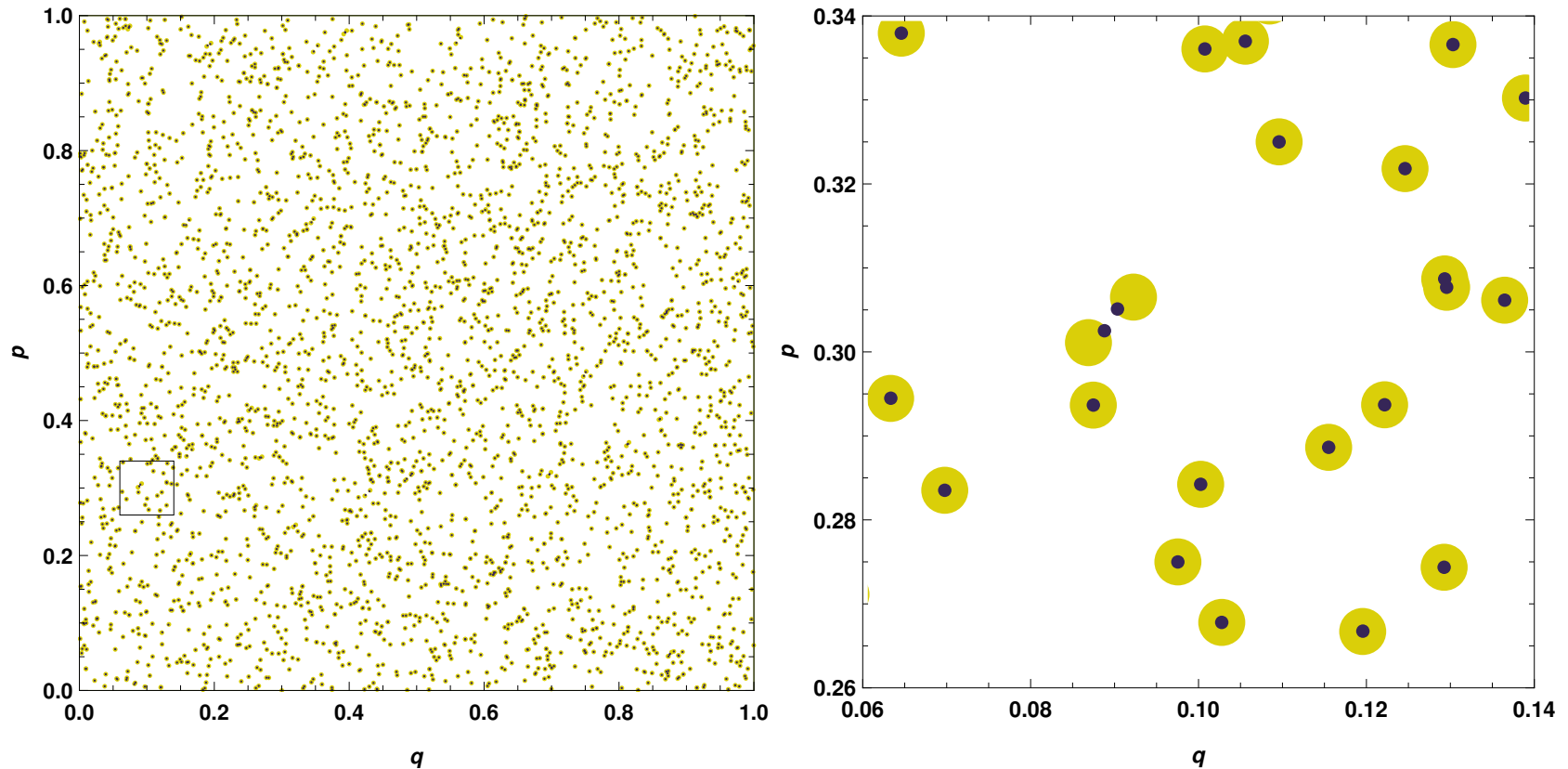
# Example of Partner Orbits

$$T = 50, N = 70, a = 3, b = 2$$

```
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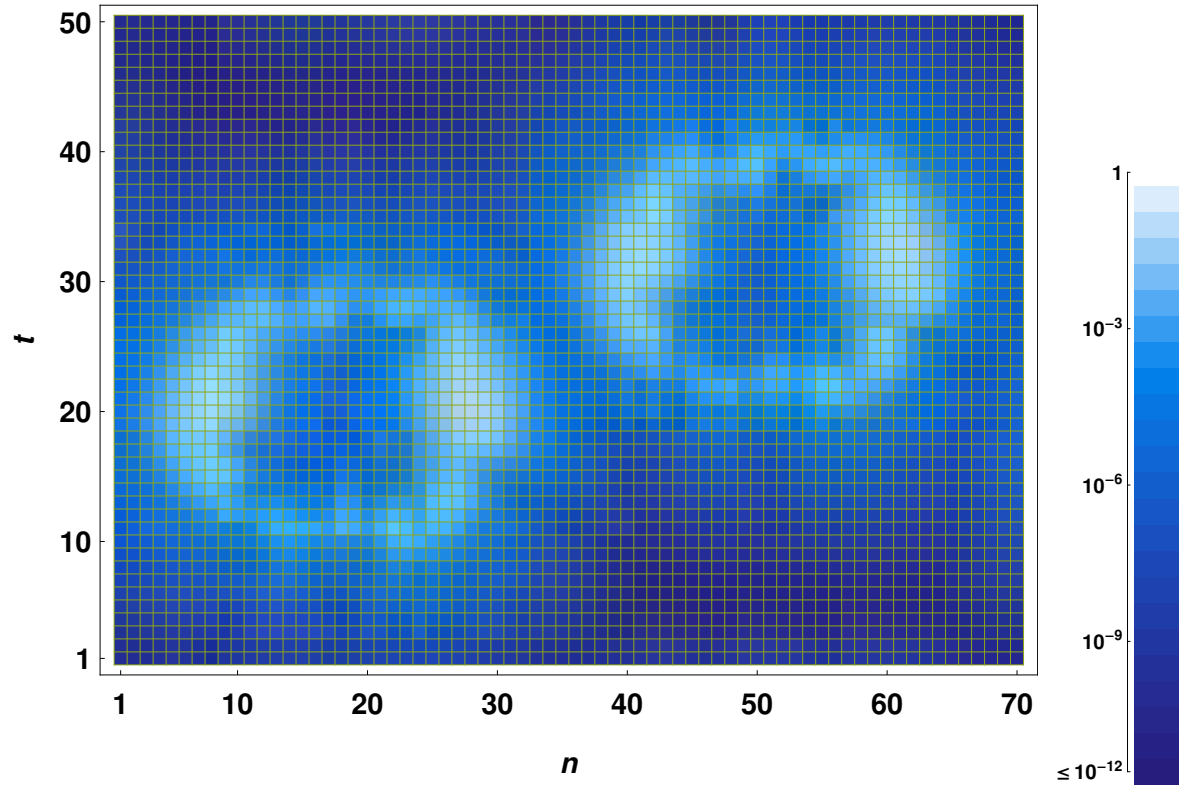
$m$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 4 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 5 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 5 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 6 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$	$\begin{pmatrix} -1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} -1 \\ -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ -3 \end{pmatrix}$	$\begin{pmatrix} -2 \\ -3 \end{pmatrix}$	$\begin{pmatrix} -2 \\ -4 \end{pmatrix}$
$a$	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

# Example of Partner Orbits



**All the points of  $\Gamma = \{(q_{n,t}, p_{n,t})\}$  and  $\bar{\Gamma} = \{(\bar{q}_{n,t}, \bar{p}_{n,t})\}$  are paired**

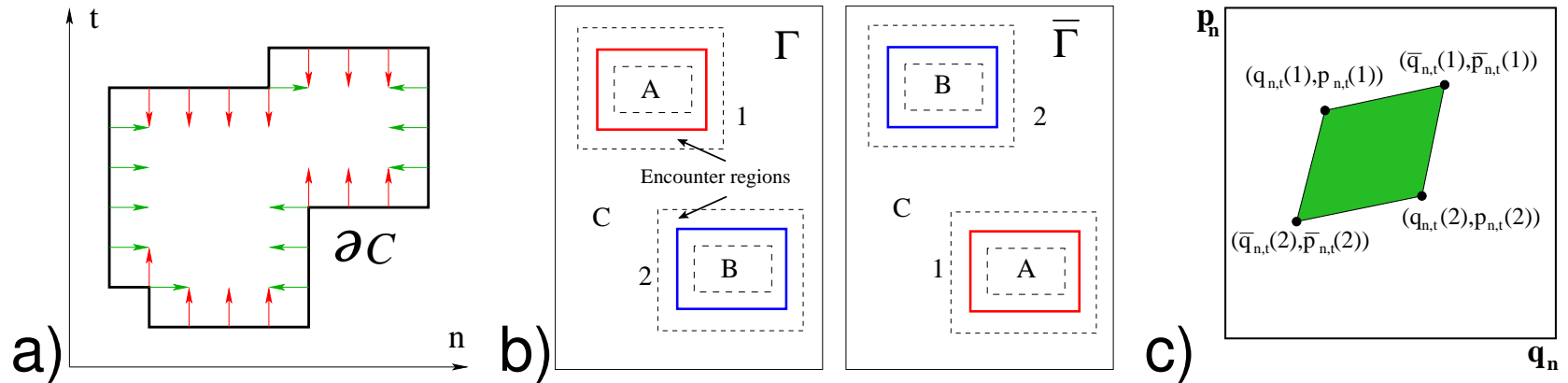
# Distances between paired points



$$d_{n,t} = \sqrt{(q_{n,t} - \bar{q}_{n',t'})^2 + (p_{n,t} - \bar{p}_{n',t'})^2},$$

**Largest distances  $\sim 2 \cdot 10^{-3}$  are between points in encounters**

# Action differences



$$S(\Gamma) - S(\bar{\Gamma}) = \sum_{(n,t) \in \partial C_{\parallel}} \Delta S_{n,t}^{\parallel} + \sum_{(n,t) \in \partial C_{\perp}} \Delta S_{n,t}^{\perp},$$

$\Delta S_{n,t}^a$   $a \in \{\parallel, \perp\}$  - symplectic area of the region formed by the points  $(q_{n,t}(k), p_{n,t}^a(k))$ ,  $(\bar{q}_{n,t}(k), \bar{p}_{n,t}^a(k))$ ,  $k = 1, 2$

$S(\Gamma) - S(\bar{\Gamma})$  independent of  $\partial C$  choice as long as it is inside of the encounter

# Quantisation

Hannay, Berry (1980); Keating (1991)

$U_N$  is  $L^N \times L^N$  unitary matrix,  $L = \hbar_{eff}^{-1}$

**Translational symmetries:**  $\implies N$  subspectra approximately of the same size  $= L^N / N$ . Almost all are paired i.e., mostly doubly degenerate levels

## Gutzwiller trace formula

Rivas, Saraceno, A. de Almeida (2000)

$$\text{Tr} (U_N)^T = |\det(\mathcal{B}_N^T - 1)|^{-\frac{1}{2}} \sum_{\Gamma \in \text{PO}} \exp(-i2\pi L S_\Gamma).$$

All entries are symmetric under exchange  $N \leftrightarrow T$

# Quantum Duality

**Particle-time duality (Quantum):**

$$\text{Tr} (U_N)^T = \text{Tr} (U_T)^N$$

$$\text{Form Factor: } K_N(T) = \frac{1}{2L^N} \left\langle \left| \text{Tr} (U_N)^T \right|^2 \right\rangle$$

For short times  $T < n_E = \lambda^{-1} \log L$ ,  $N \sim L^T$

**Regime dual to universal:**

$$K_N(T) = L^{T-N} K_\beta(TN/L^T)$$

In particular for very short times  $L^T/T < N$ ,  $K_\beta \approx 1$

$$K_N(T) \approx L^T / L^N$$

**Short time exponential growth instead of linear  $TN/L^N$**

# Many-particle Semiclassics

$$\left(\frac{L^N}{NT}\right) K_N(T) = \mathcal{K}_{\text{diag}}(N, T) + \mathcal{K}_{\text{off}}(N, T).$$

**Diagonal:**  $\mathcal{K}_{\text{diag}}(N, T) = 2/\beta$

**Off-diagonal:**

$\chi \in \left\{ \begin{array}{c} \text{Diagram: An oval containing two horizontal paths with four green circles and two red dashed arcs.} \\ \text{All possible structures} \end{array} \right\}$

$$\mathcal{K}_{\text{off}} = \sum_{\chi} \sum_{\Gamma, \bar{\Gamma} \in \chi} |A_{\Gamma}|^2 e^{i \frac{S_{\Gamma} - S_{\bar{\Gamma}}}{\hbar_{\text{eff}}}}$$

For a given encounter type  $\omega$ :

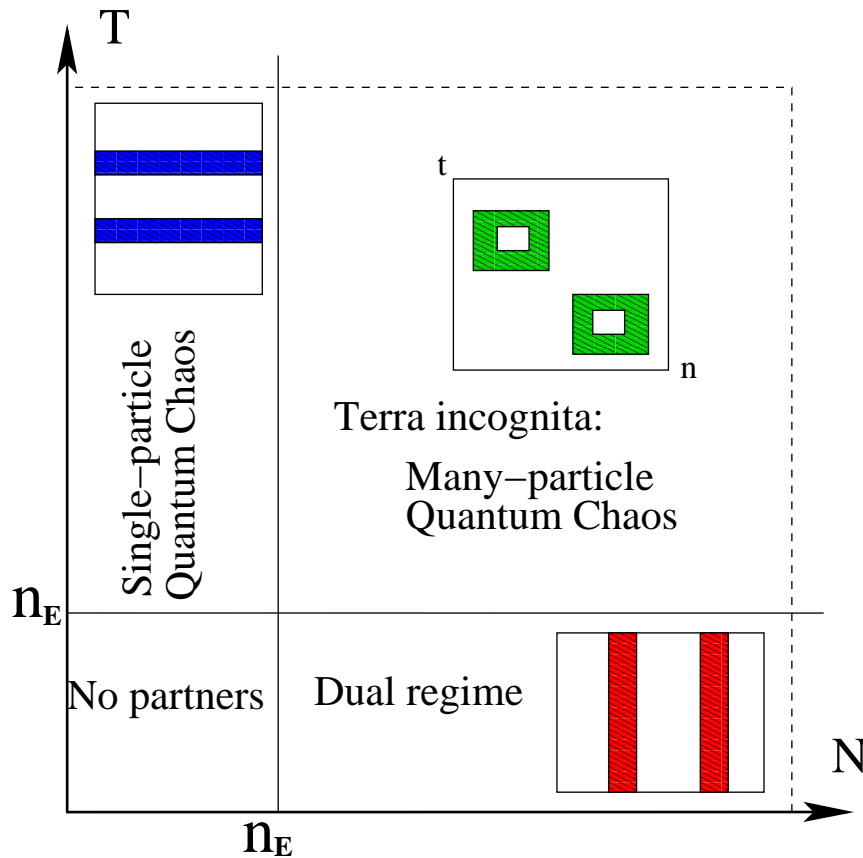
$$\mathcal{K}_{\text{off}}^{(\omega)}(N, T) = \sum_{k=1}^{\infty} \alpha_{\omega}^{(k)} \left(\frac{NT}{L^{d_{\omega}}}\right)^k$$

The scale  $L^{d_{\omega}}$  is defined by length of encounter:

$$d_{(0,1)} = N, \quad d_{(1,0)} = T, \quad d_{(0,0)} = n_E$$

For non-interacting particles  $d = 1$ .

# Summary 1



$$\mathcal{K} = \frac{1}{TN} \left\langle \left| \text{Tr} (U_N)^T \right|^2 \right\rangle$$

**Duality:**

$$\mathcal{K}(N, T) = \mathcal{K}(T, N)$$

**Challenge:** Contributions from new partners. Applications beyond spectral correlations. Extension to:

- Hamiltonian flows (continues  $T$ )
- Quantum Field Theory (continues  $T, N$ )



# Summary 2

“Sadly, searching for periodic orbits will never become as popular as a week on Côte d’Azur, or publishing yet another log-log plot in Phys. Rev. Letters.”

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— *P. Cvitanović, et al., Chaos: Classical and Quantum*

Preprint: [arXiv:1503.02676](https://arxiv.org/abs/1503.02676)