Quantum chaos: fundamentals and applications

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Complex dynamics in normal form Hamiltonian systems

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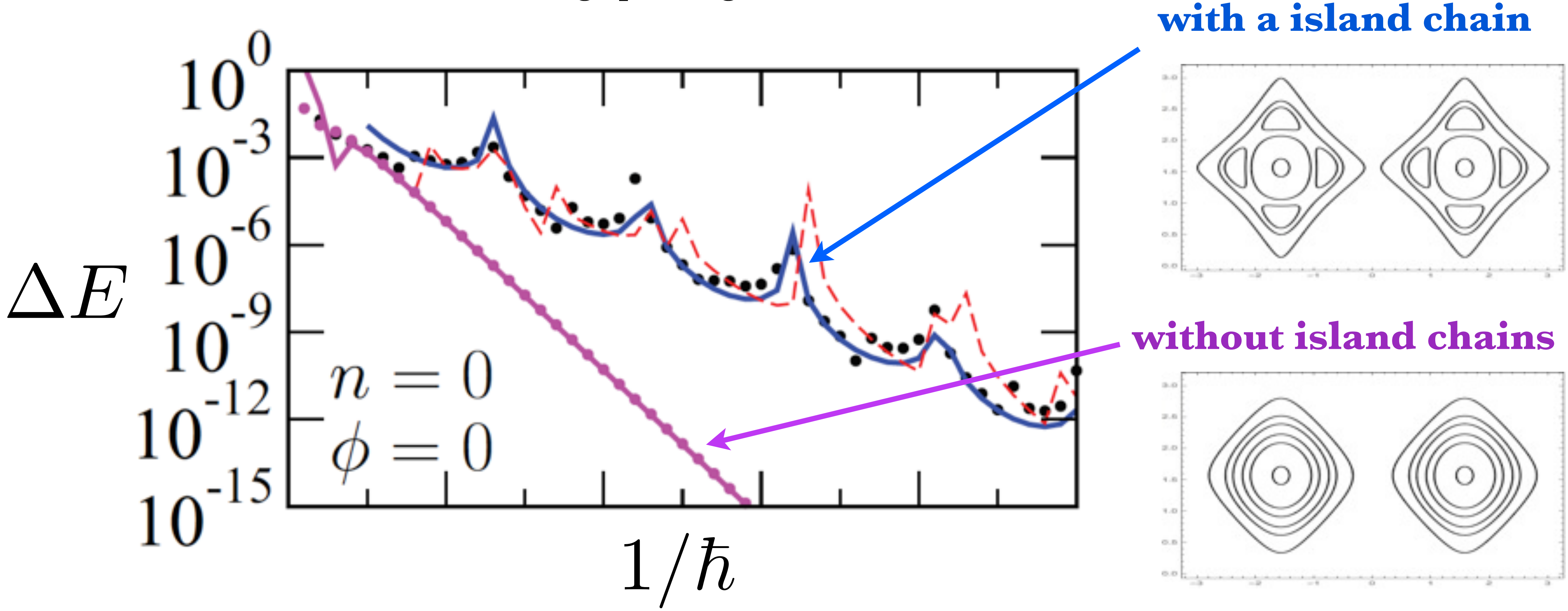
Motivation

Resonance-assisted tunneling in a normal form system

tunneling splitting vs 1 /hbar



J. Le Deunff, A. Mouchet, and P. Schlagheck, Phys. Rev. E **88**, 042927 (2013).

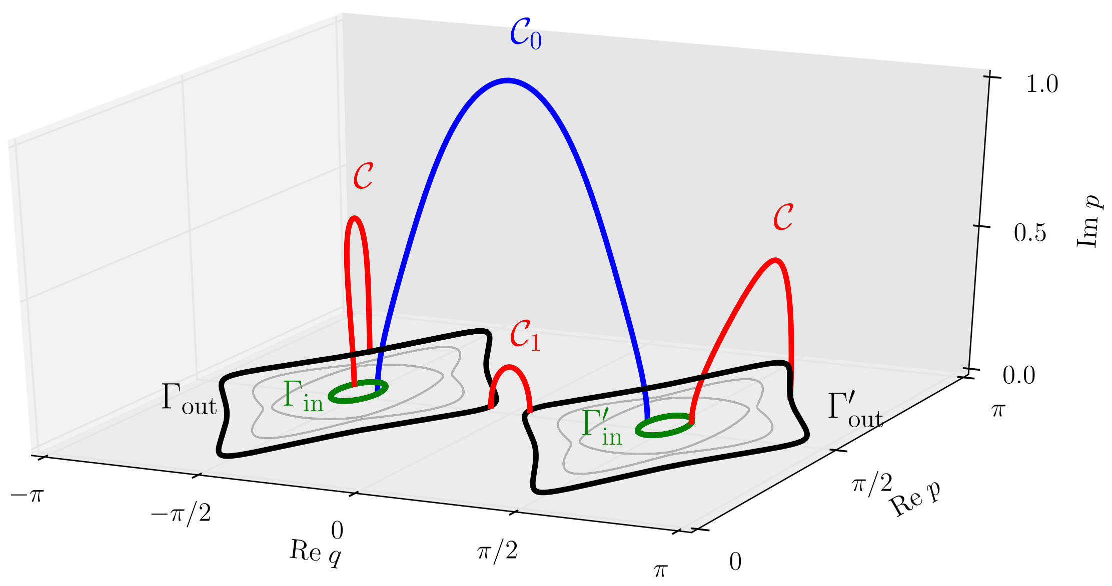
Motivation

Resonance-assisted tunneling in a normal form system

A semiclassical formula for tunneling splitting:

$$\Delta E = |A_{\mathcal{T}}|^2 \delta E$$

$$A_{\mathcal{T}} = \frac{e^{-\sigma/2\hbar}}{2\sin((S_{\text{in}} - S_{\text{out}})/2l\hbar)}$$



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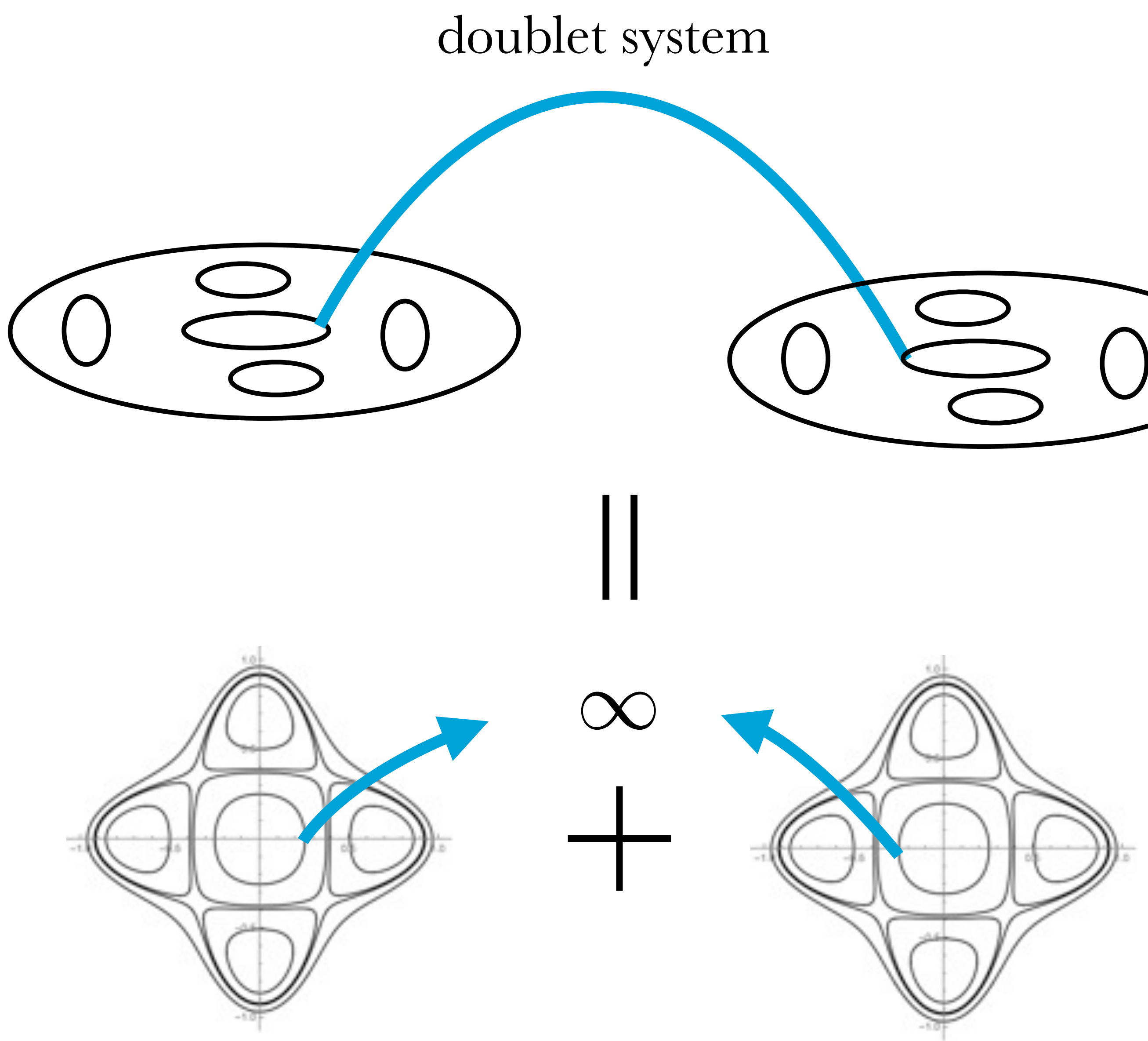
Necessary to know the topology and the imaginary action of complex trajectories.

Simpler model

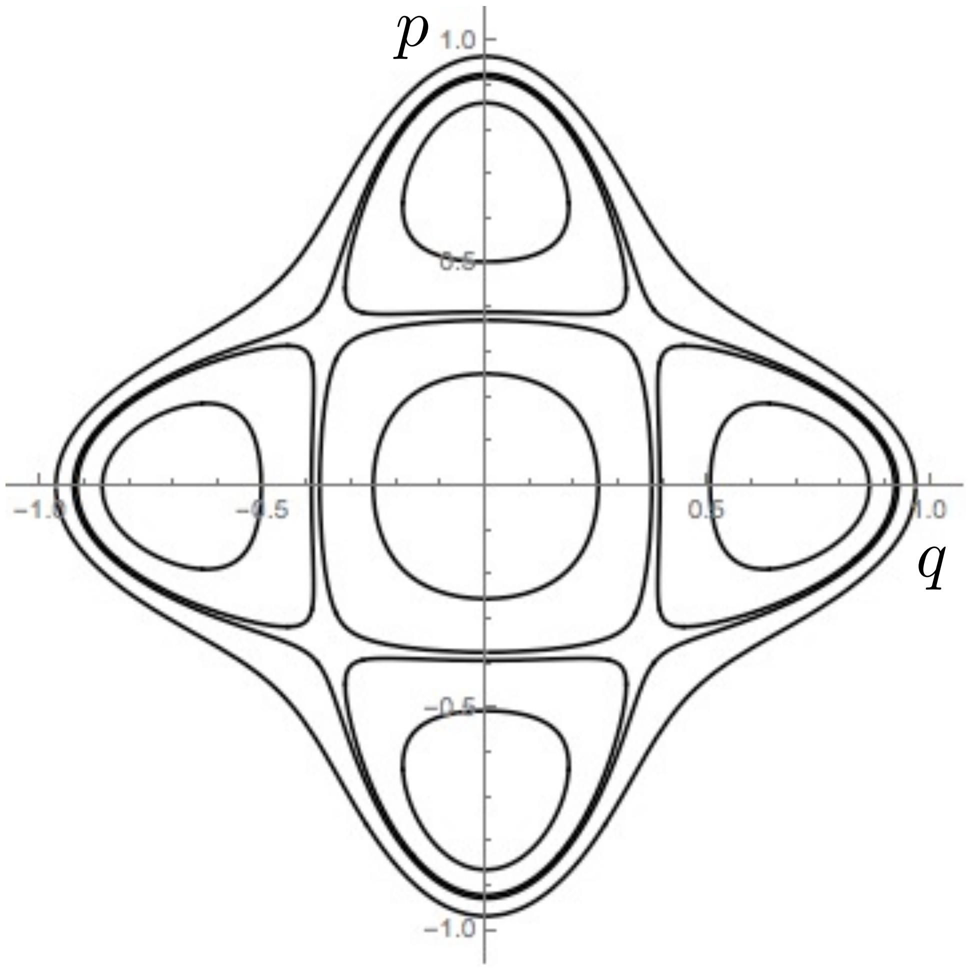
Divide connected wells into two separated wells and focus on a single well case first.



Exact analysis of a normal form system

Hamiltonian :

$$H = \frac{p^2}{2} + \frac{q^2}{2} + \epsilon \left(\frac{p^2}{2} + \frac{q^2}{2} \right)^2 + \eta p^2 q^2 .$$



Phase space with $\epsilon = \eta = -2$.

Exact analysis of a normal form system

New coordinate : $P := p^2, Q := q^2,$

Hamilton's equations : $\dot{Q} = (2 + 2\epsilon(Q + P) + 4\eta Q)\sqrt{PQ},$
$\dot{P} = (-2 - 2\epsilon(P + Q) - 4\eta P)\sqrt{PQ}.$

$$\frac{\dot{Q} + \dot{P}}{4\eta(Q - P)} = \sqrt{PQ} = \frac{\dot{Q} - \dot{P}}{4 + 4\epsilon(Q + P) + 4\eta(Q + P)}.$$

This yields

$$4(Q + P) + (2\epsilon + 2\eta)(Q + P)^2 = 2\eta(Q - P)^2 + C,$$

where C is an integration constant.

Exact analysis of a normal form system

The form of solution :

$$q = \pm\sqrt{\frac{1}{2}\left(\frac{A^{1/2}}{\epsilon+\eta}\sin\theta(t) + \left(\frac{A}{-\eta(\epsilon+\eta)}\right)^{1/2}\cos\theta(t) - \frac{1}{\epsilon+\eta}\right)},$$

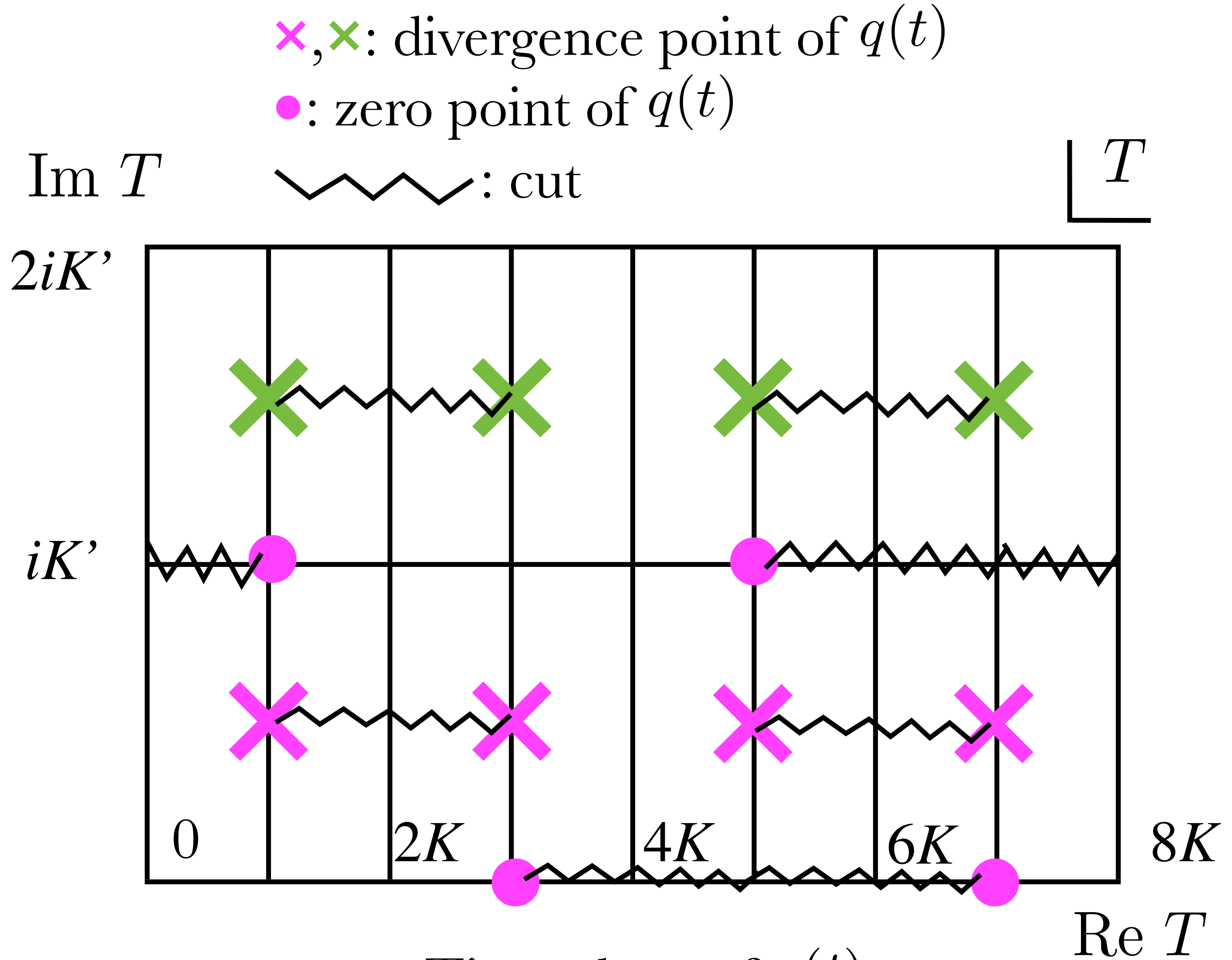
$$p = \pm\sqrt{\frac{1}{2}\left(\frac{A^{1/2}}{\epsilon+\eta}\sin\theta(t) - \left(\frac{A}{-\eta(\epsilon+\eta)}\right)^{1/2}\cos\theta(t) - \frac{1}{\epsilon+\eta}\right)}.$$

$$\theta(t) = \arcsin\left(\frac{\alpha+\beta\mathrm{sn}^2(t,k)}{\gamma\mathrm{sn}^2(t,k)-\delta} - \frac{\eta}{\epsilon A^{1/2}}\right).$$

$$A := 1 + (\epsilon+\eta)C$$

$\mathrm{sn}(t,k)$: Jacobi elliptic sn function

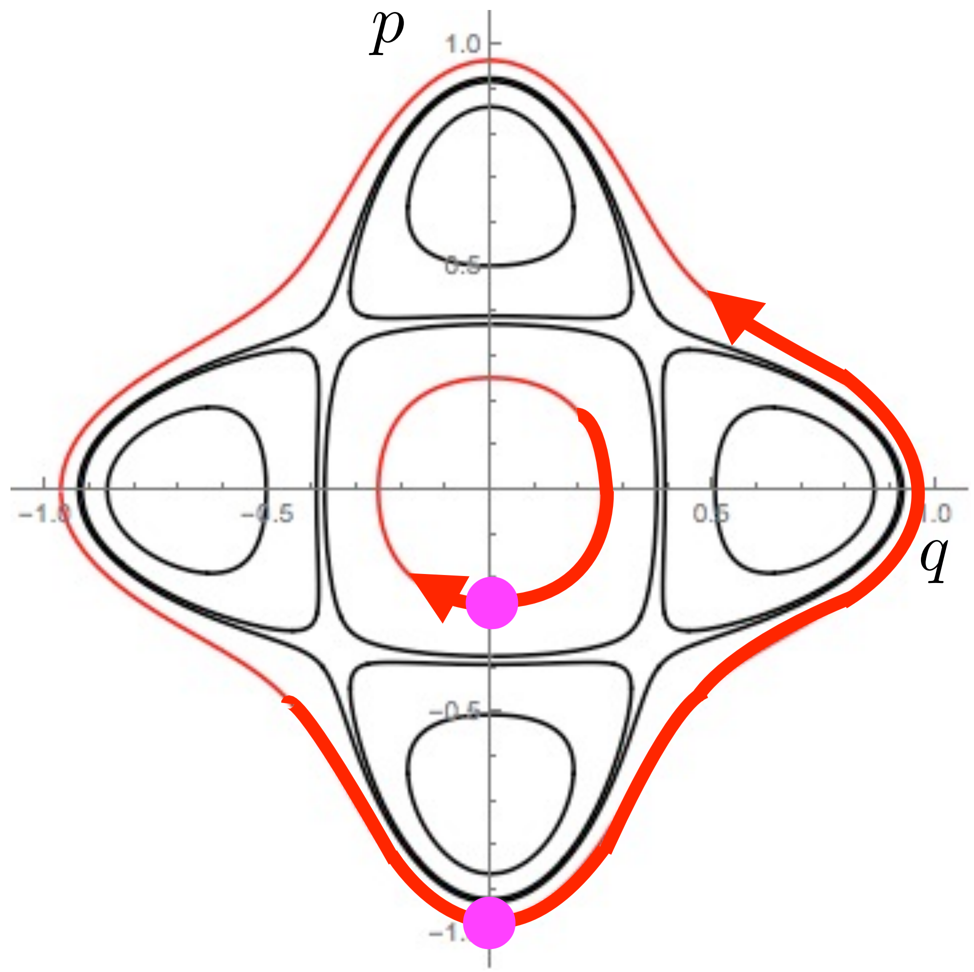
Singularity structure(Riemann sheet)



Time plane of $q(t)$

K and *K*' are the periods of sn function.

Topology of trajectory (single island chain case)



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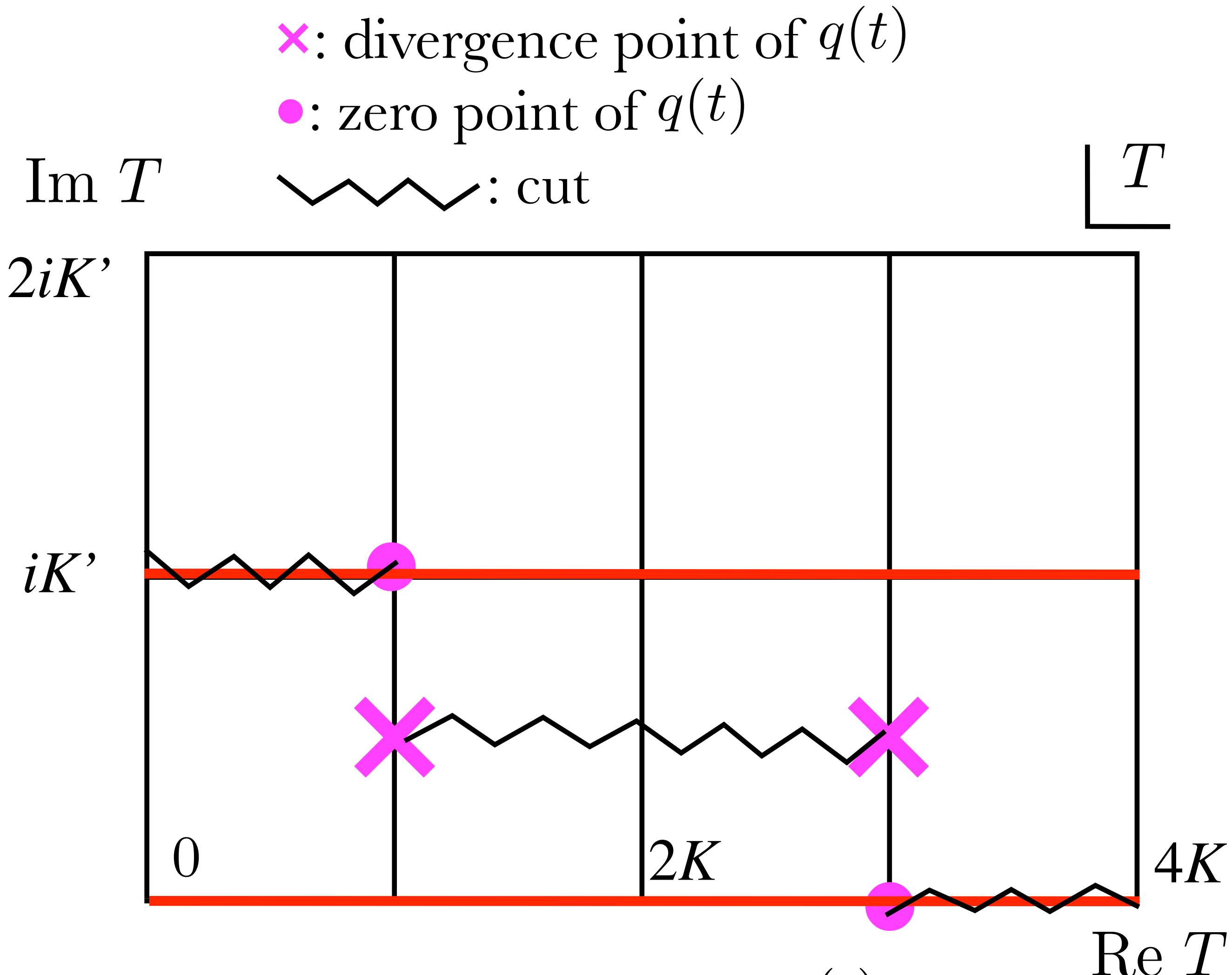

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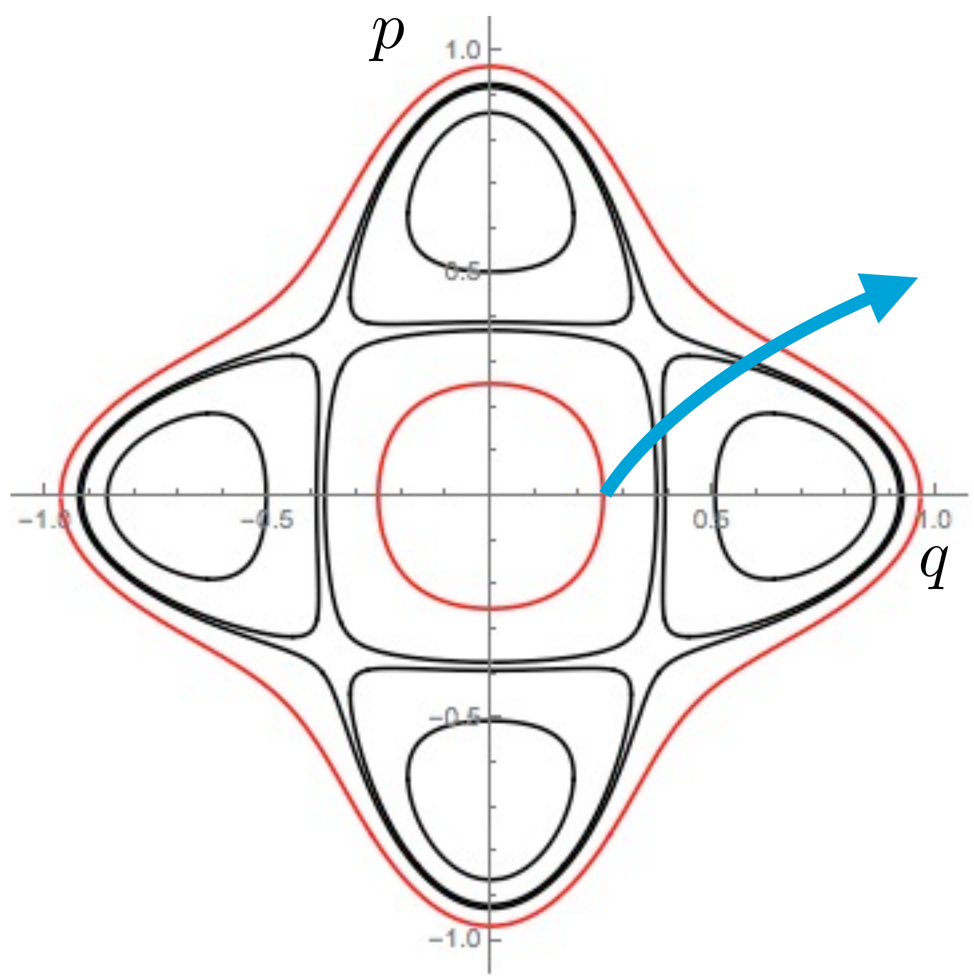

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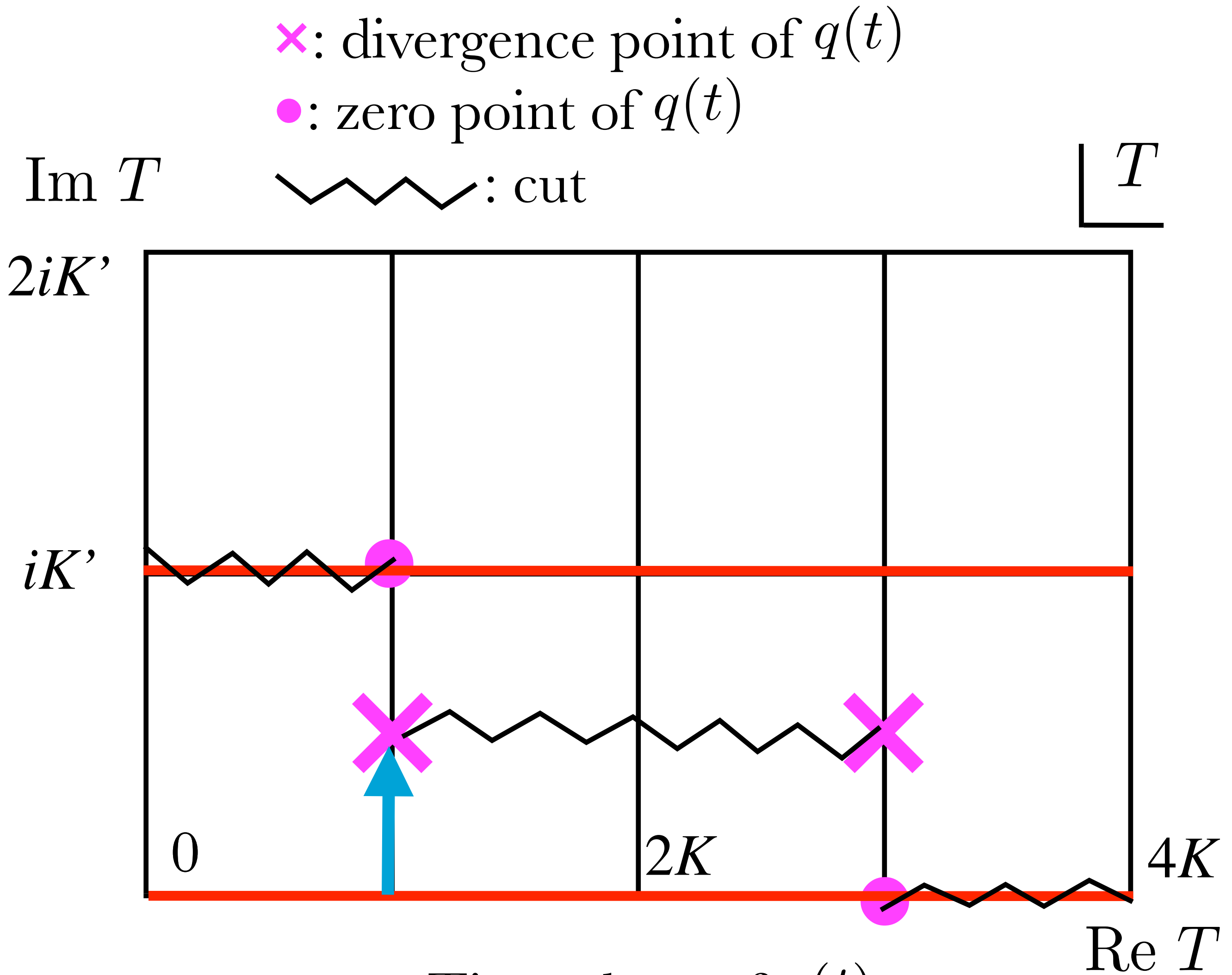

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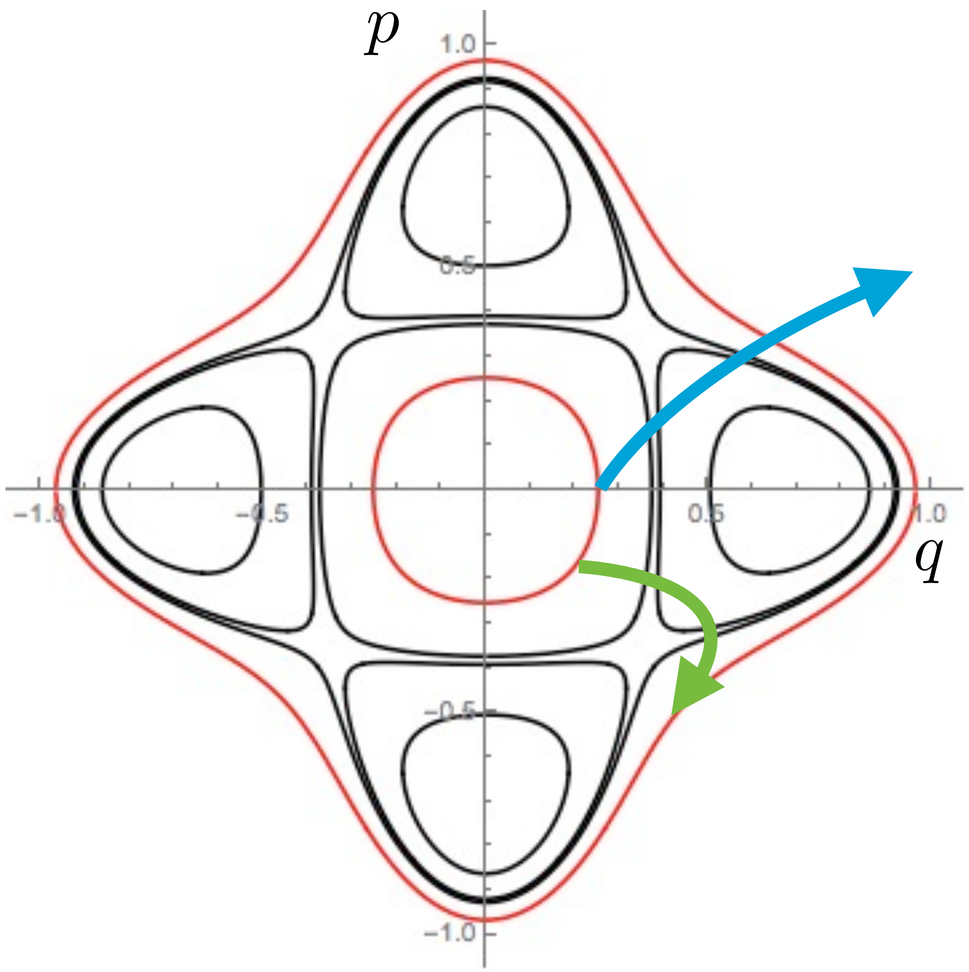

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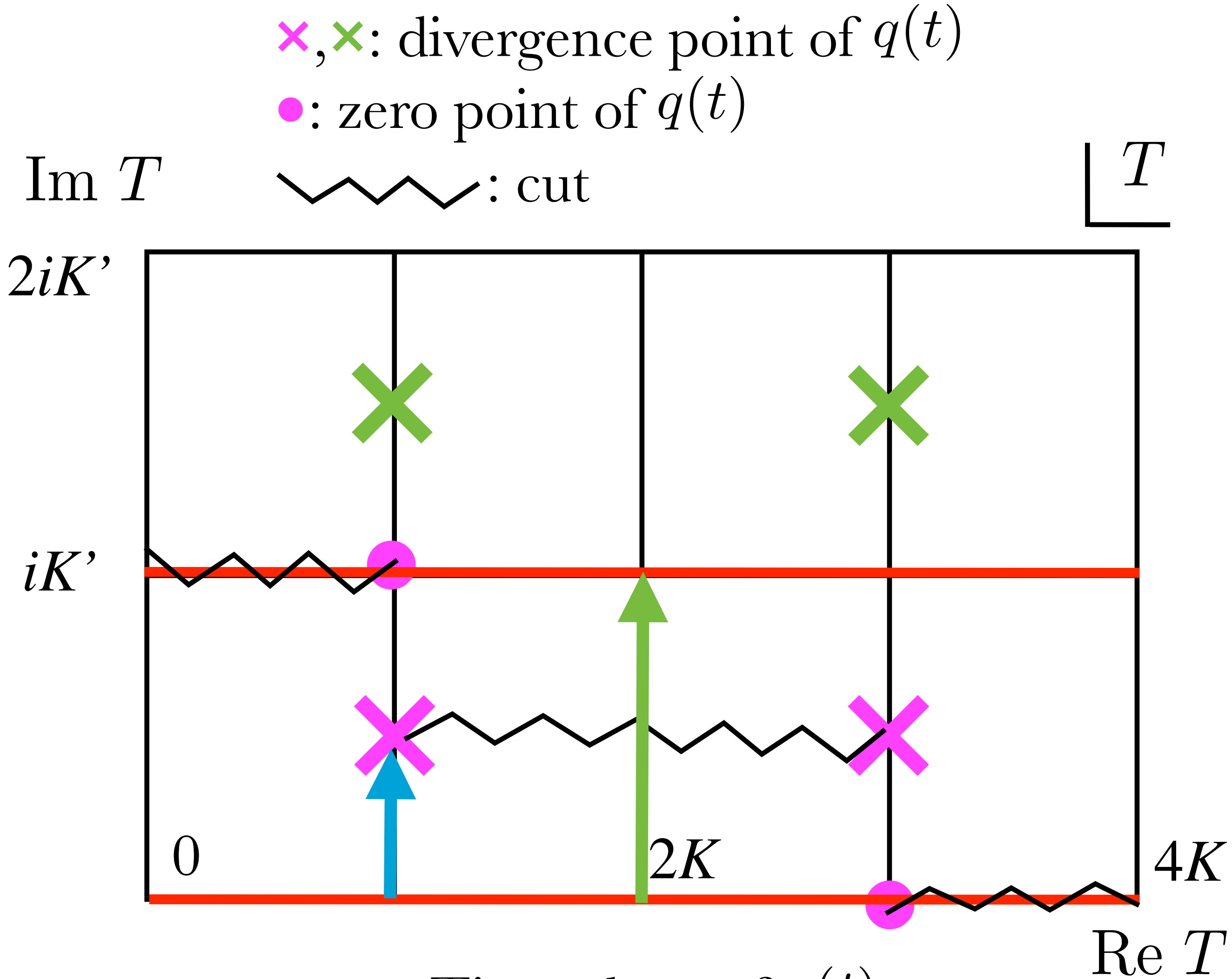

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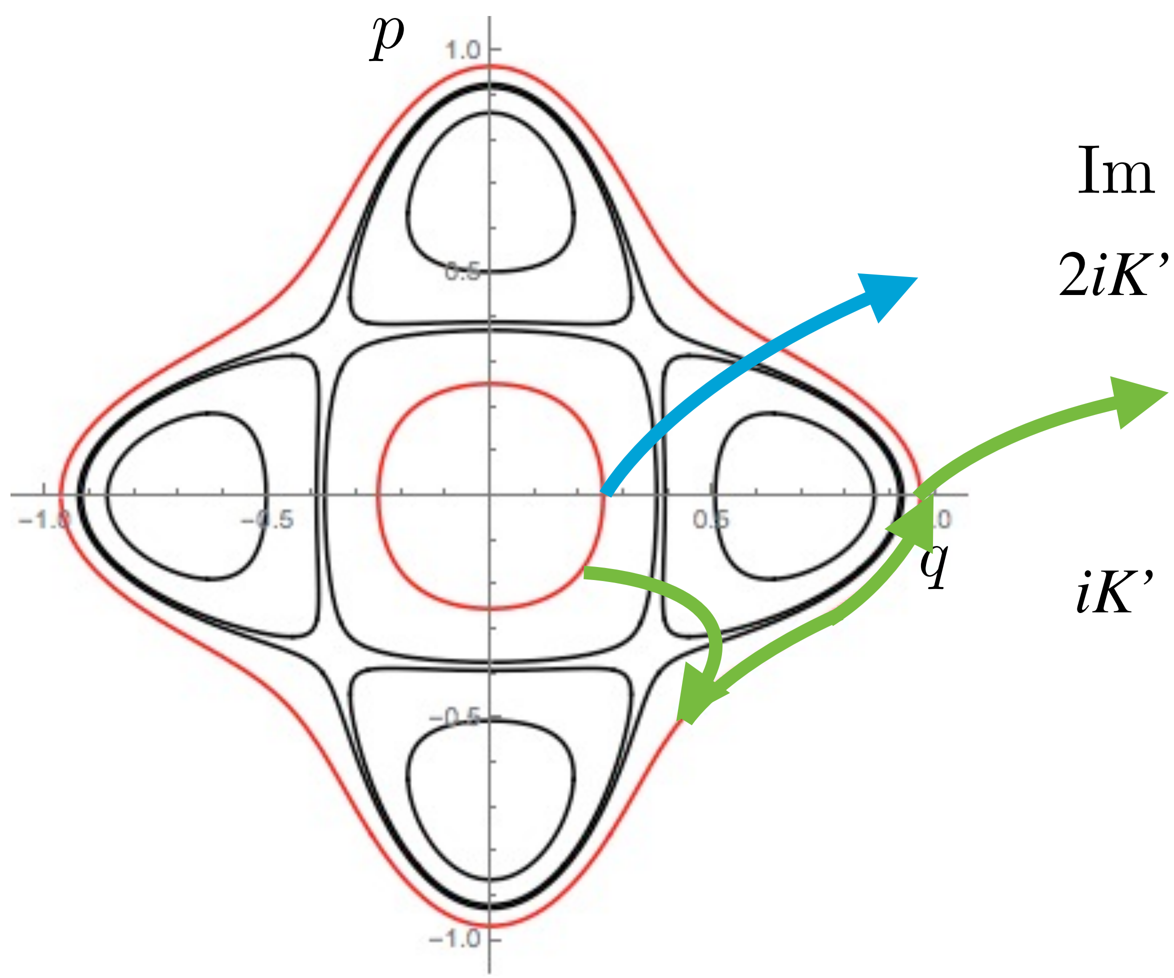

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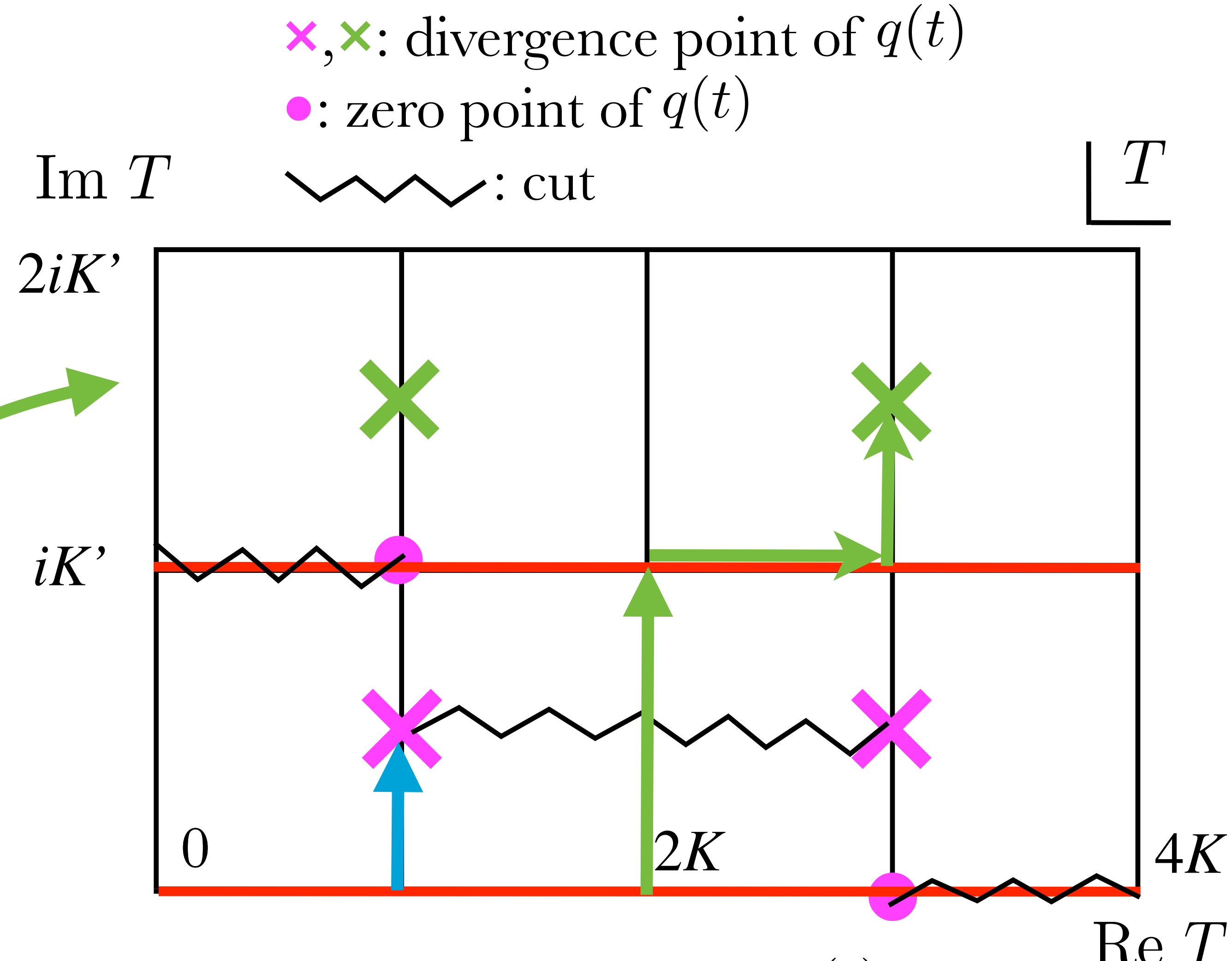
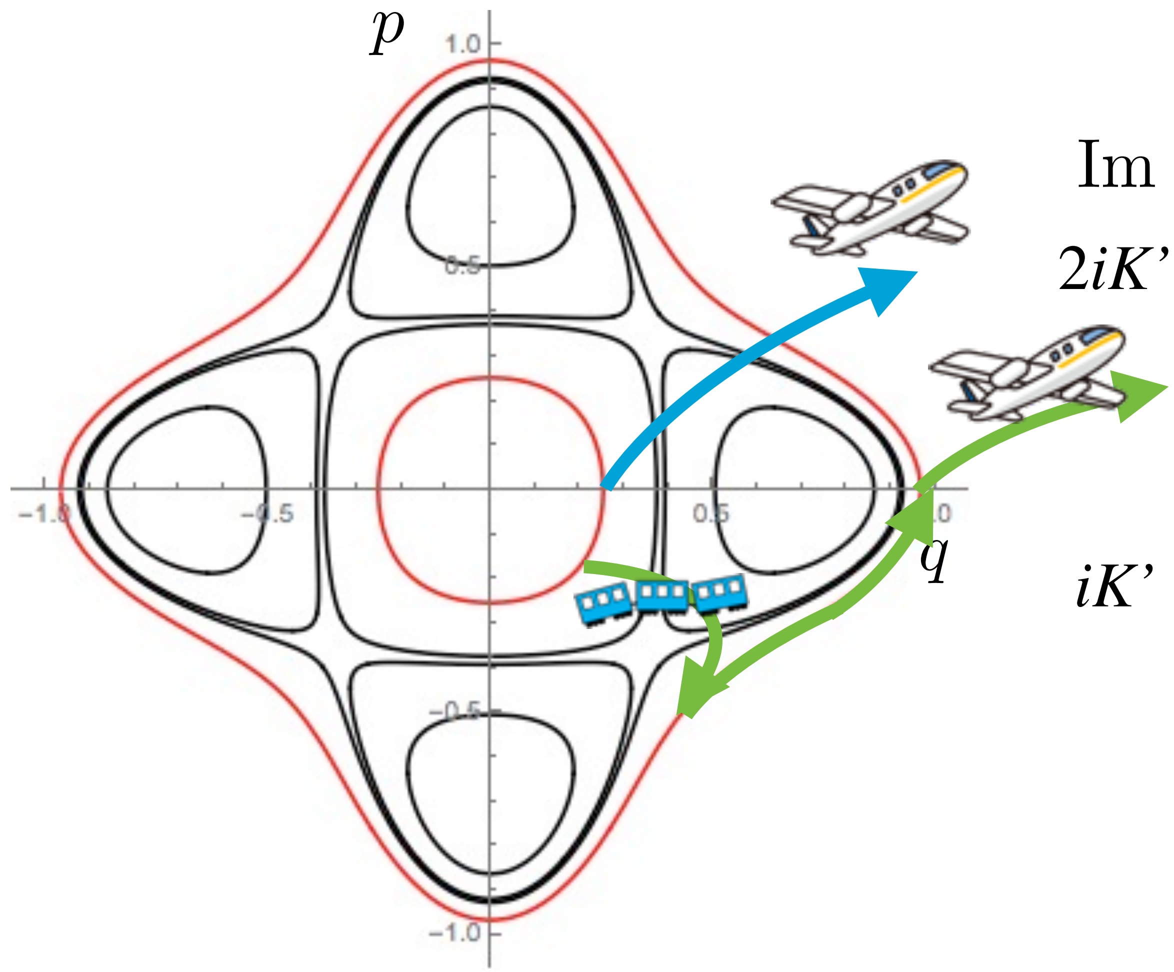


Time plane of $q(t)$

Imaginary actions for these topologically distinct paths are different.

Topology of trajectory (single island chain case)



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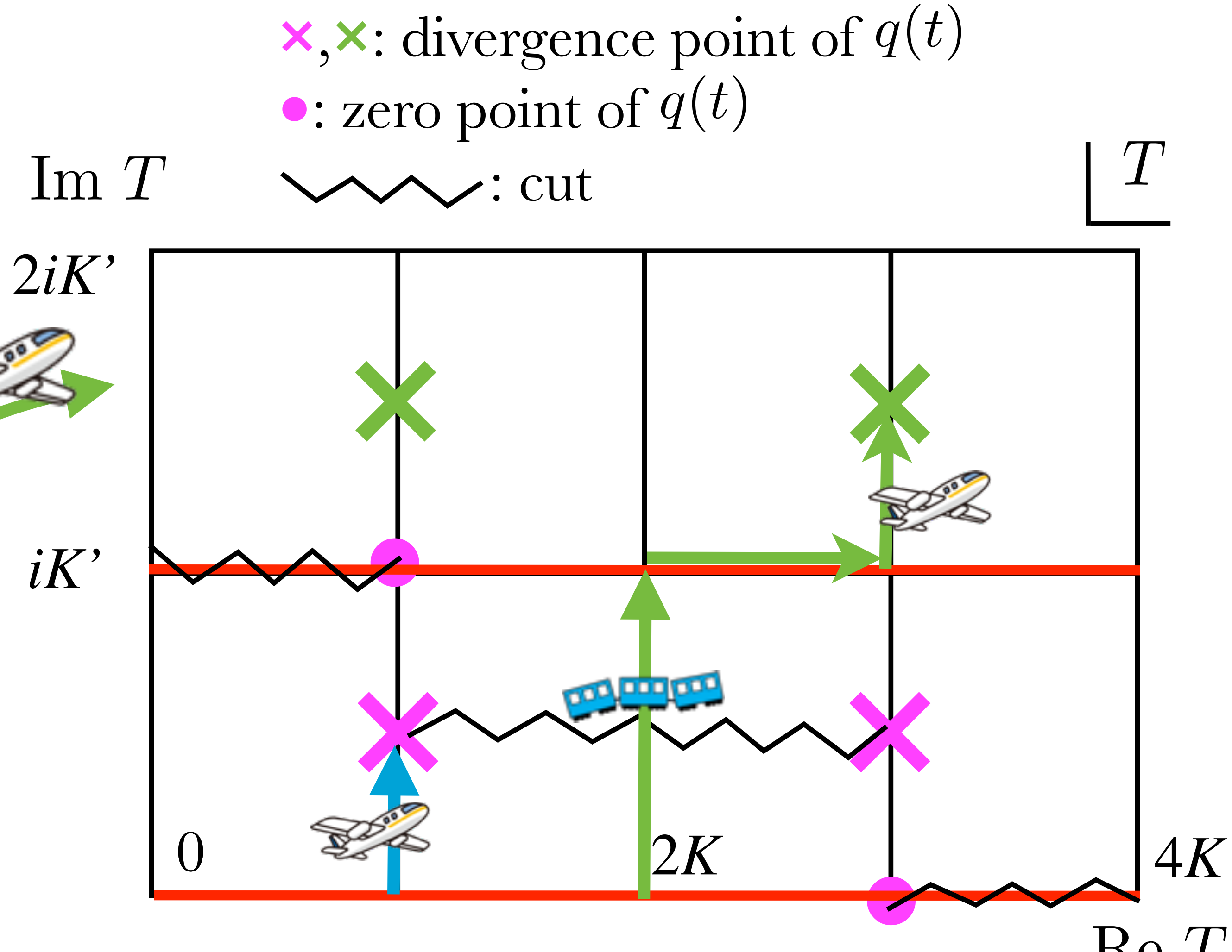


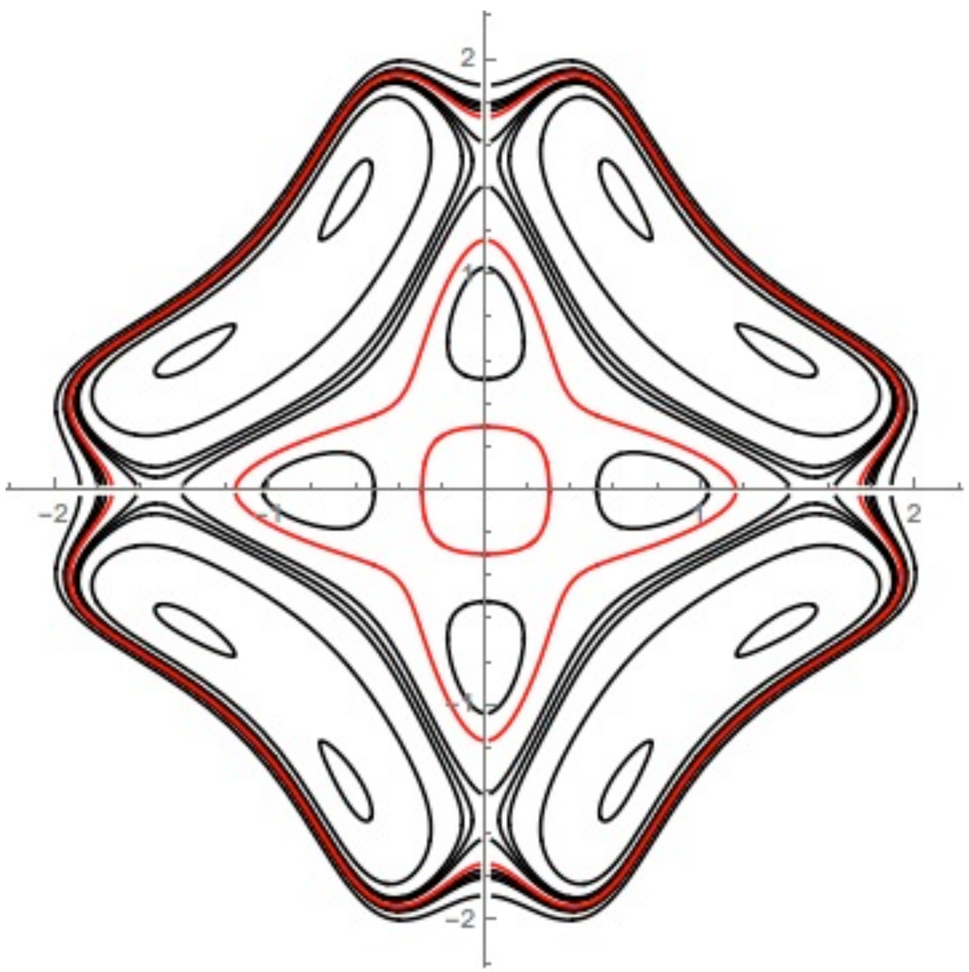
Time plane of $q(t)$

Which is cheaper?

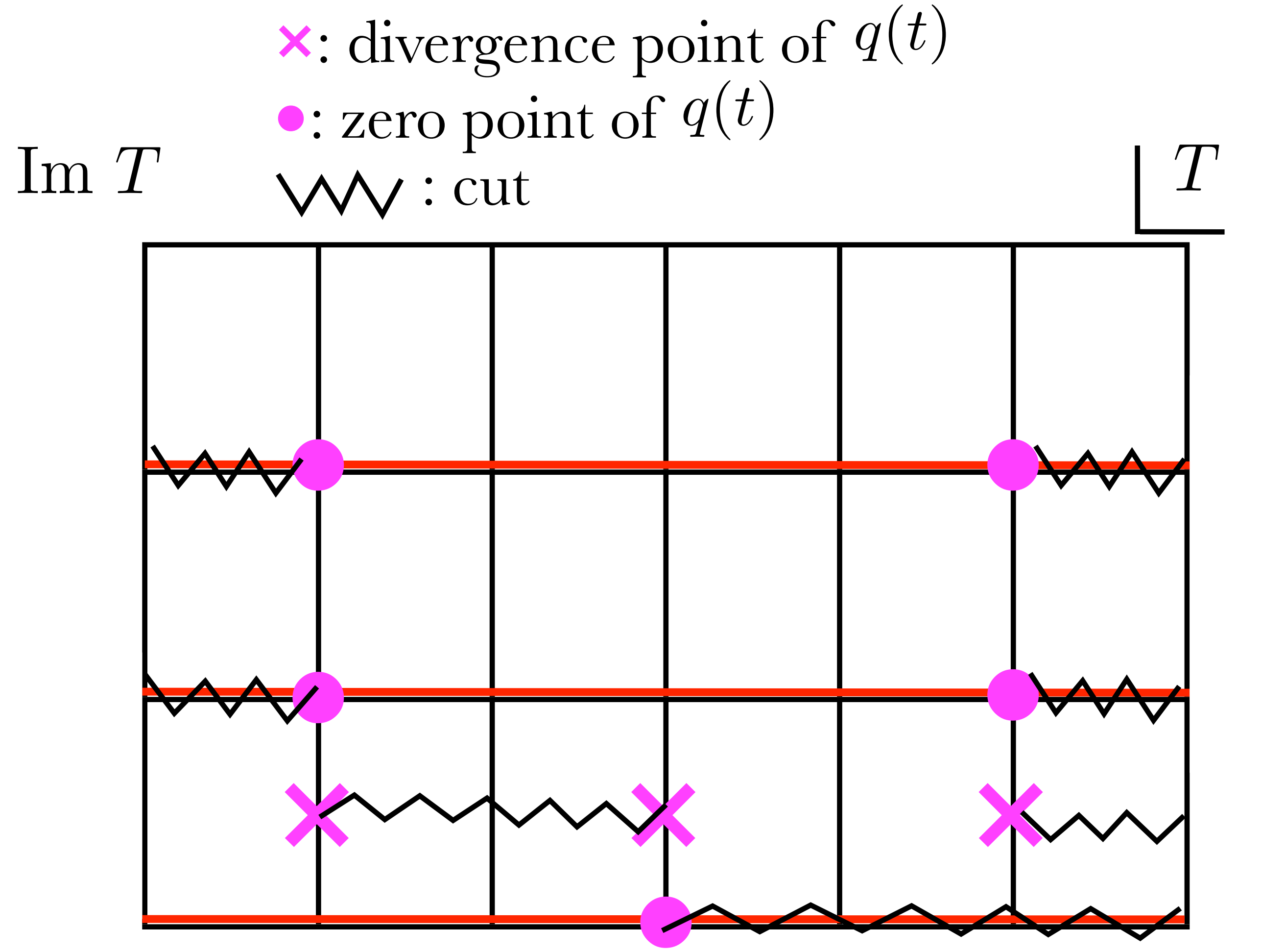
In this case, the green one is the cheaper.

Topology of trajectory(double island chain case)

$$H = \frac{1}{2}(q^2+p^2) + \frac{\epsilon}{4}(q^2+p^2)^2 + \frac{\sigma}{8}(q^2+p^2)^3 + \eta q^2 p^2 + \omega q^4 p^4$$



Phase space with $\epsilon = -2, \eta = -2.7, \sigma = 0.9, \omega = 1.8$.



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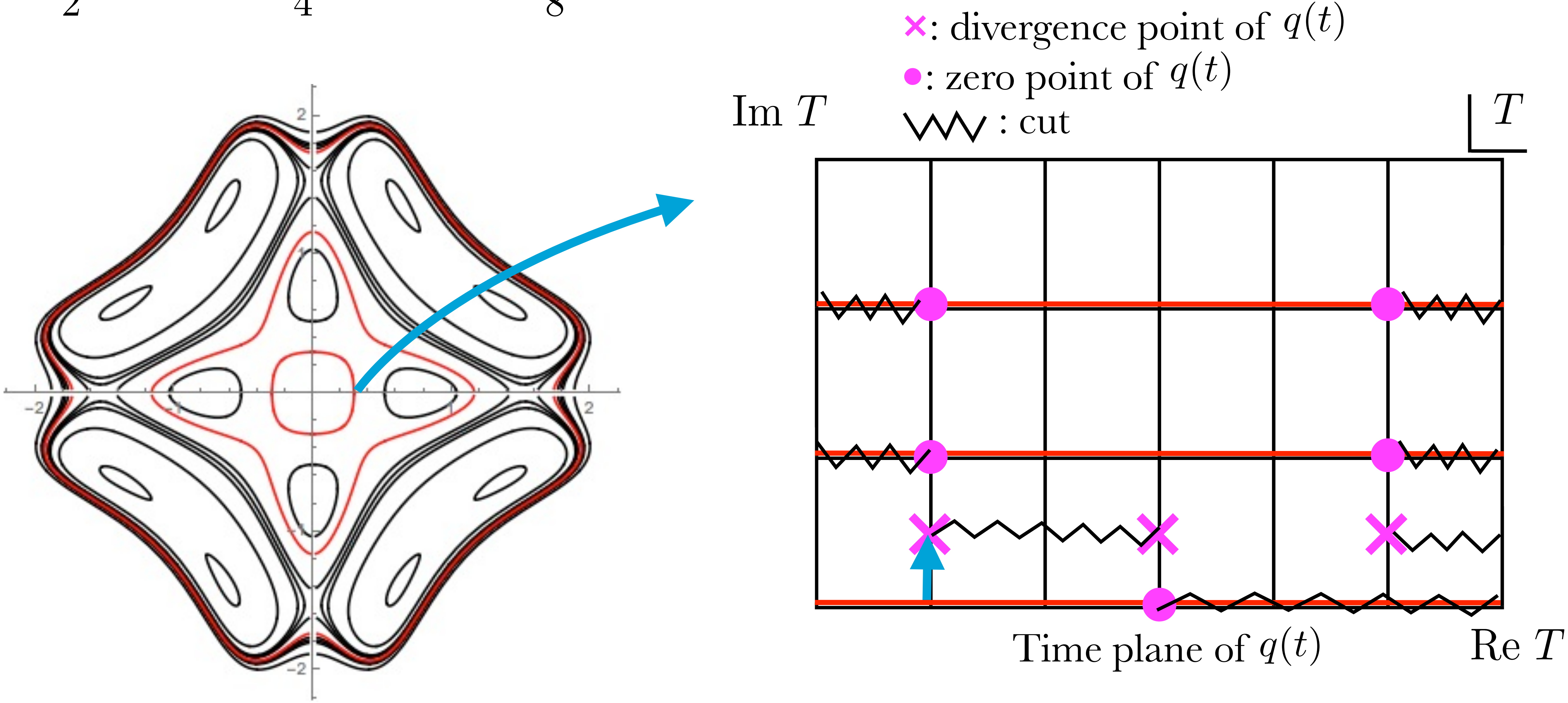

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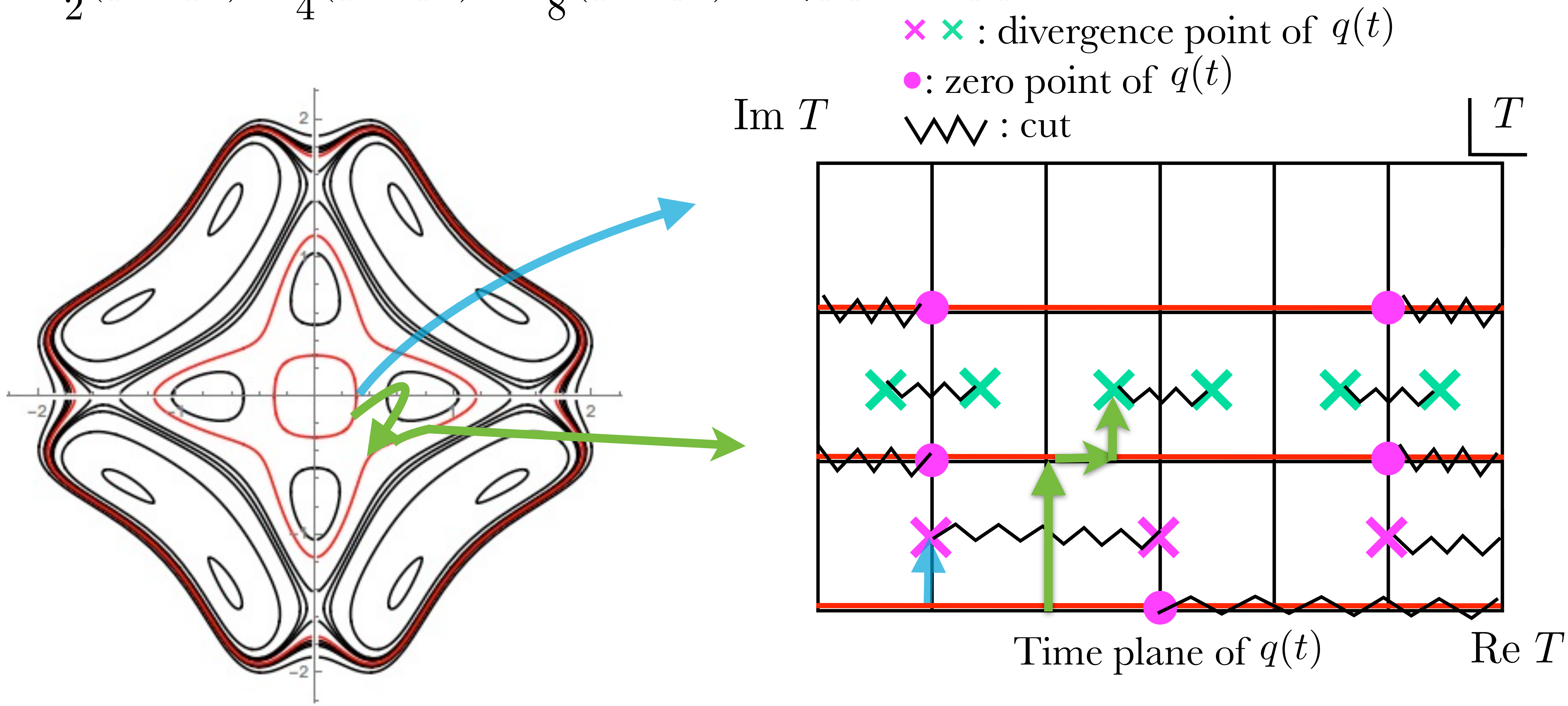

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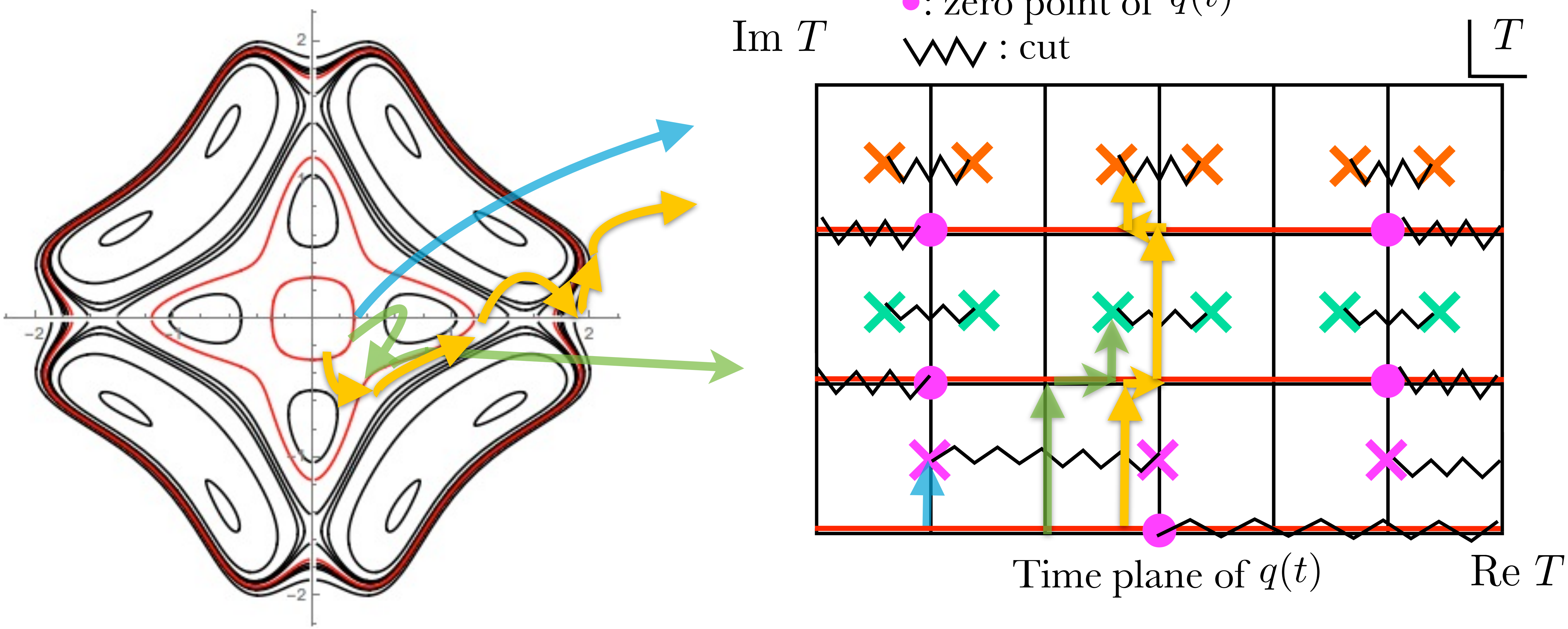
× × ×: divergence point of $q(t)$

●: zero point of $q(t)$

: cut

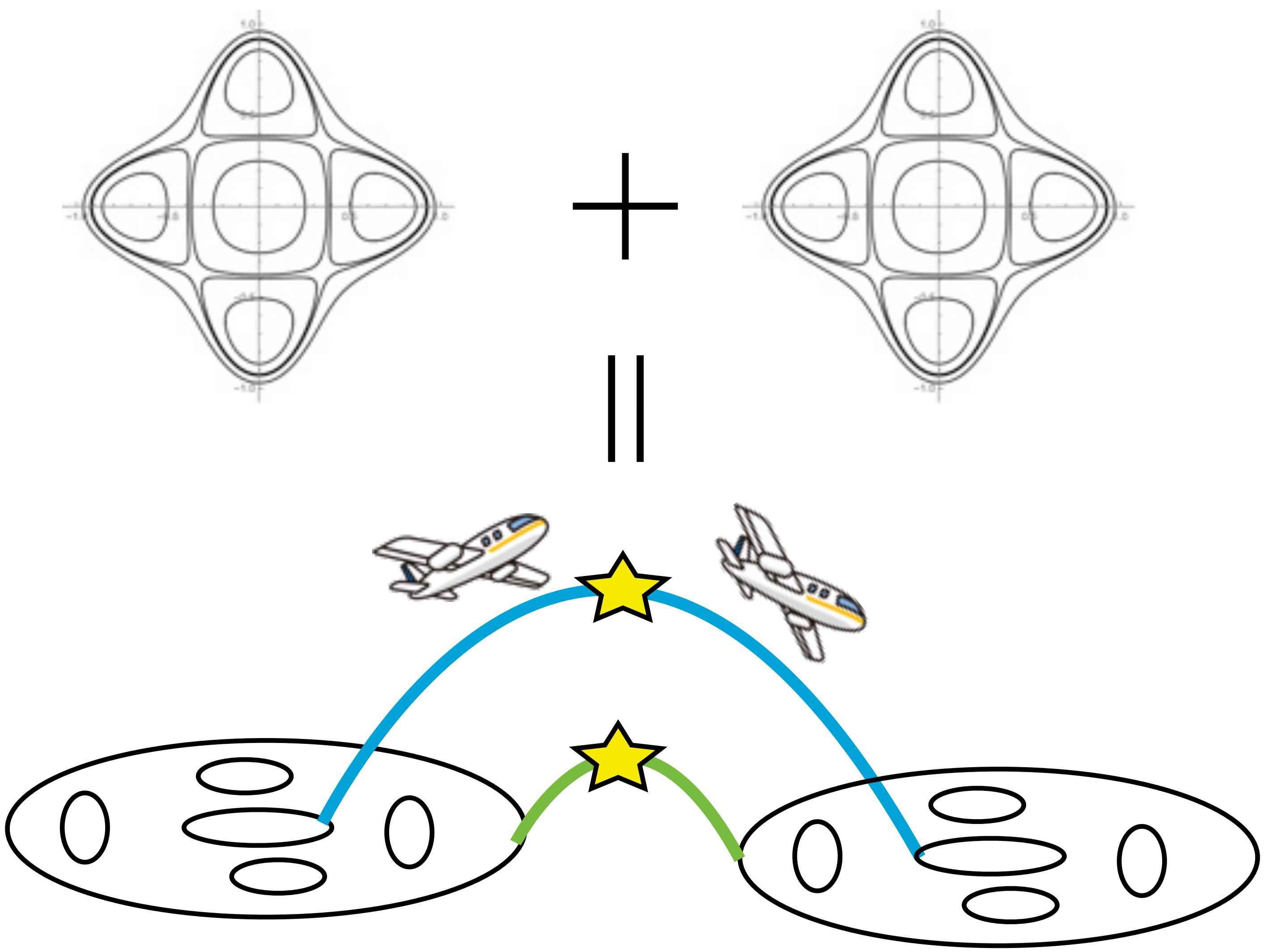


Phase space with $\epsilon = -2, \eta = -2.7, \sigma = 0.9, \omega = 1.8$.

3 possible imaginary actions.

Relation to the doublet case

If we glue two simple systems to form a doublet, the divergence points for a simple system may merge, and then a direct tunneling path must be created.



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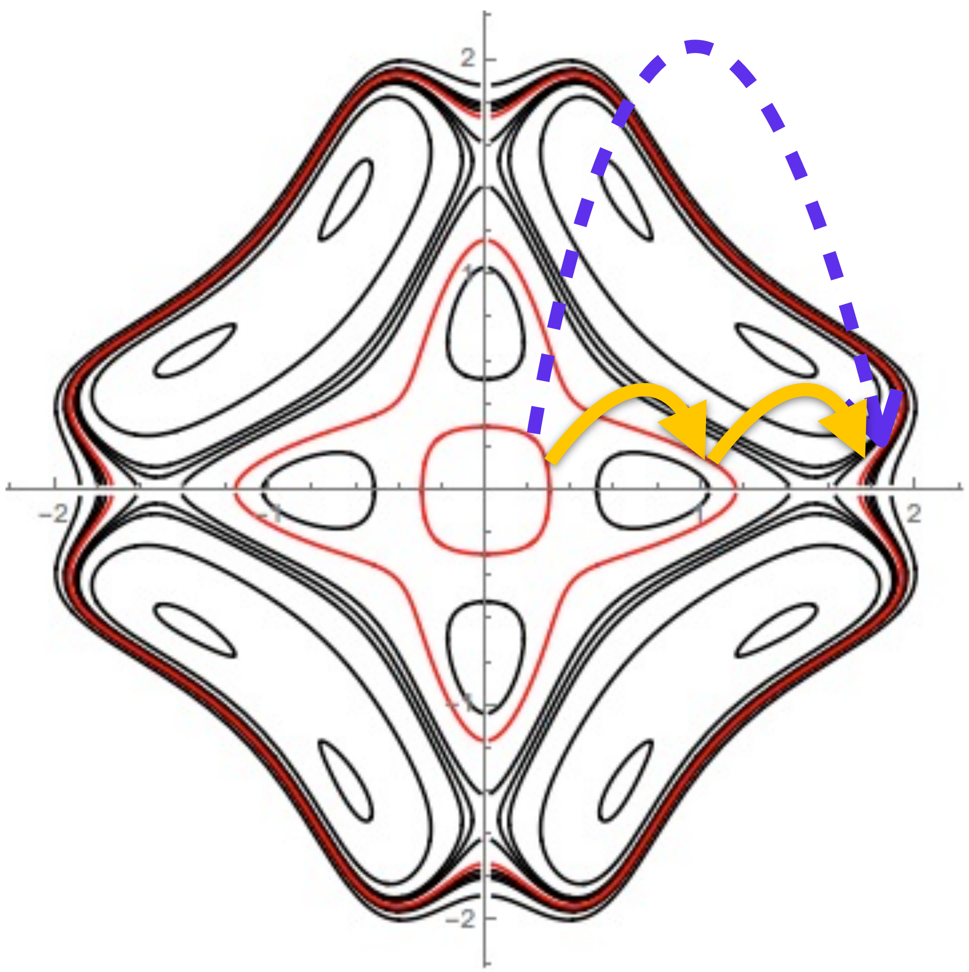
$$H = \frac{1}{2}(q^2+p^2) + \frac{\epsilon}{4}(q^2+p^2)^2 + \frac{\sigma}{8}(q^2+p^2)^3 + \eta q^2 p^2 + \omega q^4 p^4$$



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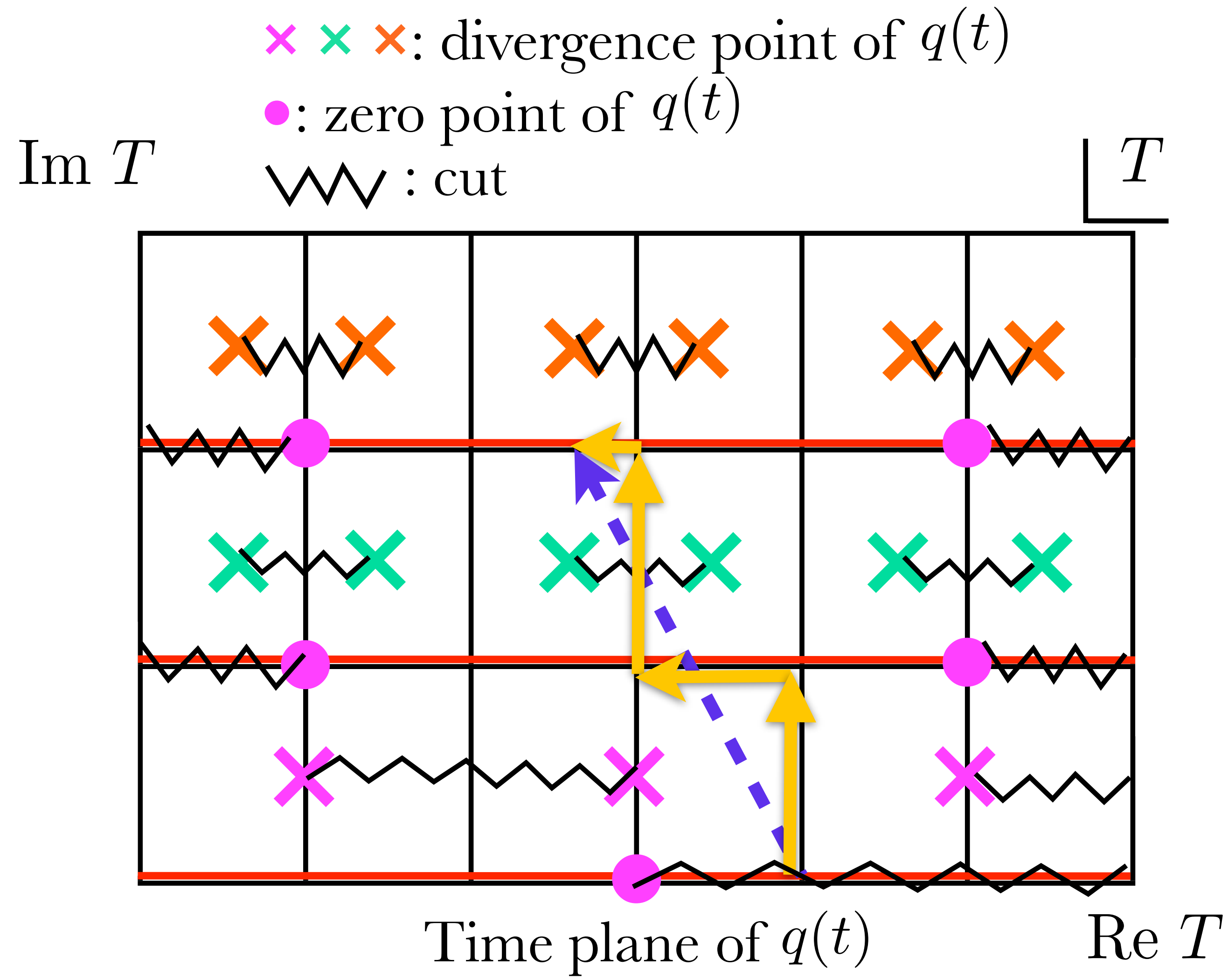

Looks different but the same topology, so the same imaginary action.

Conclusion

- We obtained the exact solution of a simple normal form Hamiltonian system, which allows us to examine the Riemann sheet structure and singularities in the complex plane analytically.

- We numerically studied complex singularity structures in more general cases, and explored how the paths with different imaginary actions appear.

Two origins of the paths with different imaginary actions:

1. paths with different topology

orbit on a torus can go to either to nearest neighboring tori or to infinity.
(take either "local train" or "plane", no "express", "Shinkansen" …)

2. resolution of degenerated paths due to symmetry breaking.