Landau-Stark states and the kicked-rotor problem

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Quantum particle in a 2D lattice

The Hamiltonian

$$\widehat{H} = \frac{(\hat{\mathbf{p}} - \mathbf{A})^2}{2M} + V(\mathbf{r}) + \mathbf{F} \cdot \mathbf{r} \,, \quad V(x+a, y) = V(x, y+a) = V(x, y)$$

- Bloch states ($A = 0$, $F = 0$)
- Landau-states ($A \neq 0$, $F = 0$)
- Wannier-Stark states ($A = 0$, $F \neq 0$)
- Landau-Stark states ($A \neq 0$, $F \neq 0$)

Applications of the Landau-Stark states:

- Cold atoms in parabolic lattices subject to a synthetic magnetic field.
- Alternative approach to Hall physics (integer quantum Hall effect, etc.)

Tight-binding approximation ($|\alpha| \ll 1/2$)

Hamiltonian for the Landau gauge $\mathbf{A} = B(0, x, 0)$

$$(\widehat{H}\psi)_{l,m} = -\frac{J_x}{2}\left(\psi_{l+1,m} + \psi_{l-1,m}\right)$$

$$-\frac{J_y}{2}\left(\psi_{l,m+1}e^{i2\pi\alpha l} + \psi_{l,m-1}e^{-i2\pi\alpha l}\right) + a(F_x l + F_y m)\psi_{l,m}$$

The spectrum:

- $E(\kappa_x, \kappa_y) = -J_x\cos(a\kappa_x) - J_y\cos(a\kappa_y)$ (Bloch states)
- $-0.5J_x(b_{l+1} + b_{l-1}) - J_y\cos(2\pi\alpha l + a\kappa)b_l = E_n(\kappa)b_l$ (Landau states)
- $E_{n,k} = aF_x n + aF_y k$ (Wannier-Stark states, $F_x, F_y \neq 0$)
- What is the spectrum of the Landau-Stark states? The answer: The spectrum is either continuous or discrete depending on the rationality of the parameter $\beta = F_x/F_y$.

The proof

- Consider Schrödinger equation $i\dot{\psi}_{l,m} = (\widehat{H}\psi)_{l,m}$
- Use the ansatz $\psi_{l,m}(t) \sim e^{i(\kappa - F_y t)m} b_l(t)$
- We have

$$i\dot{b}_l = -\frac{J_x}{2}(b_{l+1} + b_{l-1}) - J_y \cos(2\pi\alpha l + \kappa - F_y t) b_l + F_x l b_l \equiv \left(\widehat{H}_{1D}(t)\mathbf{b}\right)_l$$

- Construct evolution operator $\widehat{U}^{(\kappa)} = \widehat{\exp}\left[-i\int_0^{2\pi/F_y} \widehat{H}_{1D}(t)\mathrm{d}t\right]$
- Find eigenvectors $\widehat{U}^{(\kappa)}\mathbf{b} = \lambda\mathbf{b}$
- Construct Landau-Stark states $\Psi_{l,m} = \frac{1}{2\pi}\int_0^{2\pi} b_l^{(n)}(\kappa) e^{-i\kappa(m-k)}\mathrm{d}\kappa$

If $J_x = 0$ the evolution operator is a diagonal matrix $U_{l,l}^{(\kappa)} = \exp(-i2\pi\beta l)$ where $\beta = F_x/F_y$.

Analogy with the kicked rotor problem: for vanishing kick strength $U_{l,l} = \exp(-i2\pi\eta l^2)$.

Localized Landau-Stark states (irrational β)

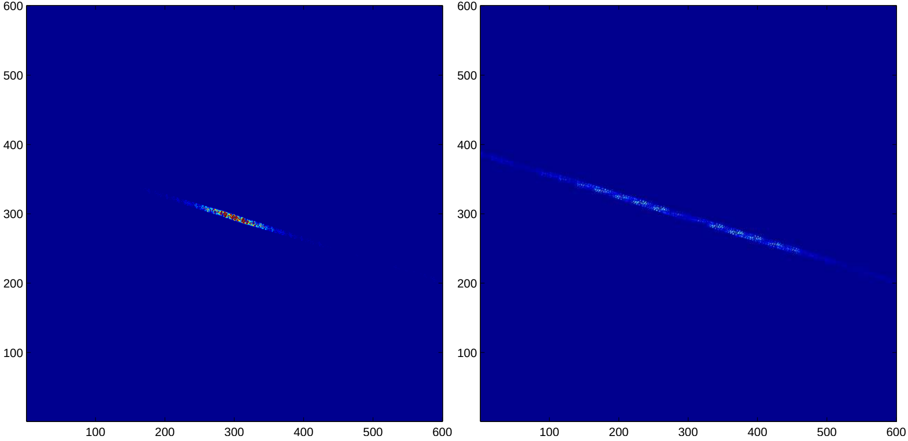


Figure: Examples of the localized Landau-Stark states for $F = 0.5$ (left) and $F = 0.4$ (right). The other parameters are $\beta = (\sqrt{5} - 1)/4 \approx 1/3$ and $\alpha \approx 1/6$. The occupation probabilities $|\psi_{l,m}|^2$ are shown as a color map.

Spectrum of the localized Landau-Stark states

If $\Psi_{l,m}$ is an eigenstate with the energy E, then the state

$$\tilde{\Psi}_{l,m} = \Psi_{l-n,m-k} e^{-i2\pi\alpha nm}$$

is also eigenstate with the energy

$$\tilde{E} = E + a(F_x n + F_y k)$$

Localization lengths

- $\xi_{\|} \sim J/aF$
- $\xi_{\perp} = \infty$ if $\beta = F_x/F_y$ is a rational number
- $\xi_{\perp} < \infty$ if β is an irrational number
- Scaling law for $\xi_{\perp}$?

Semiclassical approach (driven Harper model)

The classical Hamiltonian (quantization rule $[\hat{x}, \hat{p}] = i2\pi\alpha$)

$$H_{1D}(t) = -J'_x \cos(p - F_x t) - J'_y \cos(x + F_y t) \ , \quad J'_{x,y} = 2\pi\alpha J_{x,y}$$

This system can be studied:

- on the torus $(-\pi \le p, x < \pi)$
- on the cylinder $(-\pi \le p < \pi, \ -\infty < x < \infty)$
- in the plane $(-\infty < p, x < \infty)$

In the plane: using the canonical substitution $p' = p - F_x t$ and $x' = x + F_y t$ the Hamiltonian takes time-independent form:

$$H'_{1D} = -J'_x \cos(p') - J'_y \cos(x') + F_x x' + F_y p'$$

Phase portrait: transporting islands

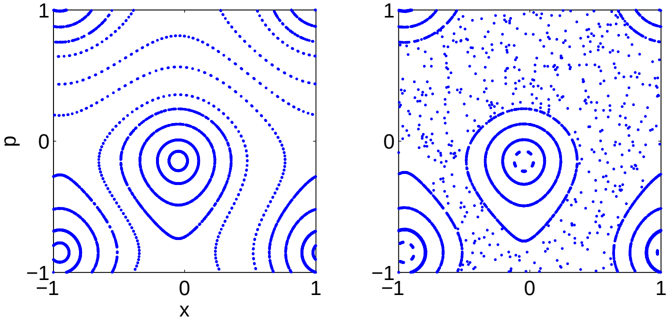
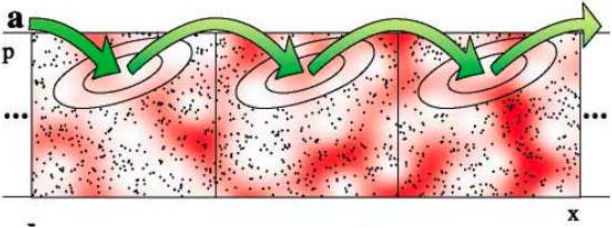


Figure: Stroboscopic map of the classical driven Harper for rational $\beta = 1/3$, left panel, and irrational $\beta = (\sqrt{5} - 1)/4 \approx 1/3$, right panel. Transporting islands disappear if $F > F_{cr}$.

Kicked Harper



[1] L. Hufnagel, R. Ketzmerick , M.-F. Otto, and H. Schanz, *Eigenstates ignoring regular and chaotic phase-space structures*, Phys. Rev. Lett. **89**, 154101 (2002); [2] A. Bäcker, R. Ketzmerick, and A. G. Monastra, *Flooding of chaotic eigenstates into regular phase space islands*, Phys. Rev. Lett. **94**, 054102 (2005)

Adopting reported results to the currently considered problem we have:

$$\xi_\perp \sim \frac{F}{\alpha} \exp\left(C \frac{S(F)}{\alpha} \right)$$

Spectrum of the extended Landau-Stark states (rational β)

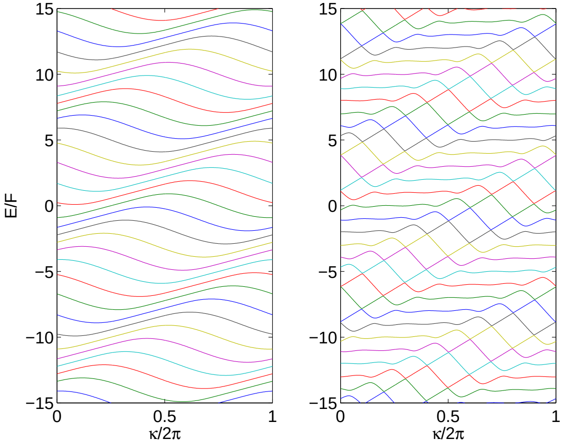


Figure: Examples of the band energy spectrum of extended Landau-Stark states. Parameters are $\alpha = 1/10$, $\beta = 0$, and $F = 1$ (left) and $F = 0.3$ (right).

Localization length: numerical results

Inverse participation ratio: $P = \left(\sum_{l,m} |\Psi_{l,m}|^4\right)^{-1}$

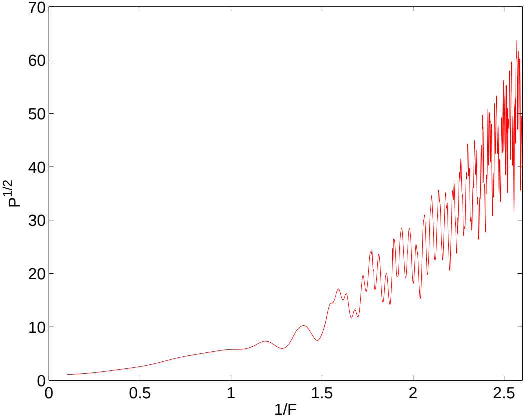


Figure: Localization length $\xi_\perp$ as the function of $1/F$. The other parameters are $\beta = (\sqrt{5}-1)/4 \approx 1/3$, and $\alpha \approx 1/6$.

Conclusions

- The spectrum of the Landau-Stark states crucially depends on the rationality condition for the parameter $\beta = F_x/F_y$.
- If β is a rational number the spectrum is continuous (band structured) and Landau-Stark states are extended Bloch-like waves in the direction orthogonal to **F**.
- If β is an irrational number the spectrum is discrete, $E = a(F_x n + F_y k)$, and Landau-Stark states are truly localized states, i.e., the localization lengths $\xi_{\|}, \xi_{\perp} < \infty$.
- Nontrivial scaling law for the localization length: $\xi_{\perp}$ blows up exponentially when F decreases below critical $F_{cr} \sim \alpha$.
- In practice this is seen as a localization-delocalization transition.

Cold atoms in parabolic lattices

The system

$$(\widehat{H}\psi)_{l,m} = -\frac{J}{2}\left(\psi_{l+1,m}e^{-i\pi\alpha m} + \psi_{l-1,m}e^{i\pi\alpha m}\right)$$
$$-\frac{J}{2}\left(\psi_{l,m+1}e^{i\pi\alpha l} + \psi_{l,m-1}e^{-i\pi\alpha l}\right) + \frac{\gamma}{2}(l^2 + m^2)\psi_{l,m}$$

where $\gamma \sim \omega_{trap}^2$ is the harmonic confinement.

Locally, at (l, m) near (l_0, m_0), we have

$$\frac{\gamma}{2}(l^2 + m^2)\psi_{l,m} \approx F_x l + F_y m$$

where $(F_x, F_y) = -\gamma(l_0, m_0)$ is the gradient force pointing the lattice origin.

Semiclassical approach ($\alpha \ll 1$)

Classical counterpart of the tight-binding Hamiltonian

$$H_{cl} = -J_x \cos(p_x - \pi\alpha y) - J_y \cos(p_y + \pi\alpha x) + \frac{\gamma}{2}(x^2 + y^2)$$

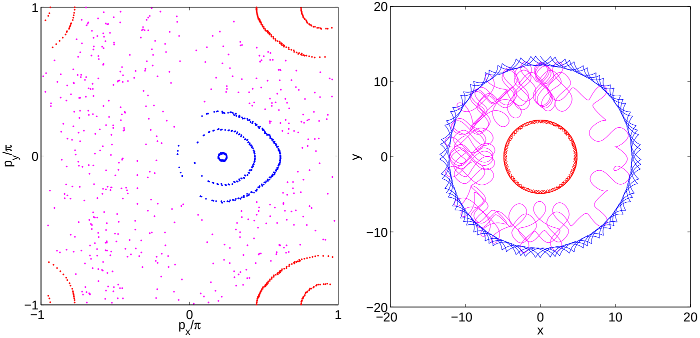
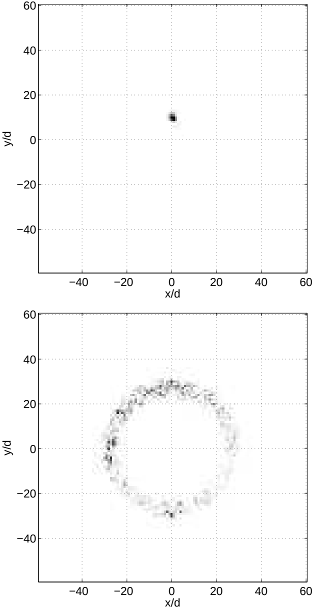


Figure: Poincare cross-section of the energy shell $E = 2.5$ (left panel) and examples of classical trajectories in the coordinate space (right panel). Encircling frequency is $\Omega = \gamma/2\pi\alpha$.

Wave packet dynamics



Acknowledgments

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Forthcoming conference