Recent advances on tunnelling in complex systems

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Fédération Denis POISSON

Pistes (more or less complex trails with fluctuations in altitudes)

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- The double-well
- Chaotic tunnelling

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- Perspective: things to do and open questions

Definition

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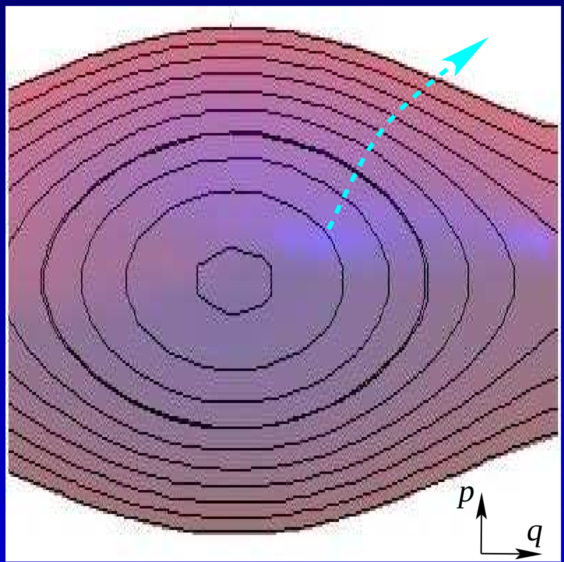
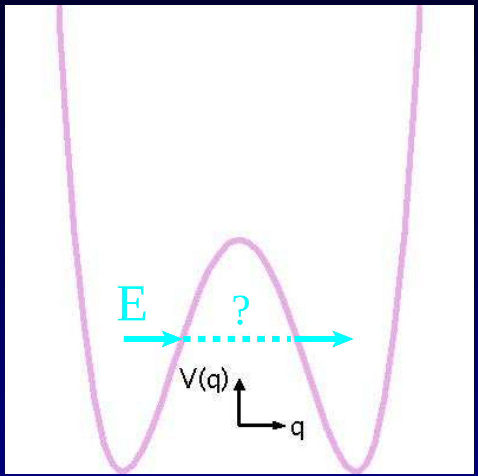


typically a crossing of (KAM) tori

The double-well



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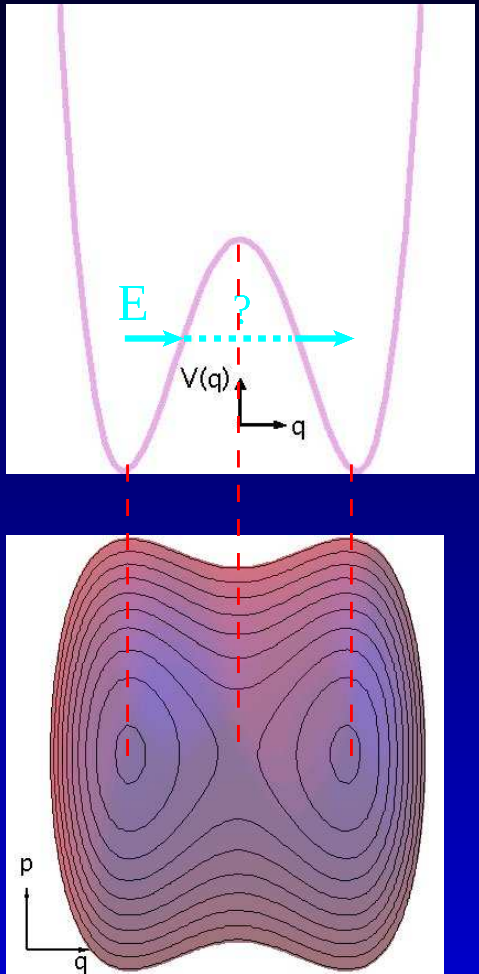

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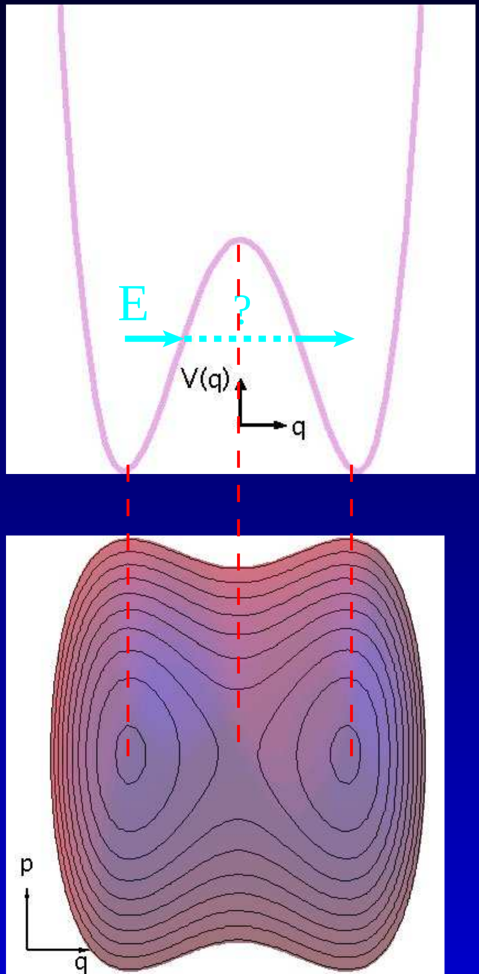

$$\Delta E = \frac{2\pi\hbar}{T} \underset{\hbar\to 0}{\sim} \alpha\hbar\, \mathrm{e}^{\mathrm{i}A/\hbar}$$

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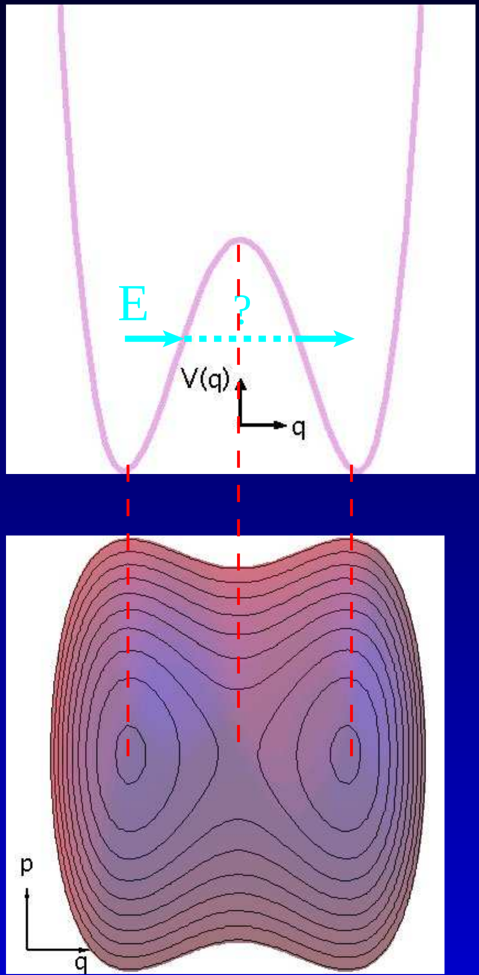

$$\Delta E = \frac{2\pi\hbar}{T} \underset{\hbar\to 0}{\sim} \alpha\hbar\, \mathrm{e}^{\mathrm{i}A/\hbar}$$

$$A = \mathrm{i} \int_{\text{below the barrier}} \sqrt{2m\big(V(q) - E\big)}\, \mathrm{d}q$$

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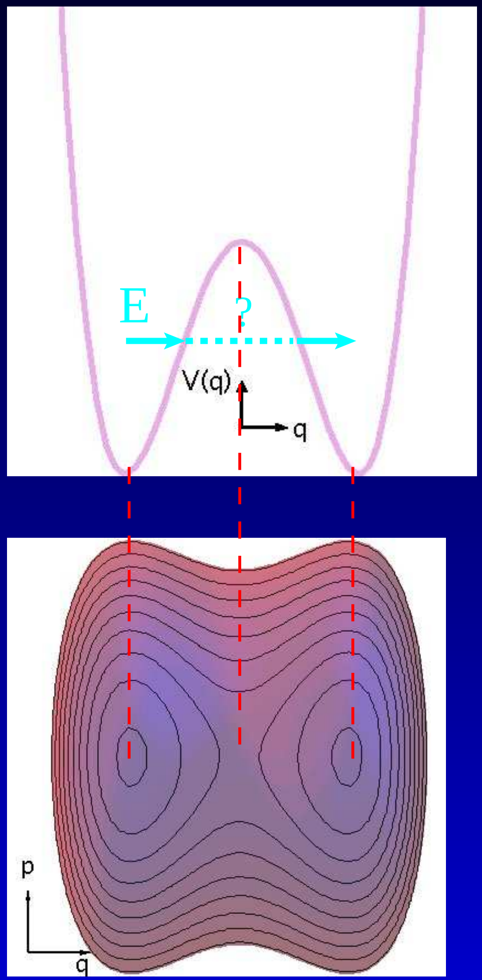

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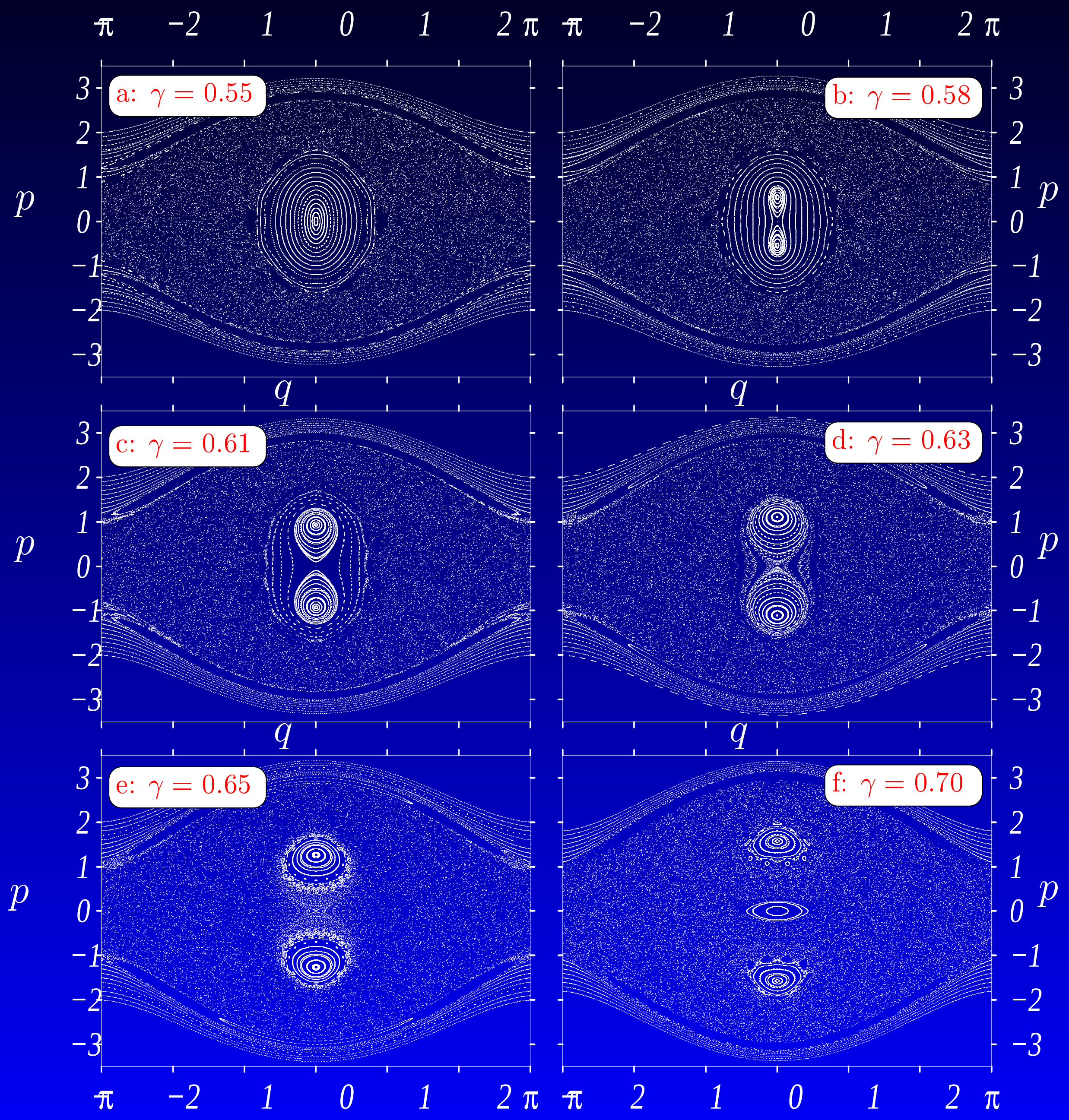
complex momentum p

Chaotic tunnelling

POINCARÉ surface of section for cold atoms ($\theta \simeq 1.7$):

$$h(p,q;t) = \frac{1}{2}\,p^2 - \gamma(\theta + \cos t)\cos q$$

Chaotic tunnelling



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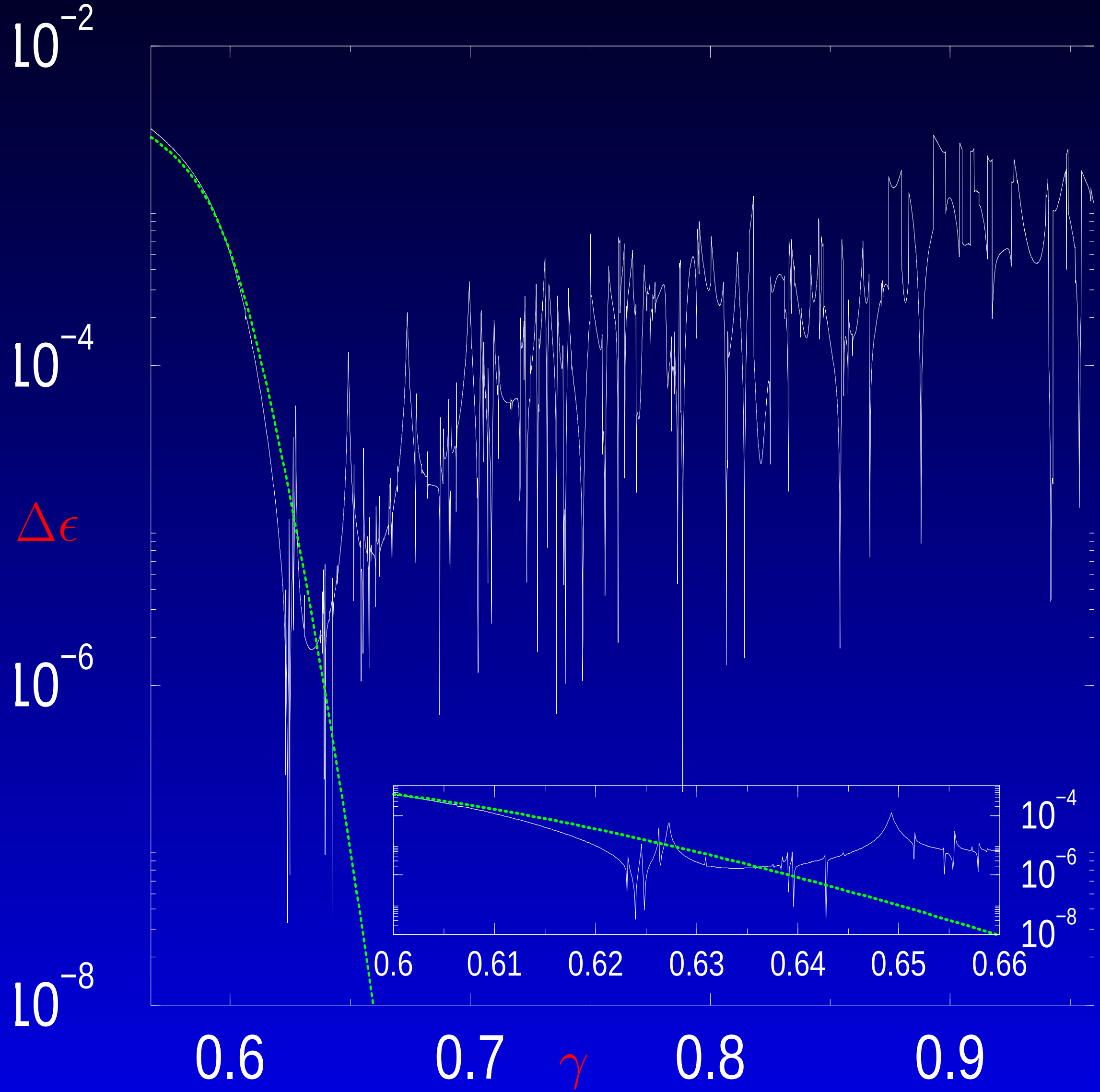
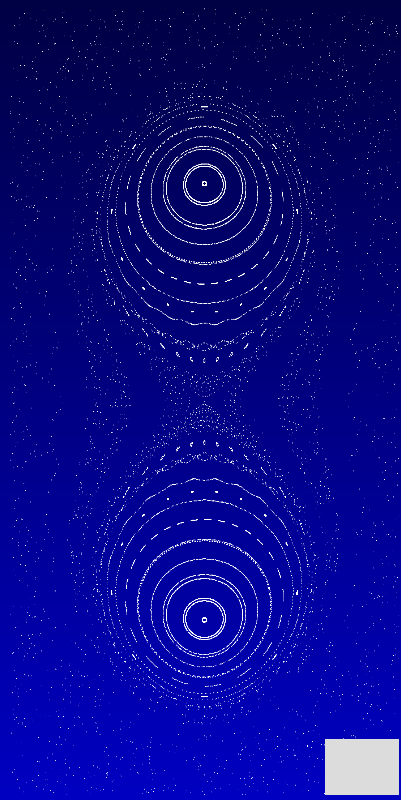


γ=.63 *1/h=13*



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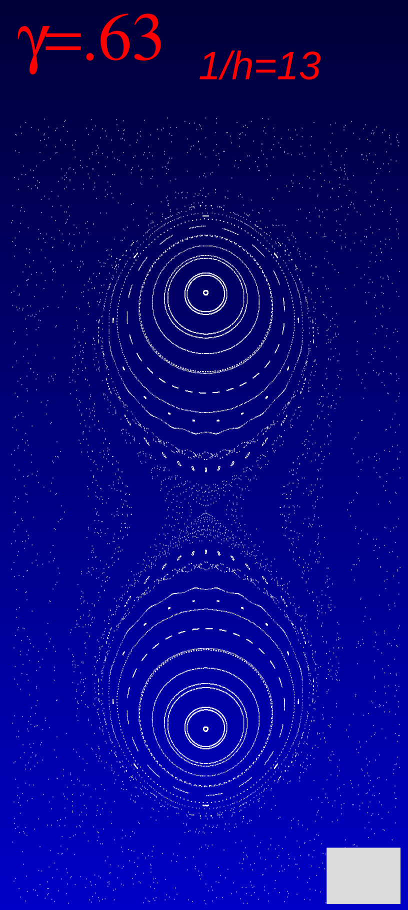
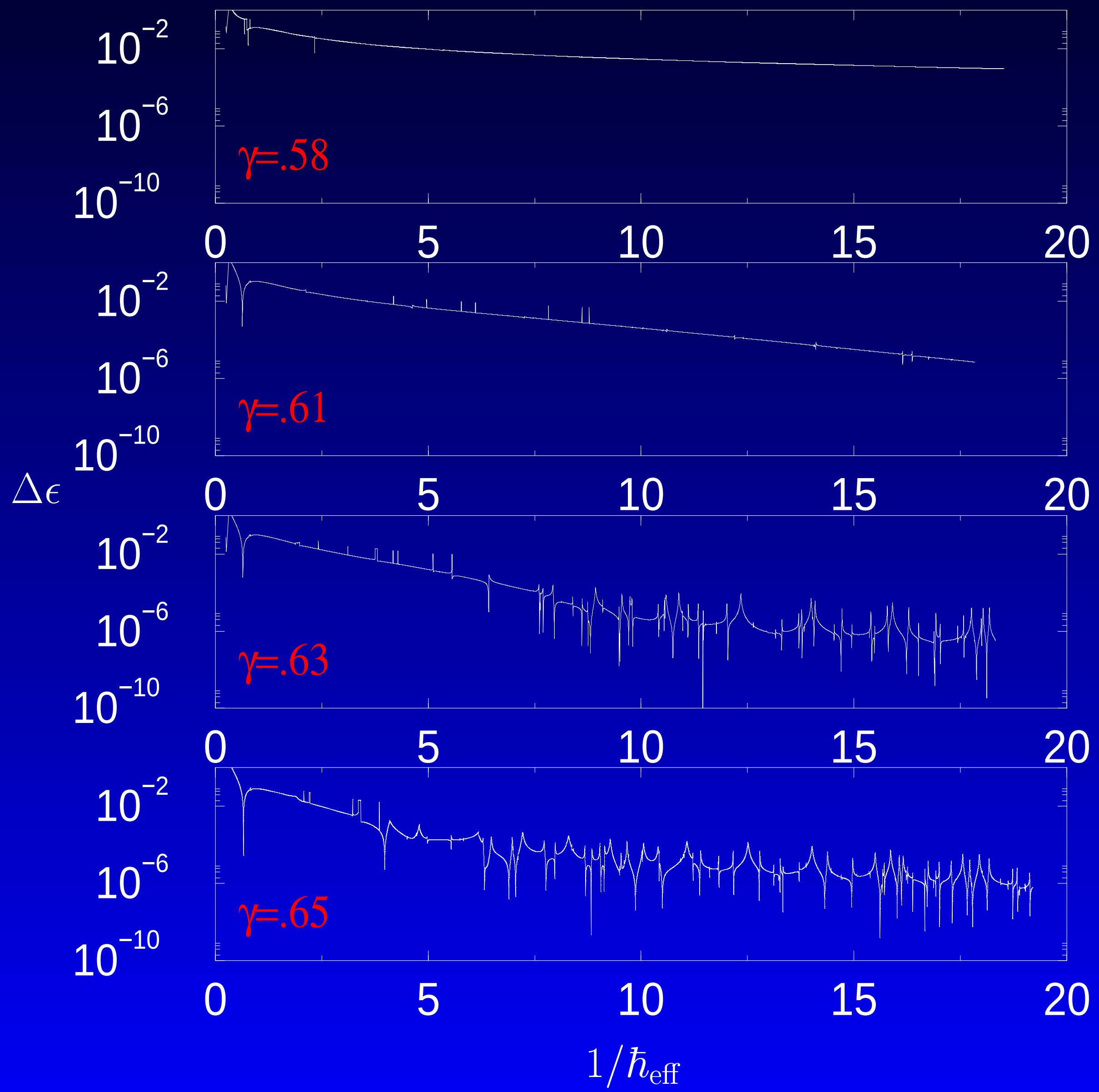
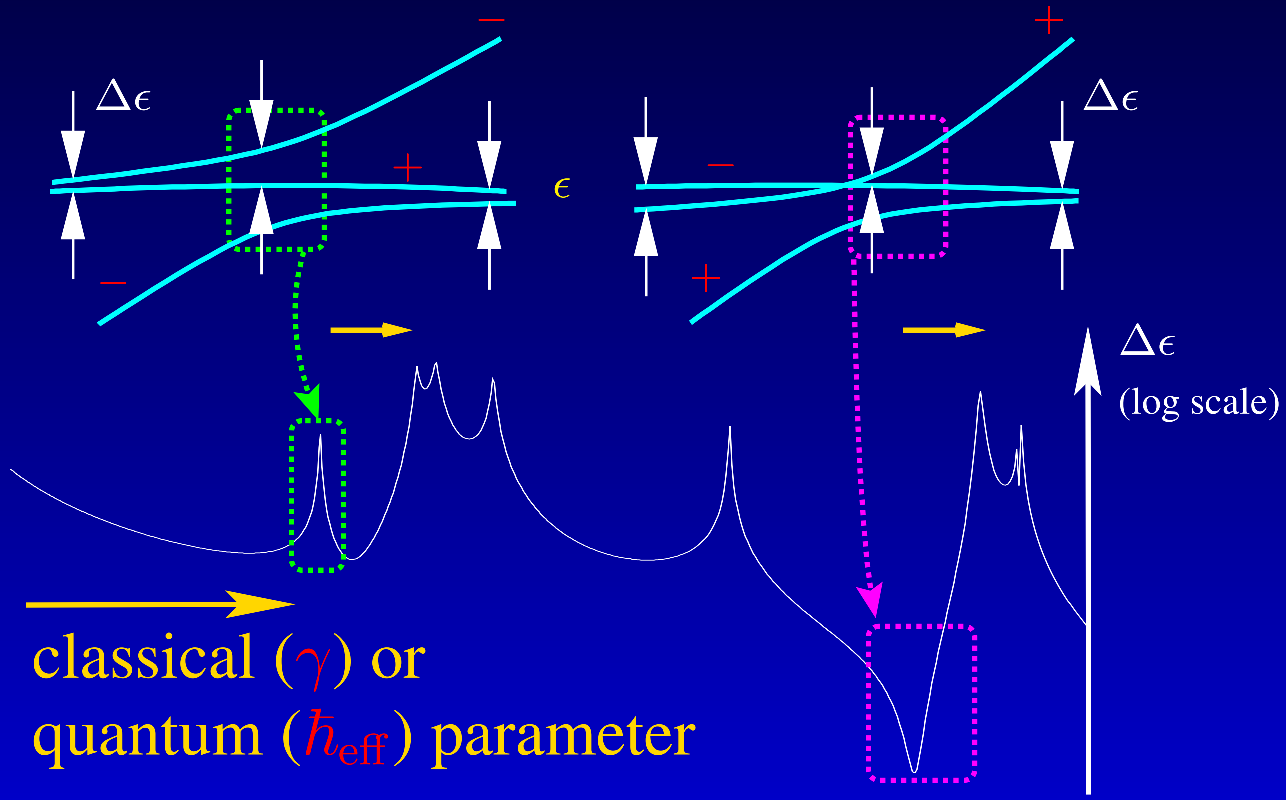




Resonance of what?

Quantum resonance: the tunnelling doublet is crossed by an intermediate level.



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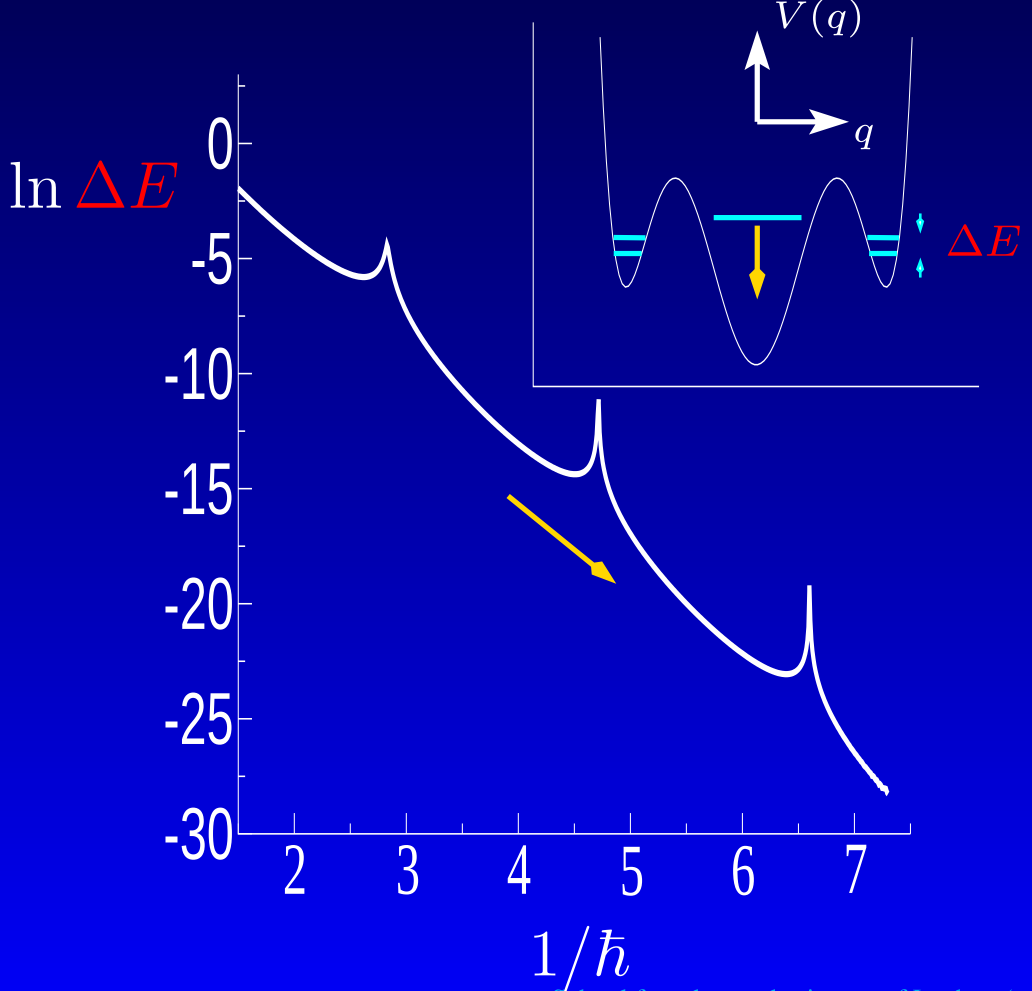
Quantum resonance: the tunnelling doublet is crossed by an intermediate level. The simplest case: $1d$ three-well potential



Resonance of what?

☞ ***Perturbative arguments*** (semiclassical I): Resonance Assisted Tunnelling (RAT) [Brodier, Schlagheck, Ullmo, 2001]

perturbation terms $\propto \dfrac{V_{w,i}}{E_{\text{well}} - E_{\text{intermediate}}}$

Semiclassical I: in the (quasi-)integrable regions, $E_w \sim n_w \hbar\omega_w$, $E_i \sim n_i \hbar\omega_i$, with (n_w, n_i) integers.

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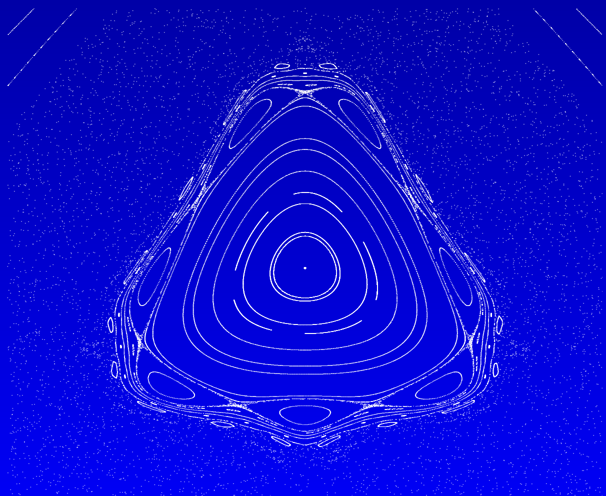
$\longrightarrow$ need for a local integrable approximation.



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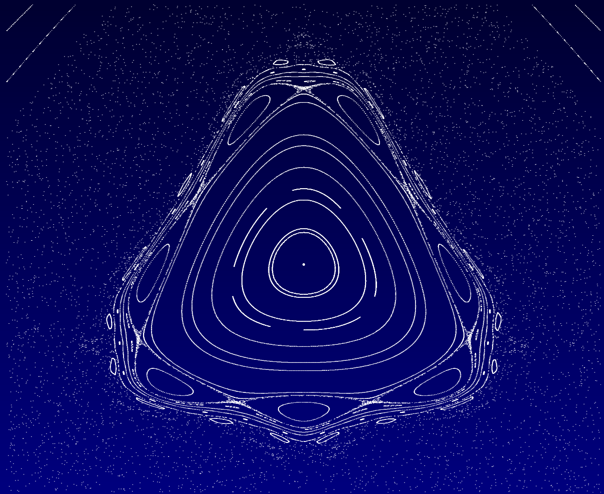

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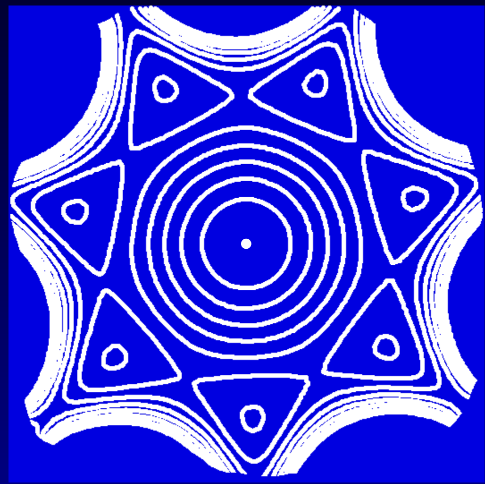

$\ell = 7$

BIRKHOFF-GUSTAVSON normal forms in action-angle variables

$$I = \tfrac{1}{2}(p^2 + q^2)\ ;\ \theta = \arctan \frac{q}{p}$$

$$H_{\text{exact}} \simeq \omega(\epsilon) I + a_2 I^2 + \cdots + a_{[\ell/2]} I^{[\ell/2]} + b I^{\ell/2} \cos(\ell\theta)$$

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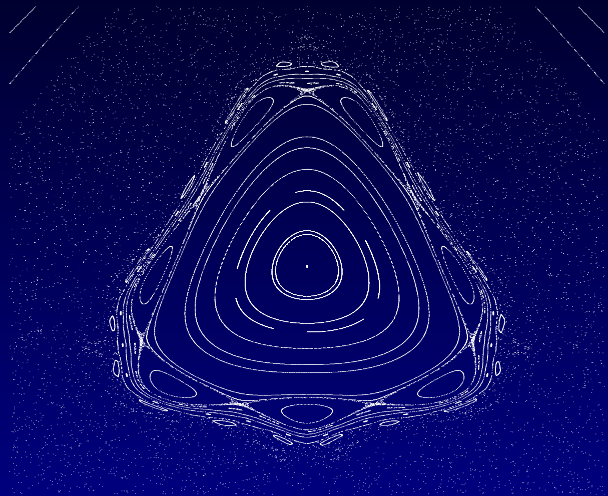

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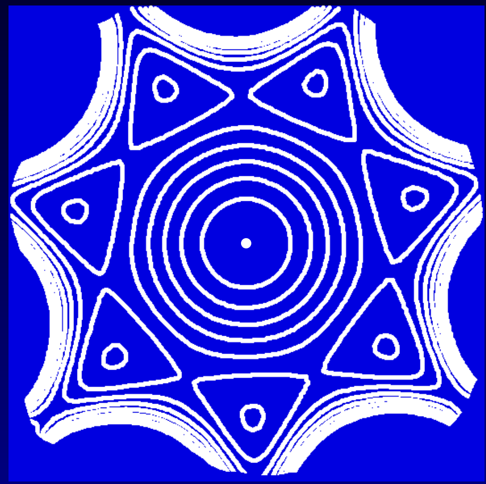

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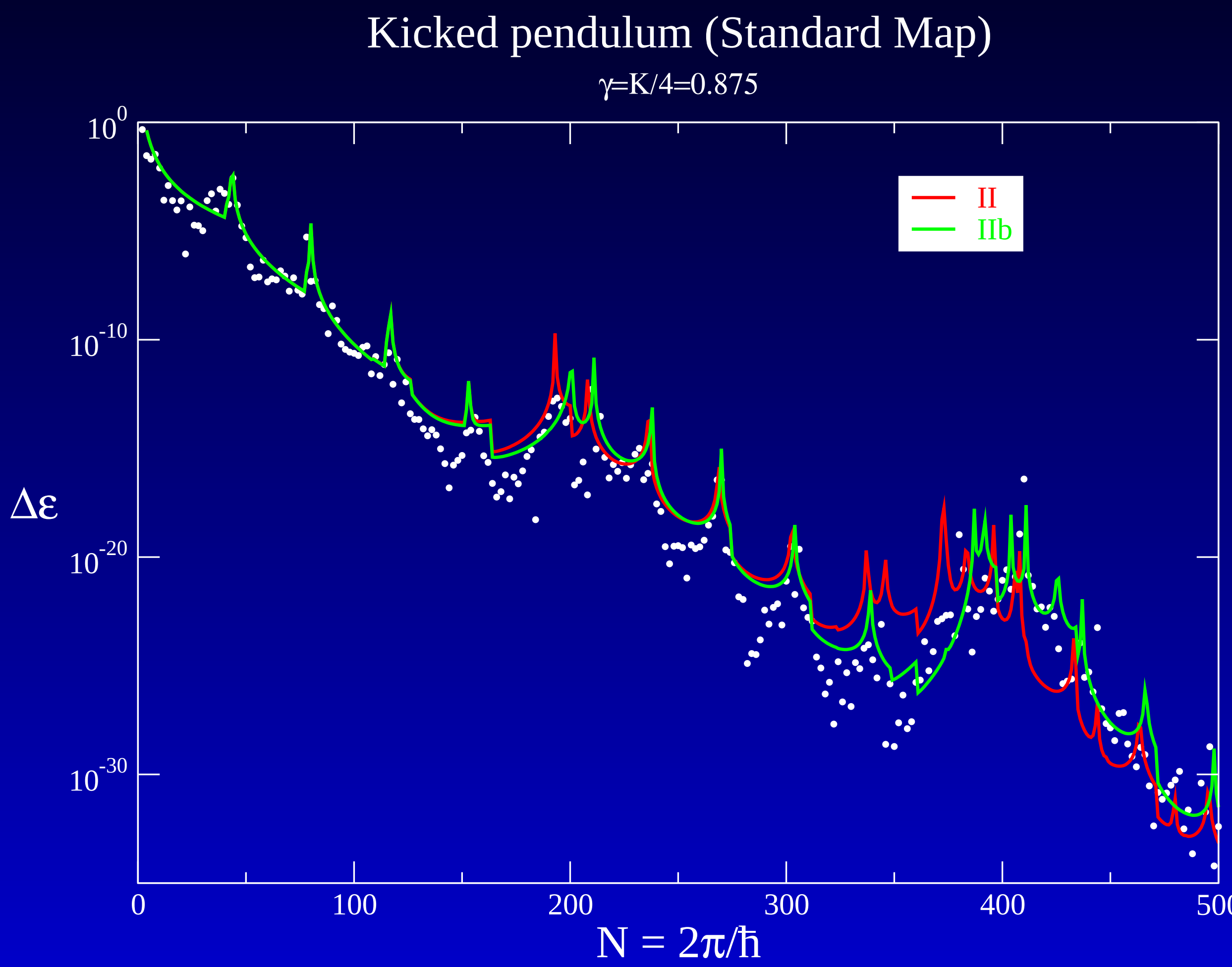
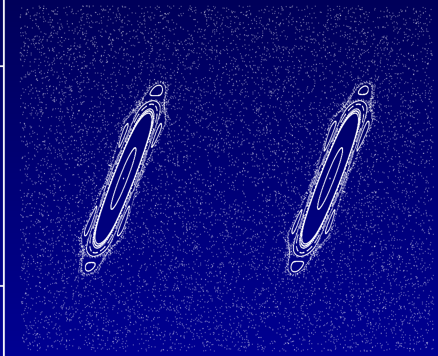
Semiclassics I: $\omega(\epsilon)$, some of the a's, and b are estimated (fitted).

$I \sim n\hbar$ and the resonant matrix elements can be computed for the quasi-modes.

Resonance of what?



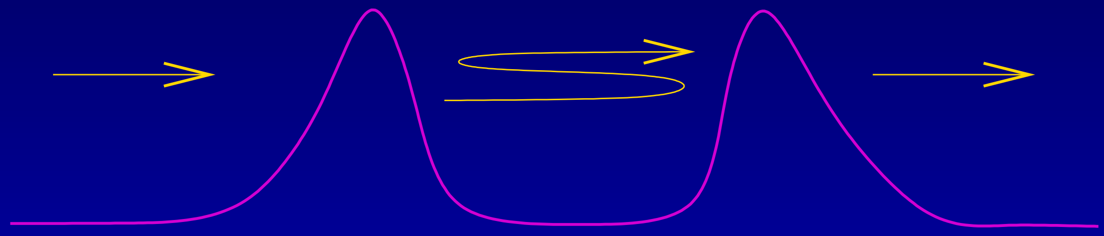


Kicked pendulum model (Standard Map $\gamma = .875$)

[SCHLAGHECK, MOUCHET, & ULLMO, 2011] [see also LÖCK, BÄCKER, KETZMERICK, & Schlagheck, 2010]

Resonance of what?

☞ ***Non-perturbative arguments*** (semiclassical II): In the energy domain $G(q_f, q_i, E) \sim \sum_{\mathfrak{p}} A_{\mathfrak{p}} \mathrm{e}^{\mathrm{i}S_{\mathfrak{p}}/\hbar}$. The intermediate states can be seen as the result of a constructive interference between classical paths (pôles of G).



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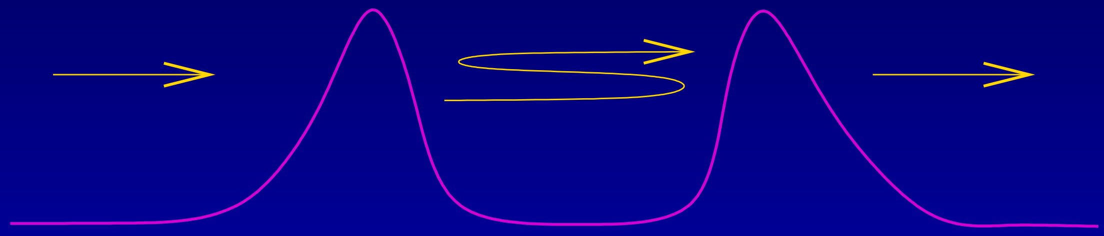

In optics: FABRY-PÉROT interferometer (1899);

In quantum physics: (BOHM, 1951); resonant tunnelling diode…

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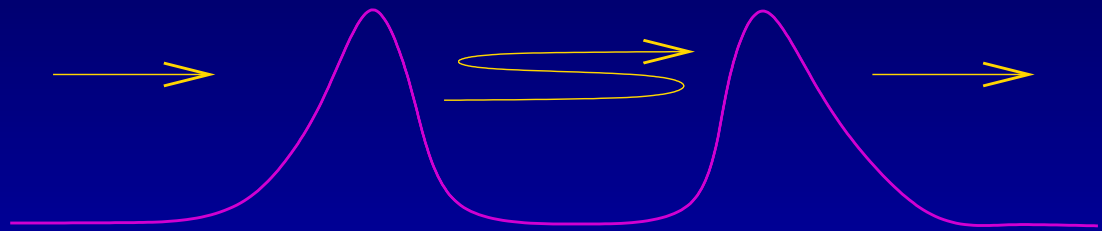
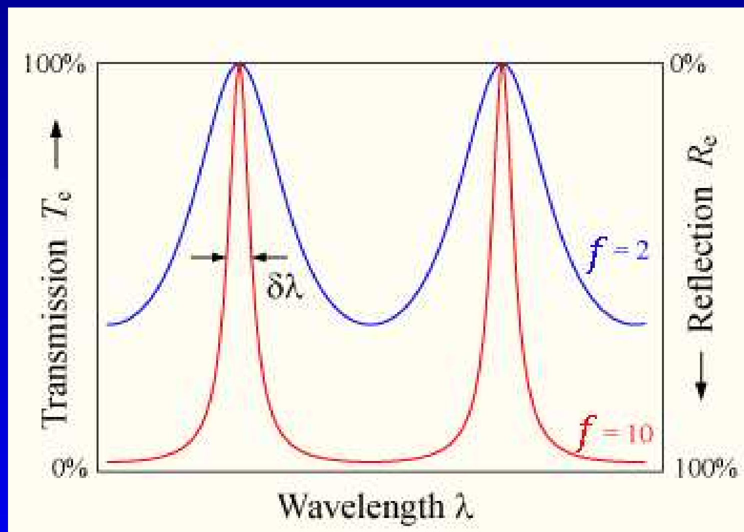


$$T_e = \frac{1}{1 + f\sin^2(\pi\delta/\lambda)}$$



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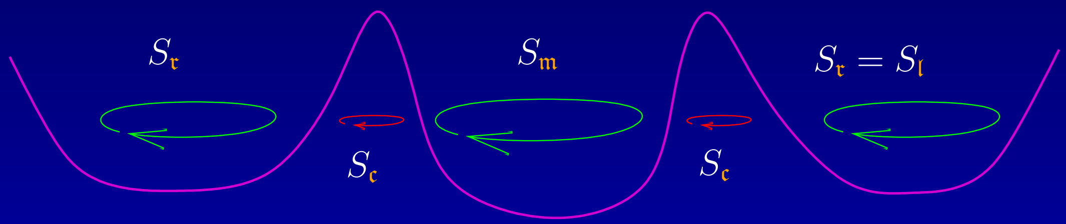

$$\Delta E \sim \mathrm{e}^{-S_{\mathfrak{c}}/\hbar} \sum_{\substack{\{w_{\mathfrak{r}},w_{\mathfrak{m}}\} \text{ pos. int. such that} \\ w_{\mathfrak{r}}T_{\mathfrak{r}}+w_{\mathfrak{m}}T_{\mathfrak{m}}=T_1}} (\cdots)\, \mathrm{e}^{\mathrm{i}w_{\mathfrak{r}}S_{\mathfrak{r}}/\hbar+\mathrm{i}w_{\mathfrak{m}}S_{\mathfrak{m}}/\hbar} \simeq \frac{T_1 \mathrm{e}^{-S_{\mathfrak{c}}/\hbar}}{\left|\sin\left((S_{\mathfrak{r}} - S_{\mathfrak{m}})/2\hbar\right)\right|}$$

Spikes when $T_{\mathfrak{r}}/T_{\mathfrak{m}} \in \mathbb{Q}$ (classical resonance)

and $S_{\mathfrak{r}} - S_{\mathfrak{m}} \sim 2n\pi\hbar$ (simultaneous EBK quantization).

The challenge of the complexification

- Quasi-integrable: natural boundaries appear when the KAM tori are analytically continued (GREEN & PERCIVAL; 1981)

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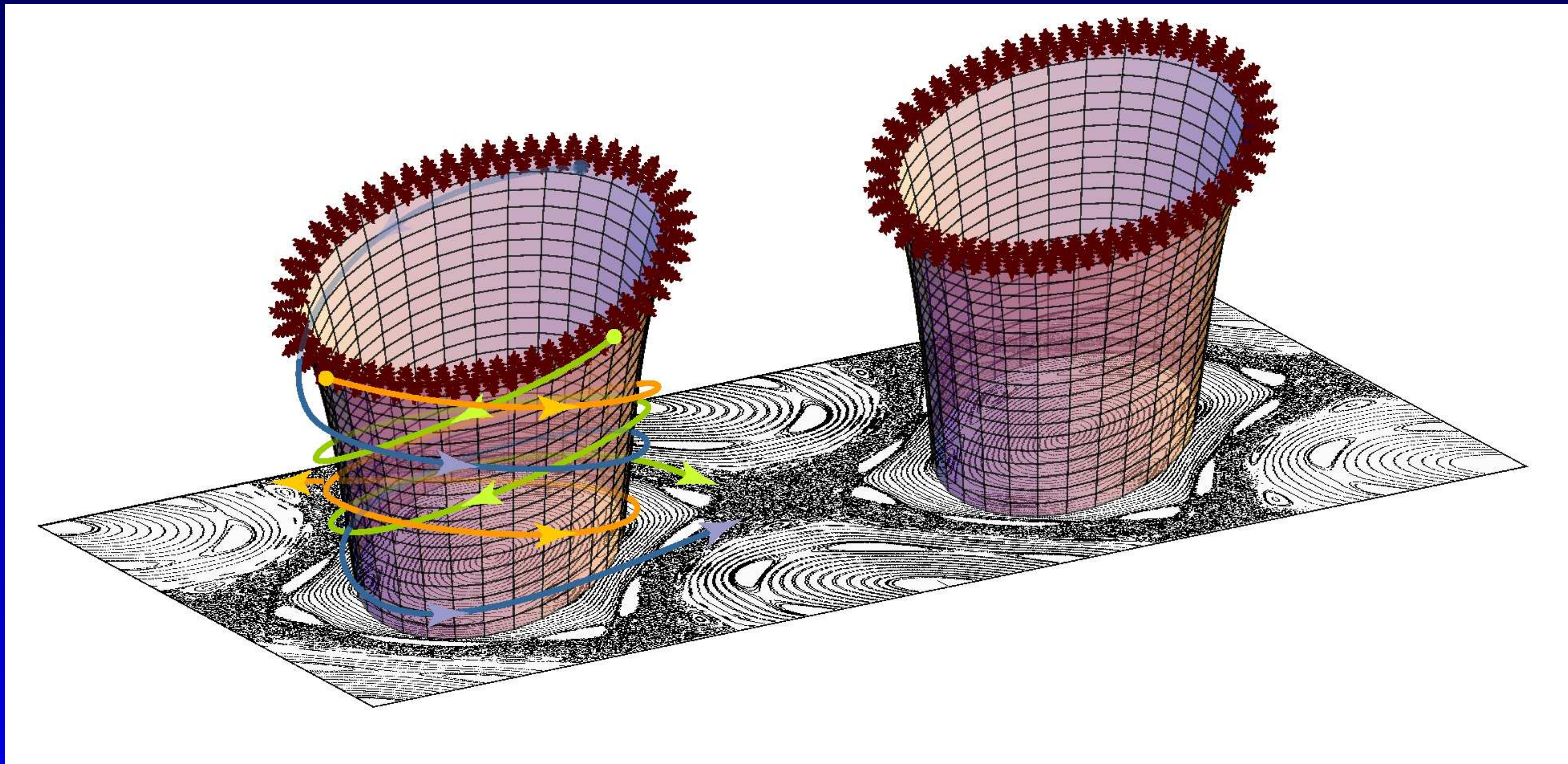

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The challenge of the complexification

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  Failure of HERRING's formula (CREAGH; 1994, 1997):

$$\Delta\epsilon \simeq \hbar^2 \int \mathrm{d}x_\perp \left[\psi_L(x_\parallel, x_\perp)\partial_\parallel \psi_R^*(x_\parallel, x_\perp) - \psi_R^*(x_\parallel, x_\perp)\partial_\parallel \psi_L(x_\parallel, x_\perp)\right]\Big|_{x_\parallel = 0}$$

beyond the natural boundaries

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- More chaos: the complex solutions of HAMILTON's equations agglomerate in JULIA set patterns

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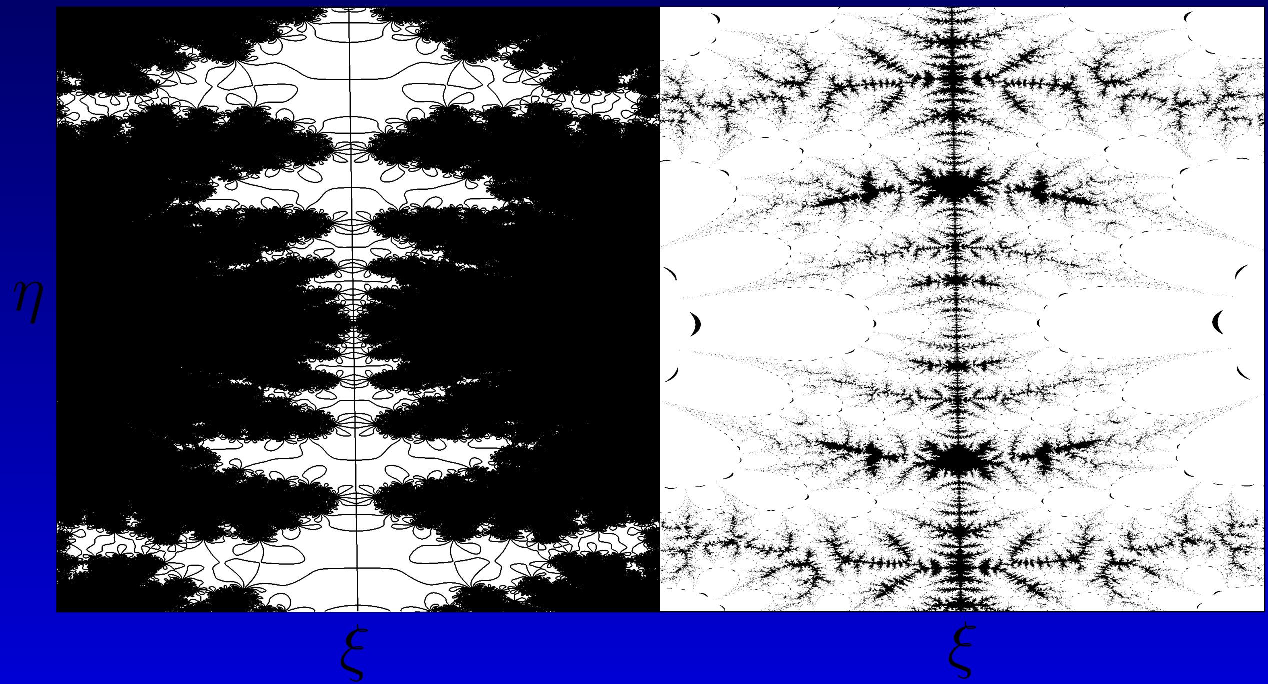

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$\rightarrow$ problem of selection of the dominant contributions to path integrals.

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- A complete WICK rotation $t \mapsto -\mathrm{i}t$ (POLYAKOV *et al.*; 1975, *cf.* also COLEMAN; 1977)

$$\frac{p^2}{2m} + V(q) \ \rightarrow \ \frac{p^2}{2m} - V(q)$$

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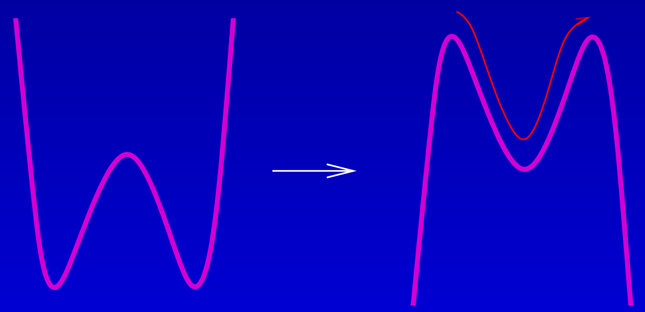

Captures the ground-state doublet only

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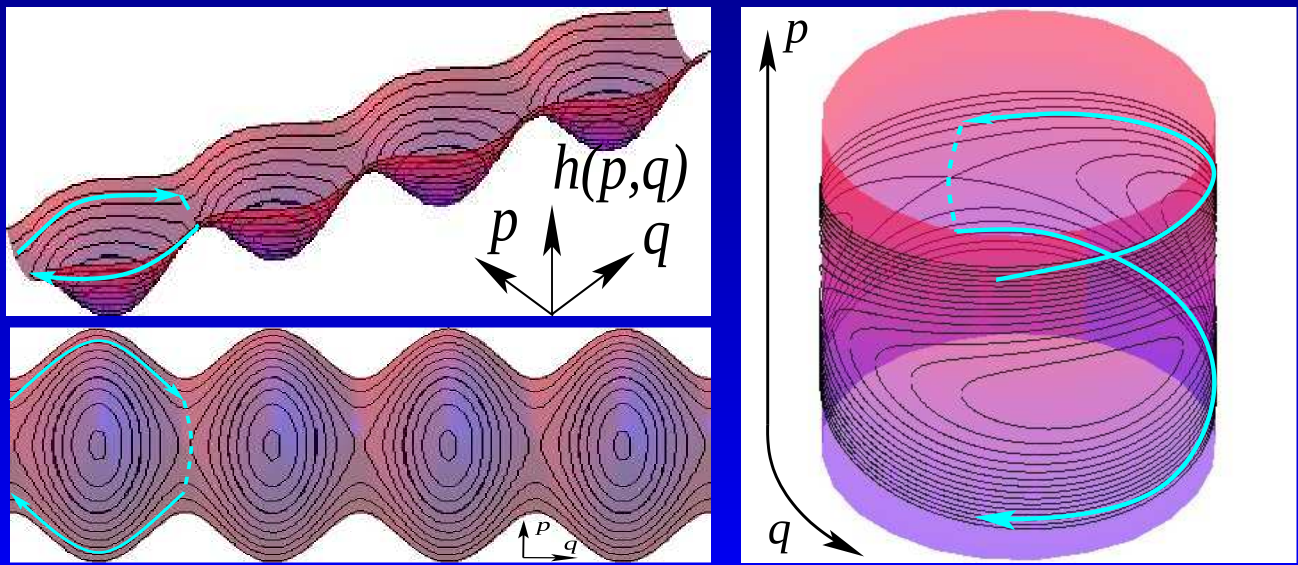

Instantons revisited: incomplete WICK rotation

[MOUCHET, 2007; LE DEUNFF & MOUCHET, 2010]

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Symmetry operator: $\hat{S}\,|\phi_n^{\pm}\rangle = \pm\,|\phi_n^{\pm}\rangle$

$$\begin{aligned} \hat{U}(T) &\stackrel{\text{def}}{=} \mathrm{e}^{-\mathrm{i}\hat{H}T/\hbar} \\ &= \sum_{n=0}^{\infty} \Big(|\phi_n^{+}\rangle \langle\phi_n^{+}|\,\mathrm{e}^{-\mathrm{i}E_n^{+}T/\hbar} + |\phi_n^{-}\rangle \langle\phi_n^{-}|\,\mathrm{e}^{-\mathrm{i}E_n^{-}T/\hbar} \Big) \end{aligned}$$

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For large enough T_2, only the ground-state doublet survives:

$$\Delta E_0 \stackrel{\text{def}}{=} E_0^- - E_0^+ \simeq \Delta_0(T) \stackrel{\text{def}}{=} \frac{2\hbar}{\mathrm{i}T}\,\frac{\mathrm{tr}\big(\hat{S}\,\hat{U}(T)\big)}{\mathrm{tr}\big(\hat{U}(T)\big)}$$

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For excited states and almost real T (small θ):

$$\Delta E_n \simeq \Delta_n(T) \stackrel{\mathrm{def}}{=} \frac{2\hbar}{\mathrm{i}T} \frac{\mathrm{tr}\big(\hat{S}\,\hat{\Pi}_n\,\hat{U}(T)\big)}{\mathrm{tr}\big(\hat{\Pi}_n \hat{U}(T)\big)}$$

Semiclassically: $\hat{\Pi}_n$ projects on one torus with energy $E_n^+ \simeq E_n^-$

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For a decay rate: $\Gamma_n \simeq -\dfrac{2}{T_2}\,\mathrm{Im}\left(\mathrm{e}^{\mathrm{i}E_nT/\hbar}\,\mathrm{tr}\big(\hat{\Pi}_n\hat{U}(T)\big)\right)$

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$$\Delta E_0 \stackrel{\text{def}}{=} E_0^- - E_0^+ \simeq \Delta_0(T) \stackrel{\text{def}}{=} \frac{2\hbar}{\mathrm{i}T} \frac{\mathrm{tr}\big(\hat{S}\,\hat{U}(T)\big)}{\mathrm{tr}\big(\hat{U}(T)\big)}$$

2. Use the semiclassical expansion of the Green functions in the (complex) time domain

$$\mathrm{tr}\big(\hat{S}\,\hat{U}(T)\big) = \int \mathrm{d}q\, G(q,-q,T) \sim \sum_{\mathfrak{o}} A_{\mathfrak{o}} \mathrm{e}^{-\mathrm{i}S_{\mathfrak{o}}/\hbar}$$

complex trajectories connecting (p,q) to $(-p,-q)$ in time T.

Instantons revisited: incomplete WICK rotation

[MOUCHET, 2007; LE DEUNFF & MOUCHET, 2010]

1. Introduce a complex time (incomplete WICK rotation):

$$T = |T|e^{-i\theta} = T_1 - iT_2$$

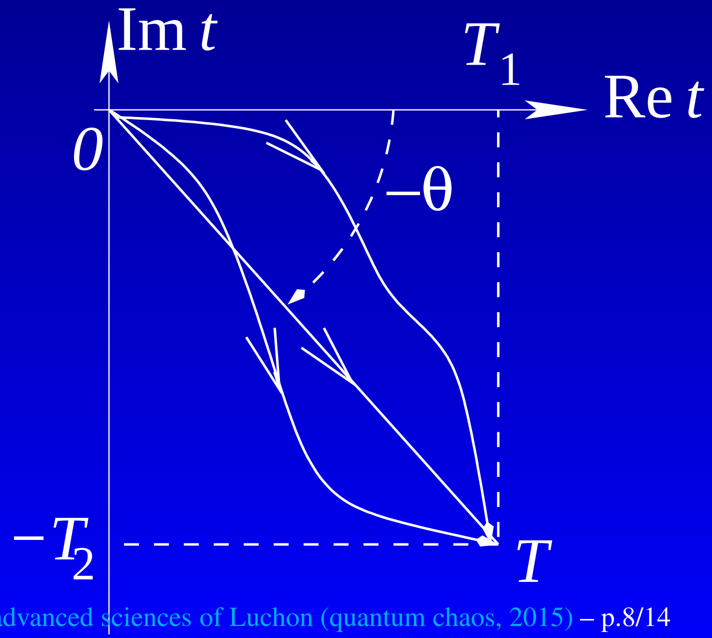
2. Use the semiclassical expansion of the Green functions in the (complex) time domain

3. Take advantage of deforming the complex time path to retain only the trajectories with real q:

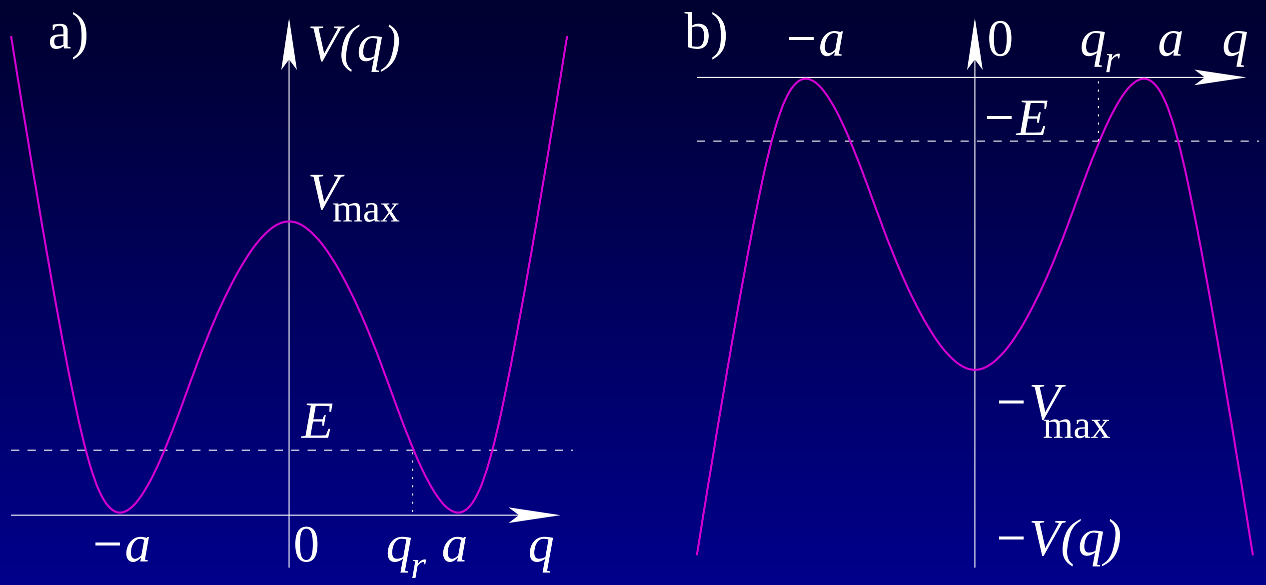
$$\int e^{iS[p(s),q(s),t(s)]/\hbar}\, D[p]D[q]$$

is independent of the choice of $s \mapsto t(s)$ provided $\operatorname{Im} t \searrow$

(MCLAUGHLIN; 1972)



The double well



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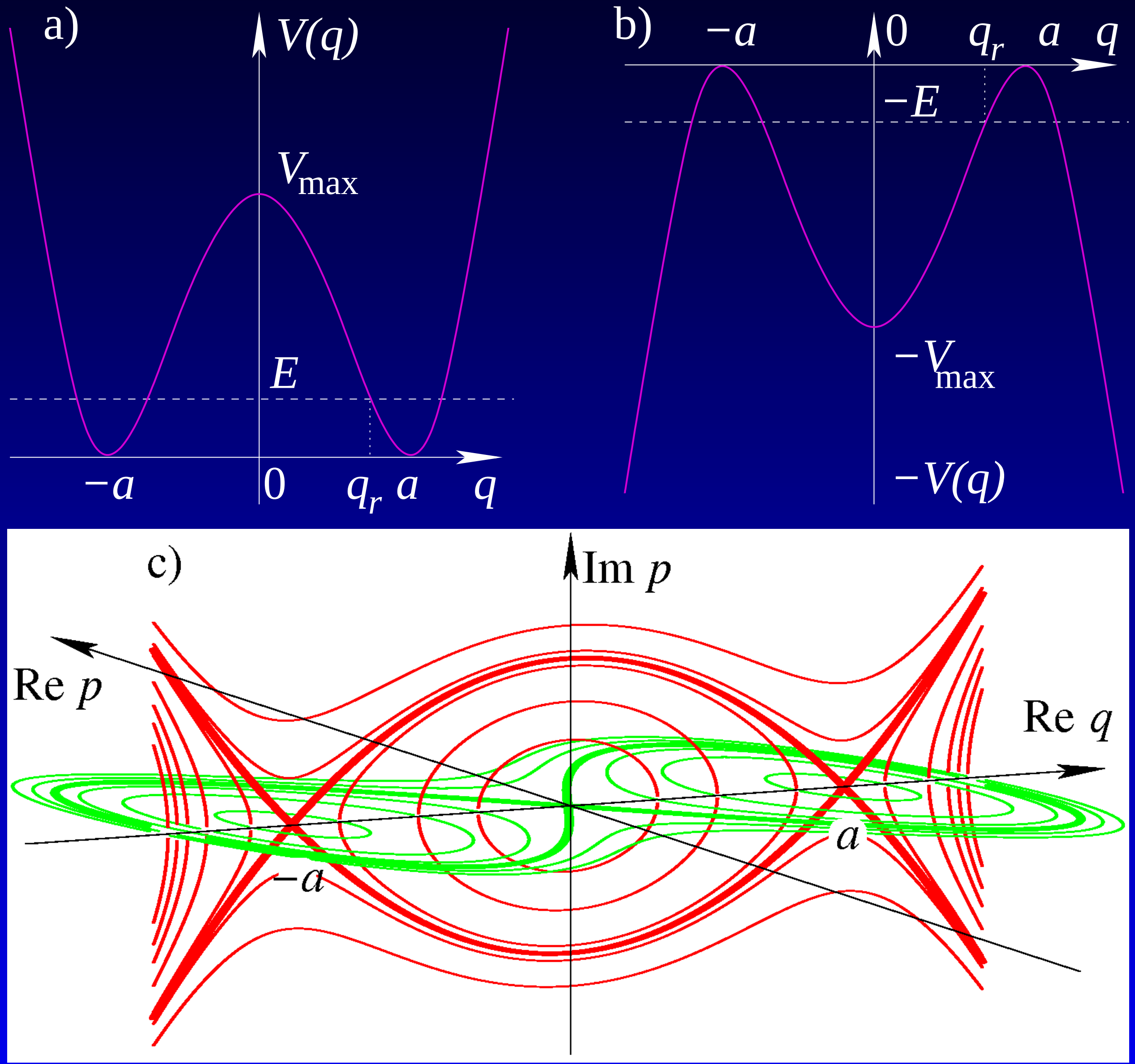

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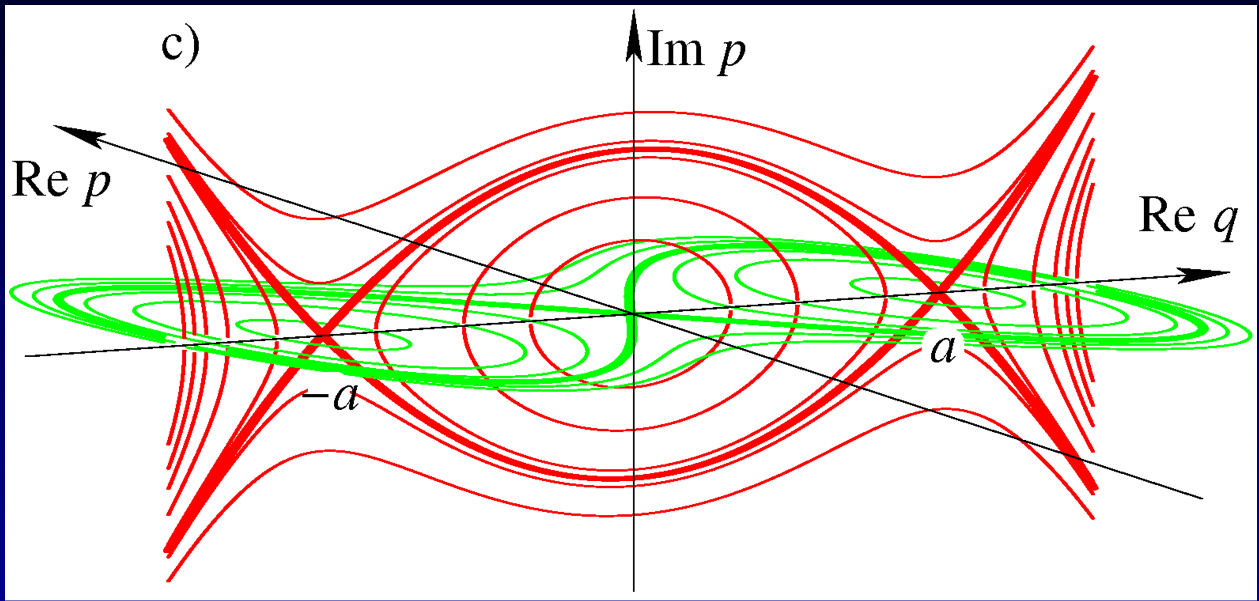

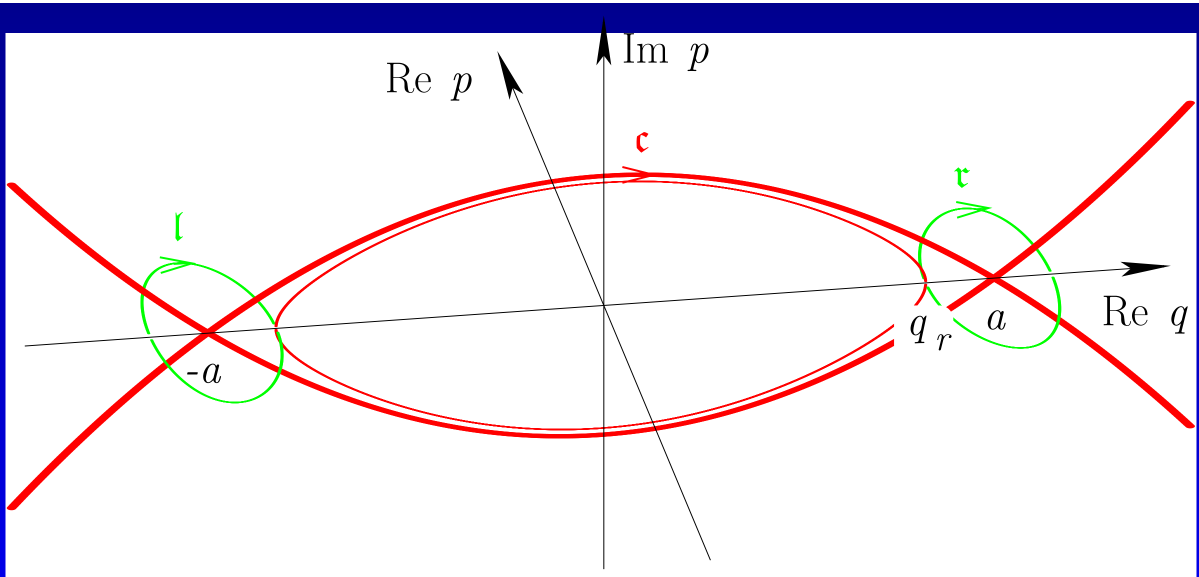

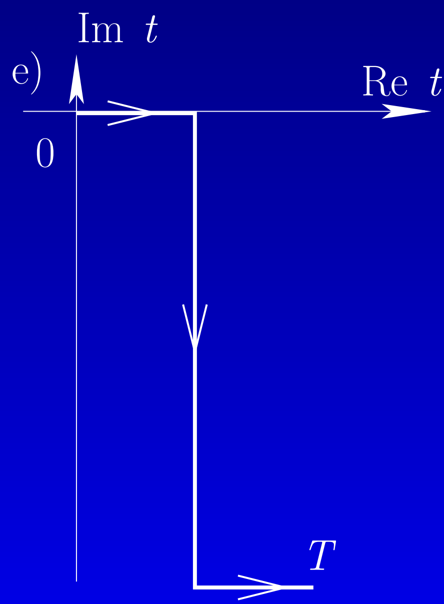

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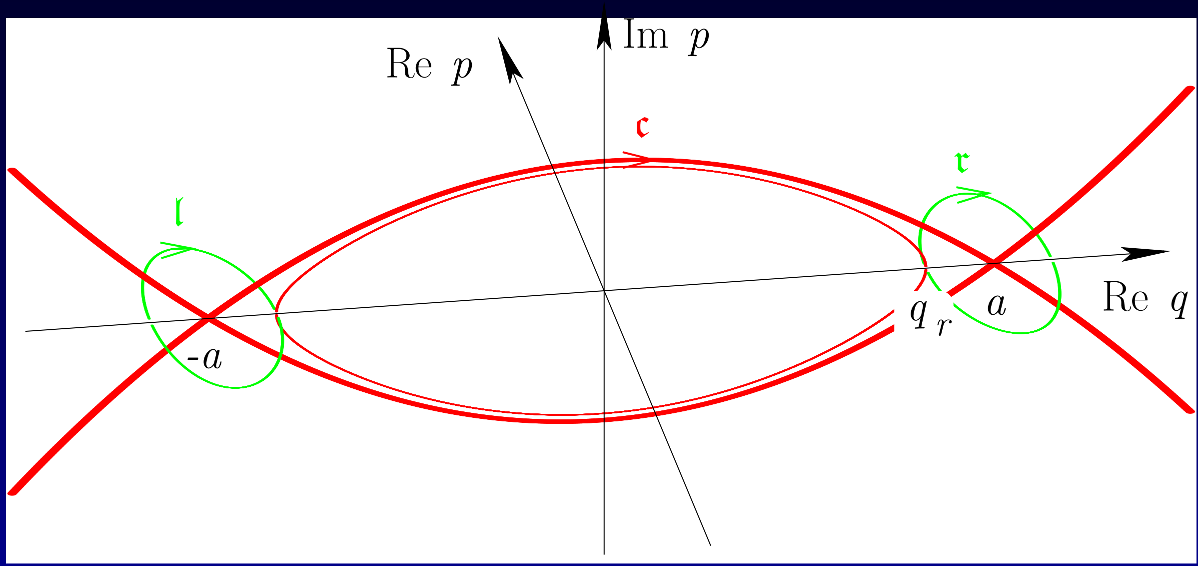
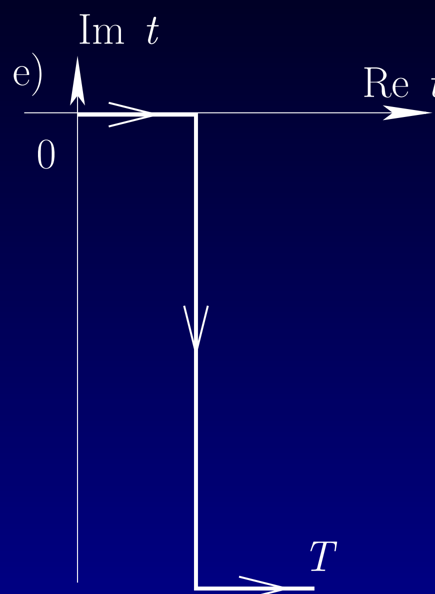


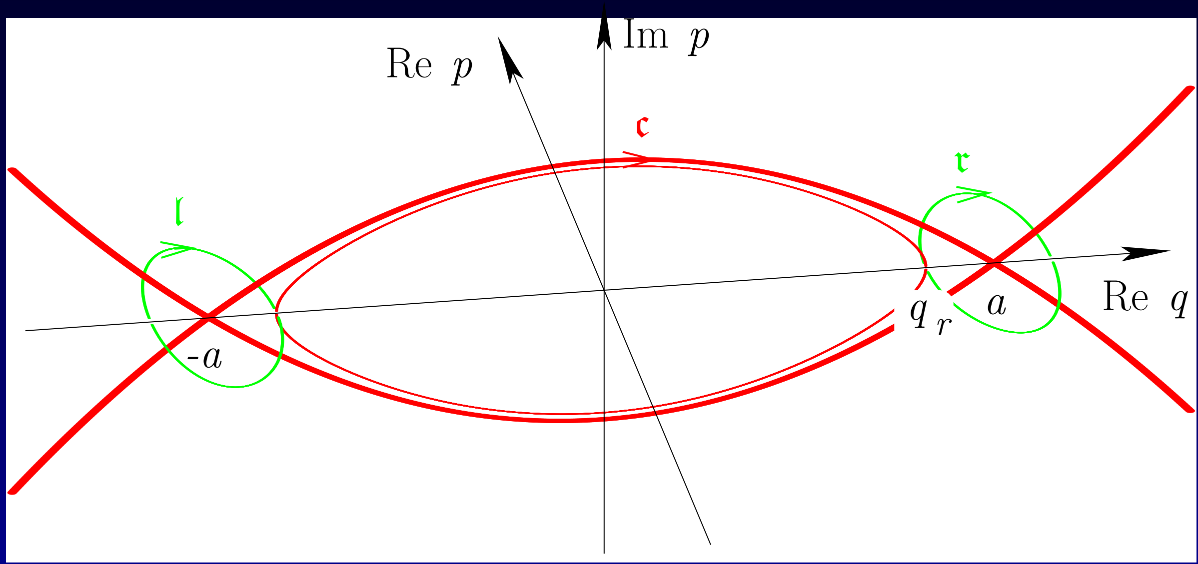


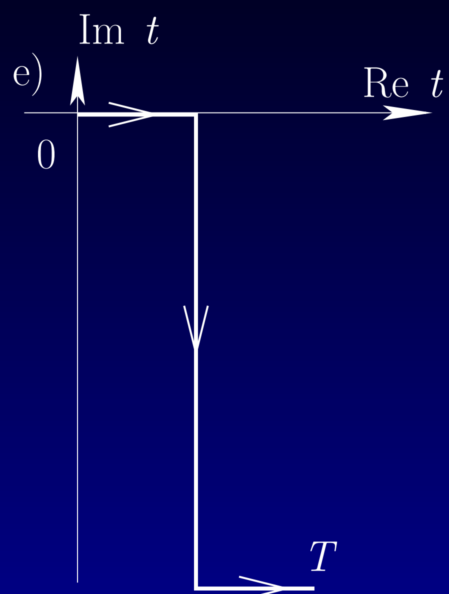
$\mathfrak{o}$ is a concatenation of three trajectories: $\mathfrak{r} \cup \mathfrak{c} \cup \mathfrak{l}$

$$\rightarrow \qquad \Delta_n \underset{\hbar \to 0}{\sim} \frac{2\hbar}{T_{\mathfrak{r}}(E_n)} \, \mathrm{e}^{-S_{\mathfrak{c}}(E_n)/(2\hbar)} \; .$$

(LANDAU & LIFSCHITZ; 1958).

The double well





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$$\rightarrow \qquad \Delta_n \underset{\hbar \to 0}{\simeq} \frac{2\hbar}{T_{\mathfrak{r}}(E_n)} \, \mathrm{e}^{-S_{\mathfrak{c}}(E_n)/(2\hbar)} \, .$$

(Landau & Lifschitz; 1958). The correct prefactor is given by $\times g_n$ (Garg; 2000) with $g_0 \simeq 1.075$, $g_1 \simeq 1.028$, …

The pendulum

Construct $\mathfrak{o}$ to connect p to $-p$ keeping p real:

$$\frac{p^2}{2} - \gamma \cos q \quad \rightarrow \quad \text{(i) } \frac{p^2}{2} - \gamma \cosh q \quad \text{or} \quad \text{(ii) } \frac{p^2}{2} + \gamma \cosh q$$

The pendulum

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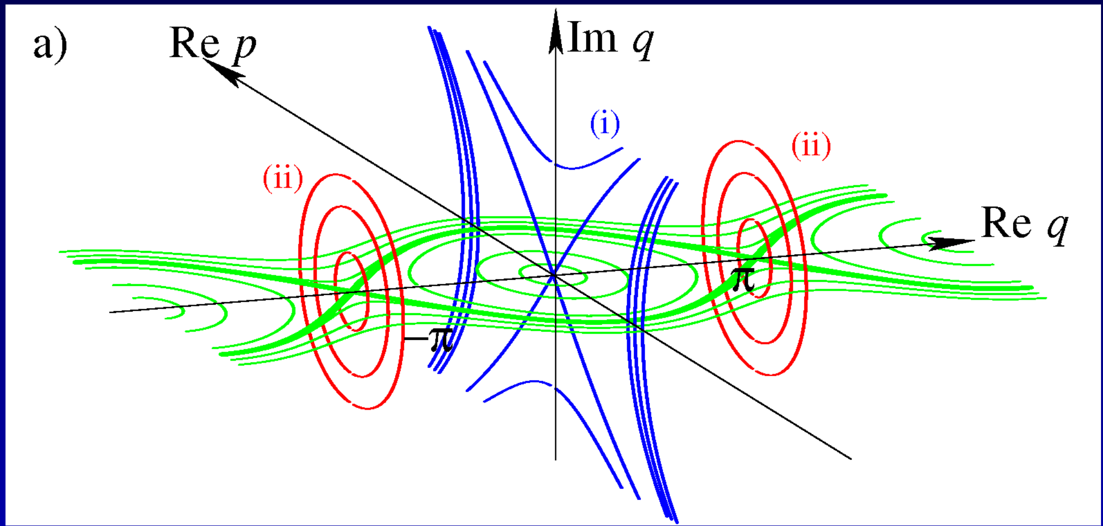

The pendulum

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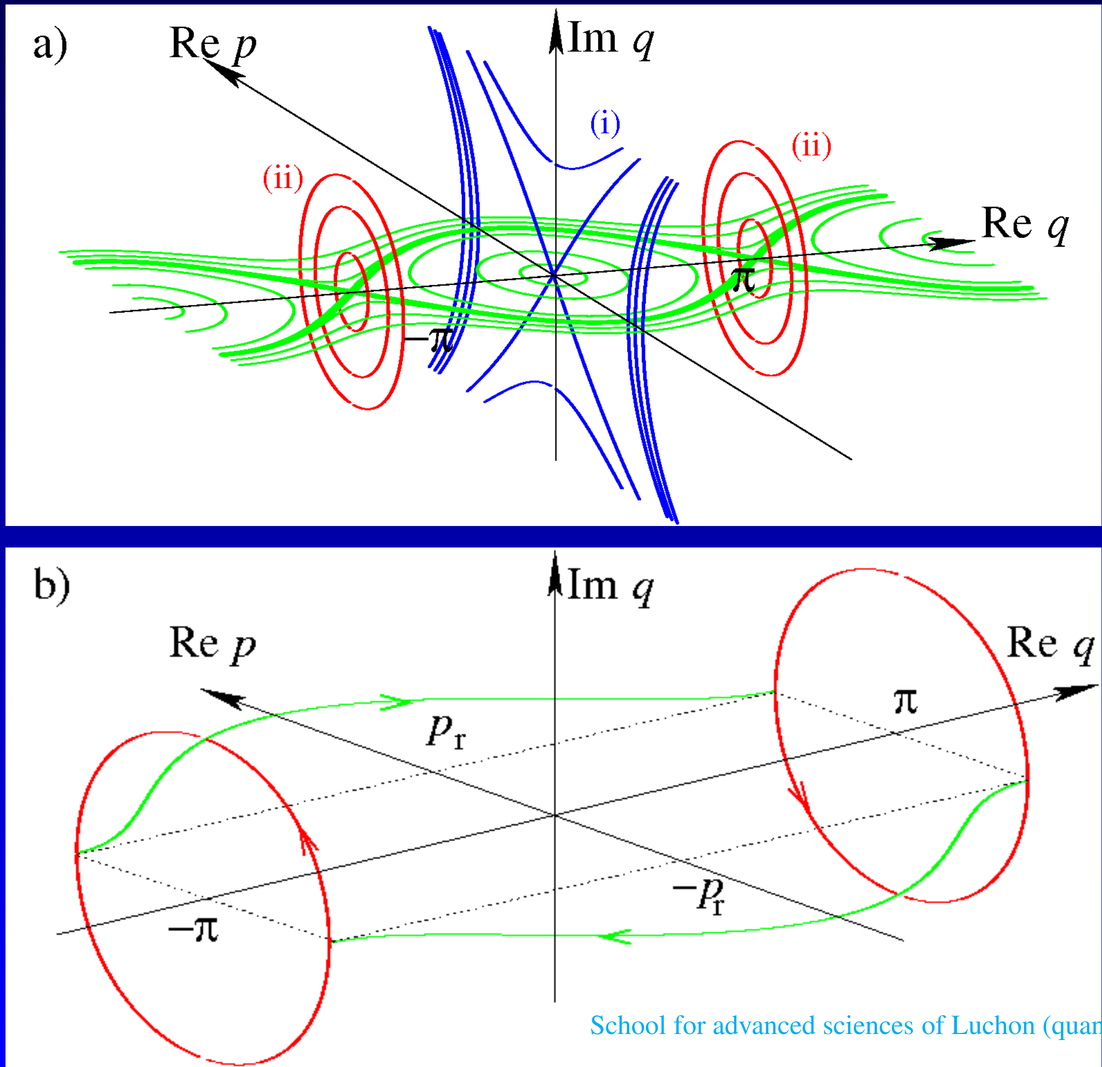
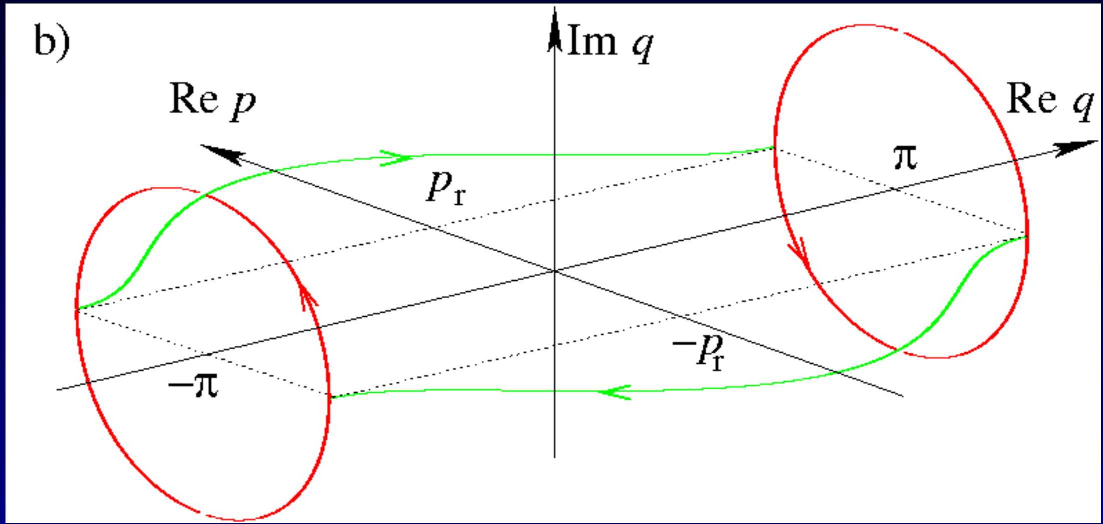


The pendulum



$$\rightarrow \qquad \Delta_n \underset{\hbar\to 0}{\sim} \frac{1}{\pi n^{4n-1}} \left(\frac{\mathrm{e}}{2}\right)^{4n} \hbar^2 \left(\frac{\gamma}{\hbar^2}\right)^{2n}$$

Remark: furnishes an asymptotic expression for differences between MATHIEU's characteristic values.

The pendulum

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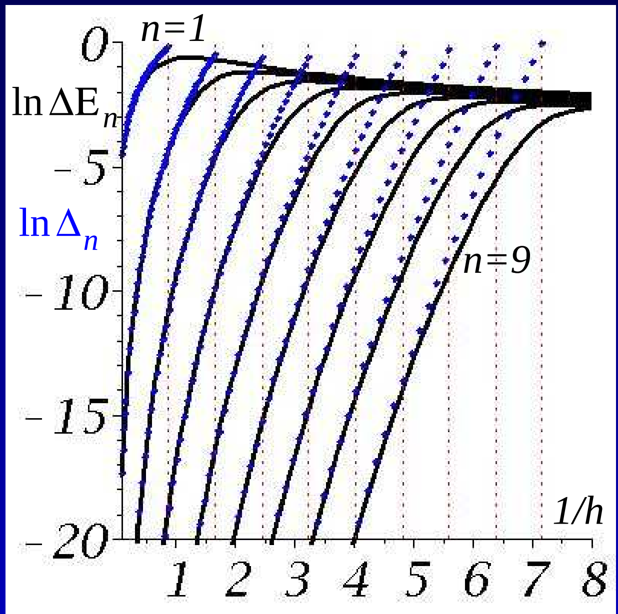
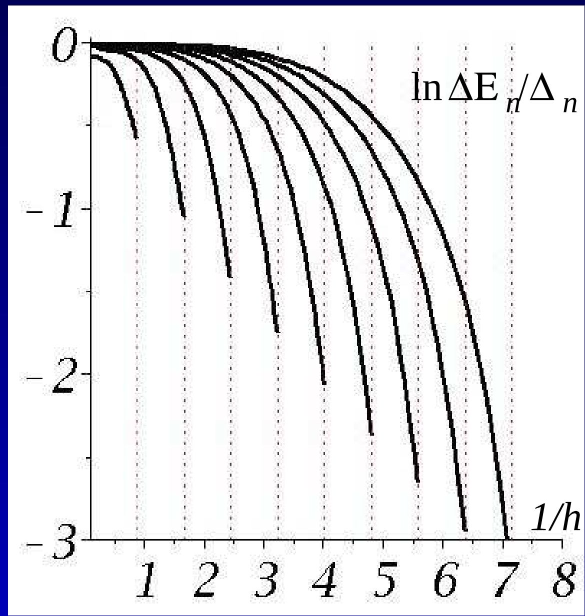
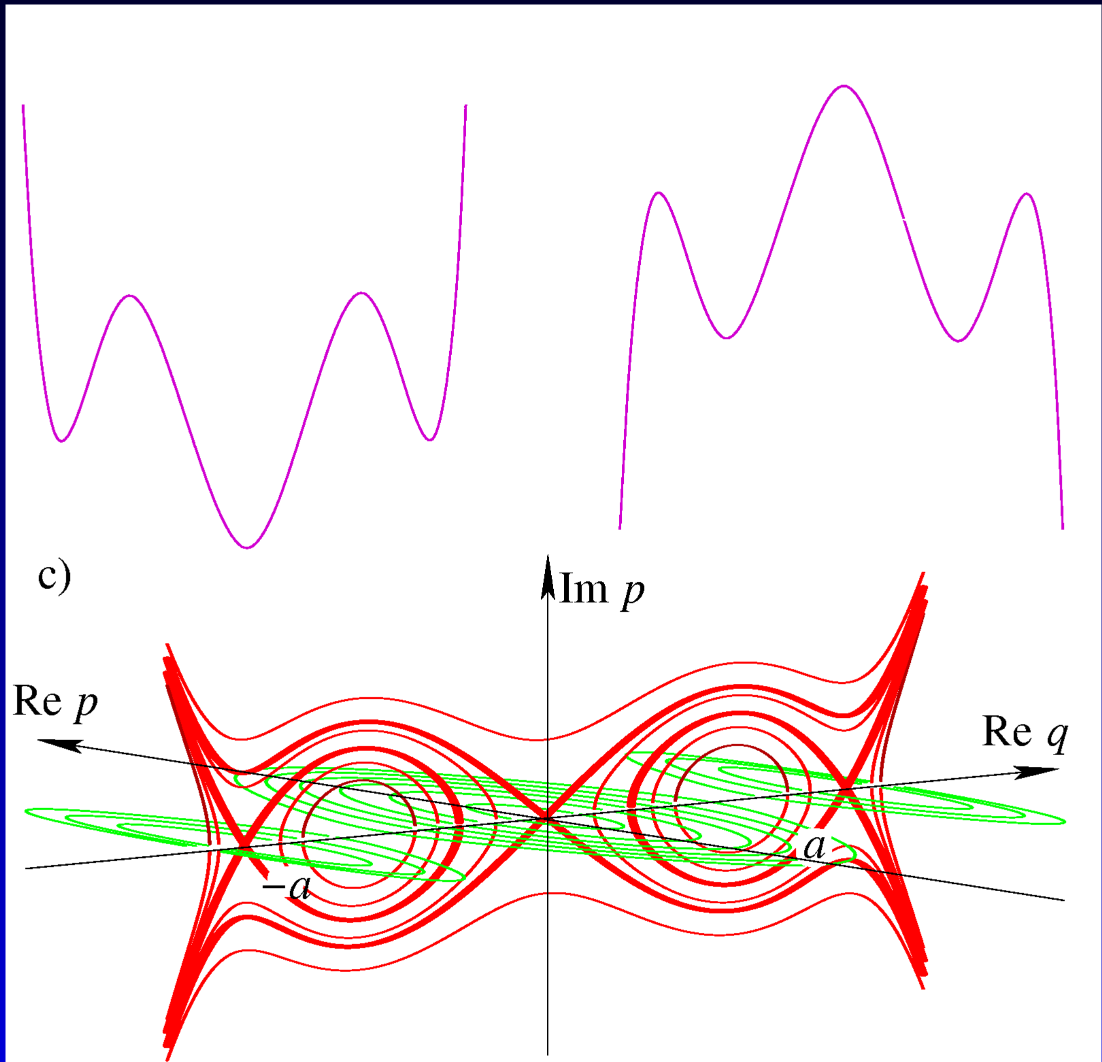




Triple well



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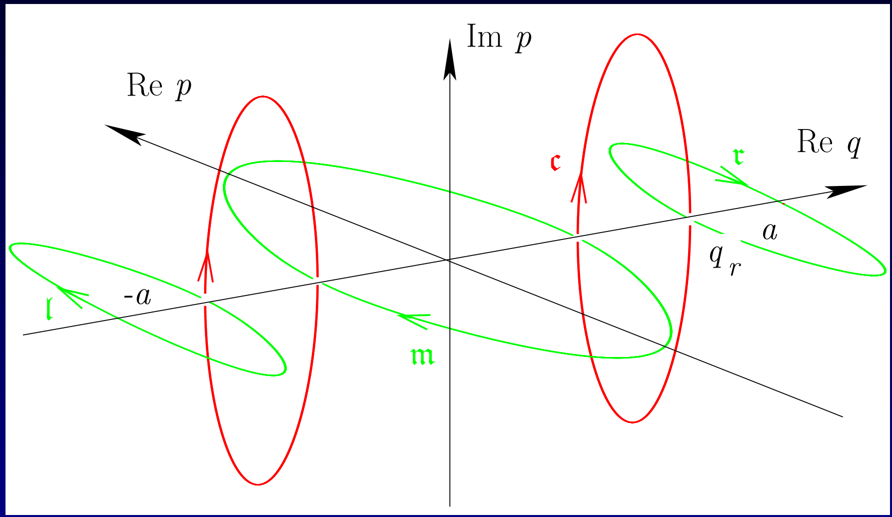
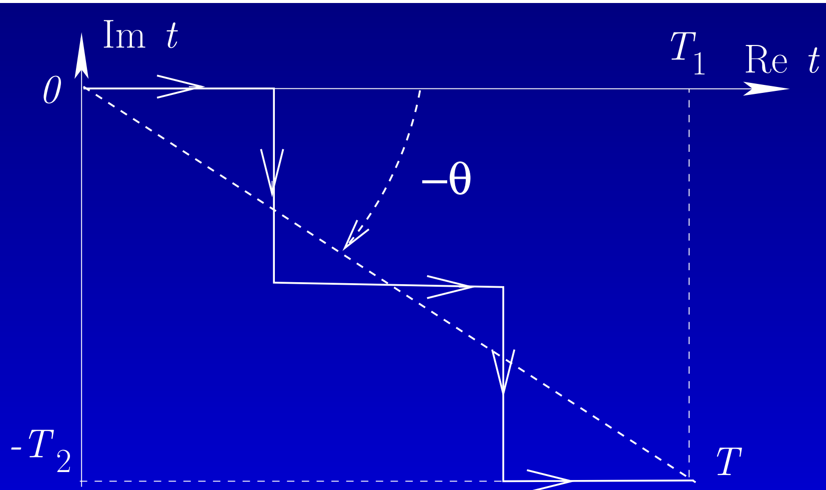




$$T = w_{\mathfrak{r}} T_{\mathfrak{r}}(E) + \left(w_{\mathfrak{m}} + \frac{1}{2} \right) T_{\mathfrak{m}}(E) - \mathrm{i} T_{\mathfrak{c}}(E)$$

Triple well

$$\Delta_n(T) \underset{\hbar\to 0}{\sim} \left|\frac{2\hbar F(T)}{T}\right| \mathrm{e}^{-S_{\mathfrak{c}}(E_n)/\hbar}$$

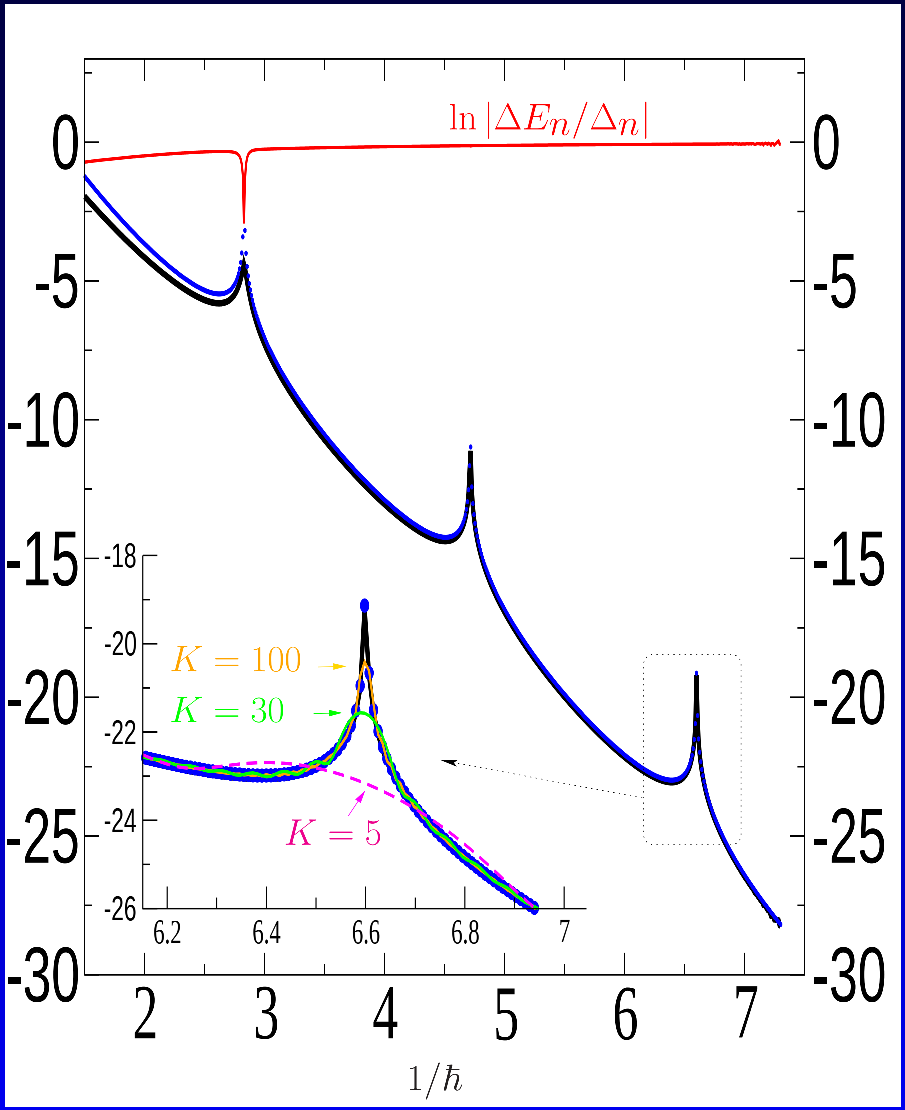
$$F = \sum_{\substack{\{w_{\mathfrak{r}}, w_{\mathfrak{m}}\} \text{ pos. int. such that} \\ w_{\mathfrak{r}} T_{\mathfrak{r}}(E_n) + w_{\mathfrak{m}} T_{\mathfrak{m}}(E_n) = T_1}} (w_{\mathfrak{r}} + 1)\, \mathrm{e}^{\mathrm{i} w_{\mathfrak{r}}[S_{\mathfrak{r}}(E_n)/\hbar - \pi] + \mathrm{i} w_{\mathfrak{m}}[S_{\mathfrak{m}}(E_n)/\hbar - \pi]}$$

$$\simeq \frac{T_1}{\left|\sin\left((S_{\mathfrak{r}} - S_{\mathfrak{m}})/2\hbar\right)\right|}$$ when $T_1 \gg T_2$ $(\theta \ll 1)$

$\dfrac{S}{2\pi\hbar} - \dfrac{1}{2}$ integer $\leftrightarrow$ EBK quantization

Triple well

Diminution of θ: $T = |T|\mathrm{e}^{-\mathrm{i}\theta} = KT_{\mathfrak{m}} - \mathrm{i}T_{\mathfrak{c}}$ with $K \nearrow$:



TAYLOR-made integrable resonant Hamiltonians

[J. Le Deunff, A. Mouchet & P. Schlagheck, 2013]

Take a ℓ-normal form:

$$h(p,q)=\frac{1}{2}\omega(p^2+q^2)+a_2(p^2+q^2)^2+\cdots+a_{[\ell/2]}(p^2+q^2)^{[\ell/2]}+b\,\mathrm{Re}[\mathrm{e}^{\mathrm{i}\phi}(p+\mathrm{i}q)^\ell]$$

and consider the doubly periodic Hamiltonian

$$H(P,Q)\stackrel{\text{def}}{=}h\left(\cos P,\cos Q\right)$$

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The phase space is then a $2d$-torus. Quantization leads to a finite dimensional Hilbert space whose dimension is

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Tunnelling between two adjacent cells (in Q or in P) with a resonance chain centered at $(P,Q)=(\pm\frac{\pi}{2},\pm\frac{\pi}{2})$ modulo 2π.

TAYLOR-made integrable resonant Hamiltonians

$\ell = 4 \qquad H(P,Q) \stackrel{\text{def}}{=} h\left(\cos P, \cos Q\right)$

$$h(p,q)=\frac{1}{2}(p^2+q^2)+a_1(p^2+q^2)^2+b\left[(p^4-6p^2q^2+q^4)\cos\phi-4(p^3q-q^3p)\sin\phi\right]$$

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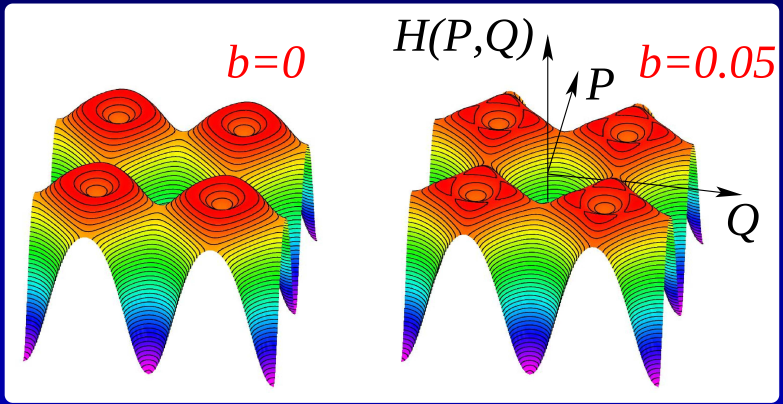
$a_1 = -.55$



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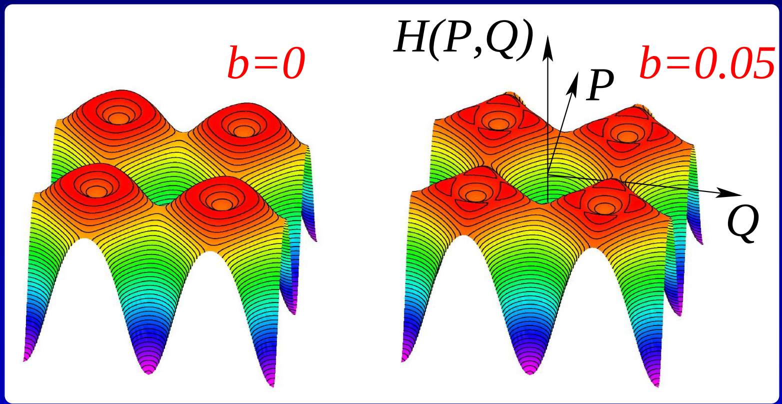

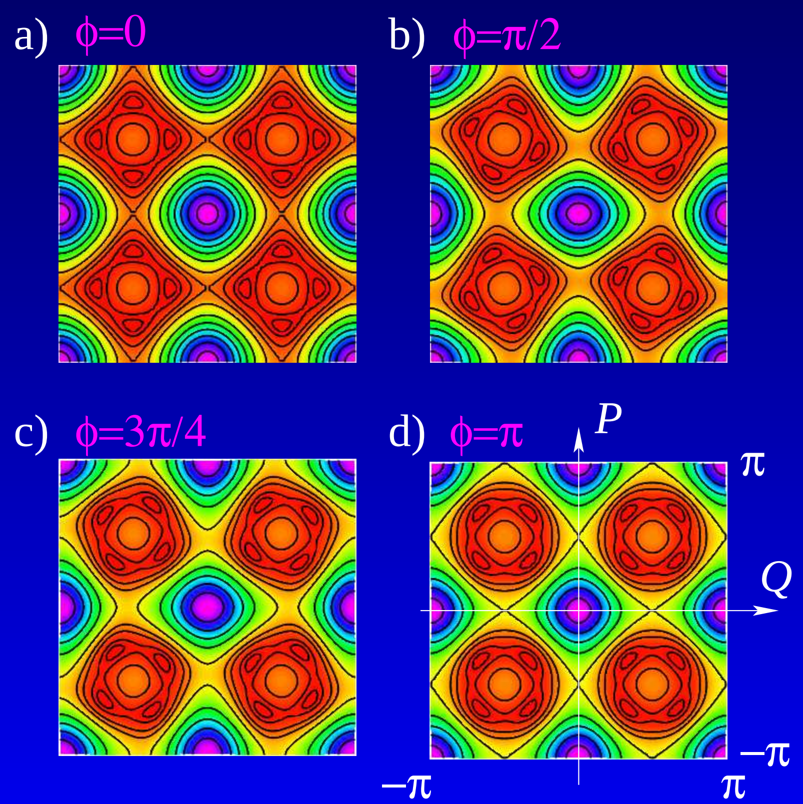

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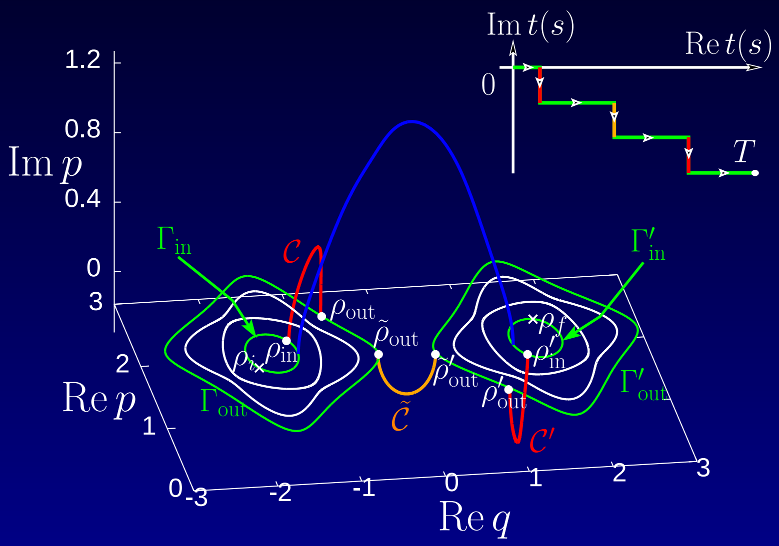

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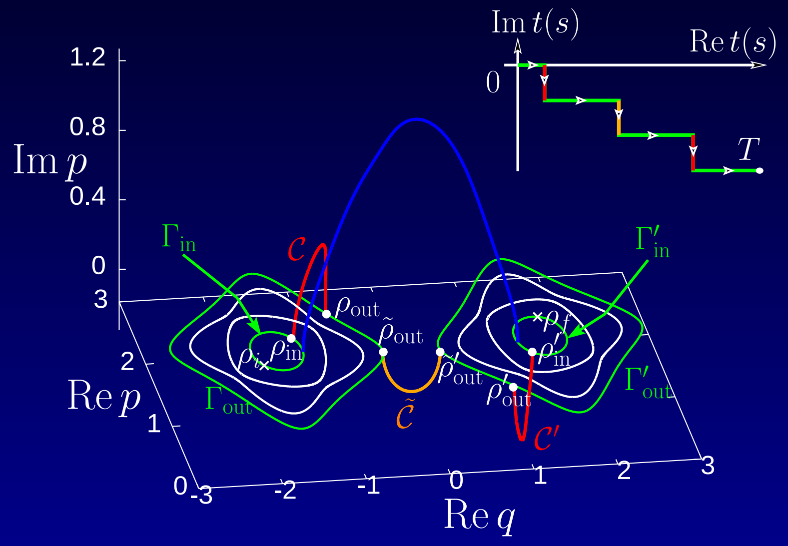

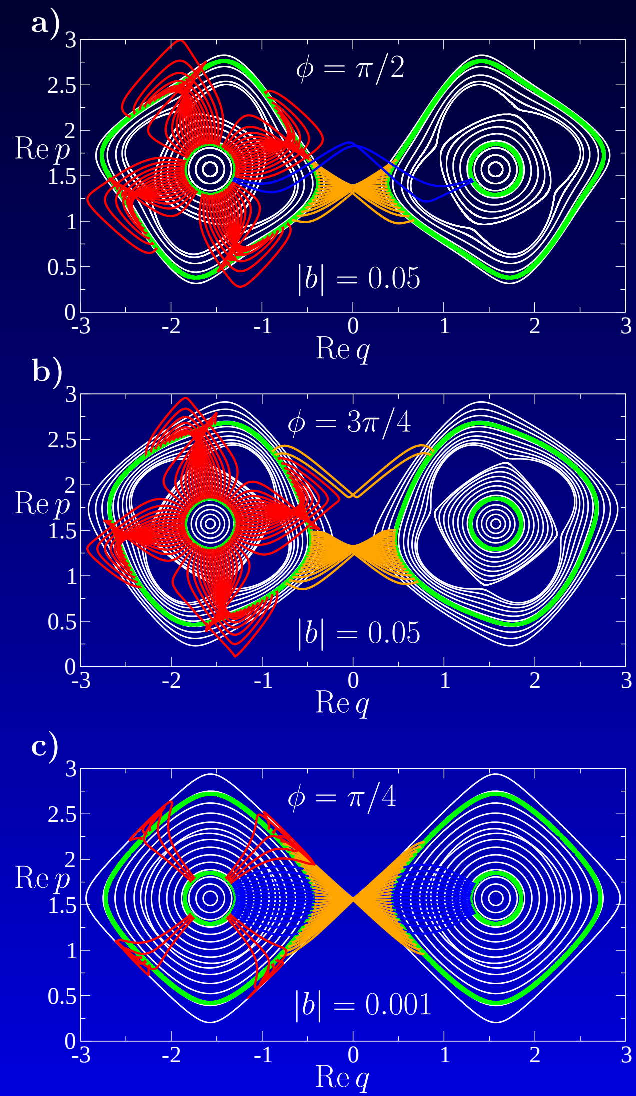

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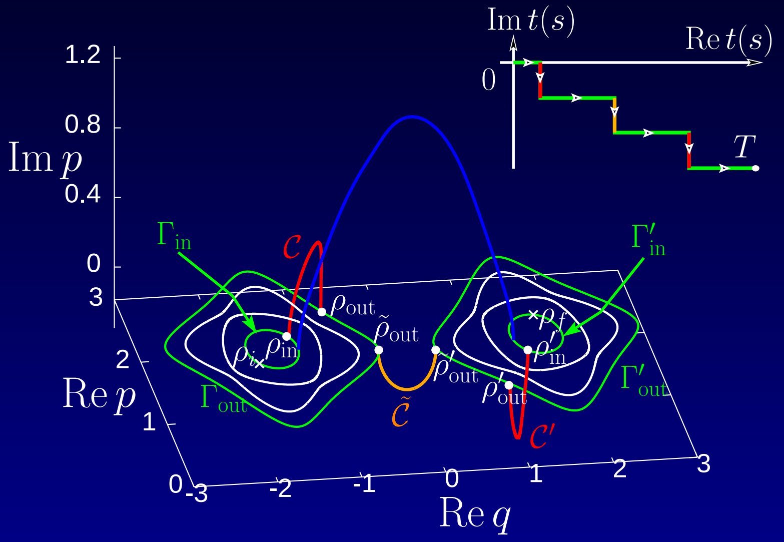

Quasimode (JWKB): $\Psi_n(\theta) \simeq \dfrac{1}{\sqrt{T_{\rm in}(E_n)|\dot\theta|}}\left[\mathrm{e}^{\mathrm{i}S_{\rm in}(\theta,E_n)/\hbar} + A_n\mathrm{e}^{\mathrm{i}S_{\rm out}(\theta,E_n)/\hbar}\right]$

avec $\quad A_n = \dfrac{\mathrm{e}^{-S_{\mathfrak{c}}(E_n)/(2\hbar)}}{2\sin\left[\big(S_{\rm in}(E_n) - S_{\rm out}(E_n)\big)/(2\ell\hbar)\right]}$

et $\quad \delta E_n = \dfrac{2\hbar\omega_{\rm out}}{\pi}\,\mathrm{e}^{-S_{\tilde{\mathfrak{c}}}(E_n)/(2\hbar)}$

$\Delta E_n = |A_n|^2\delta E_n$

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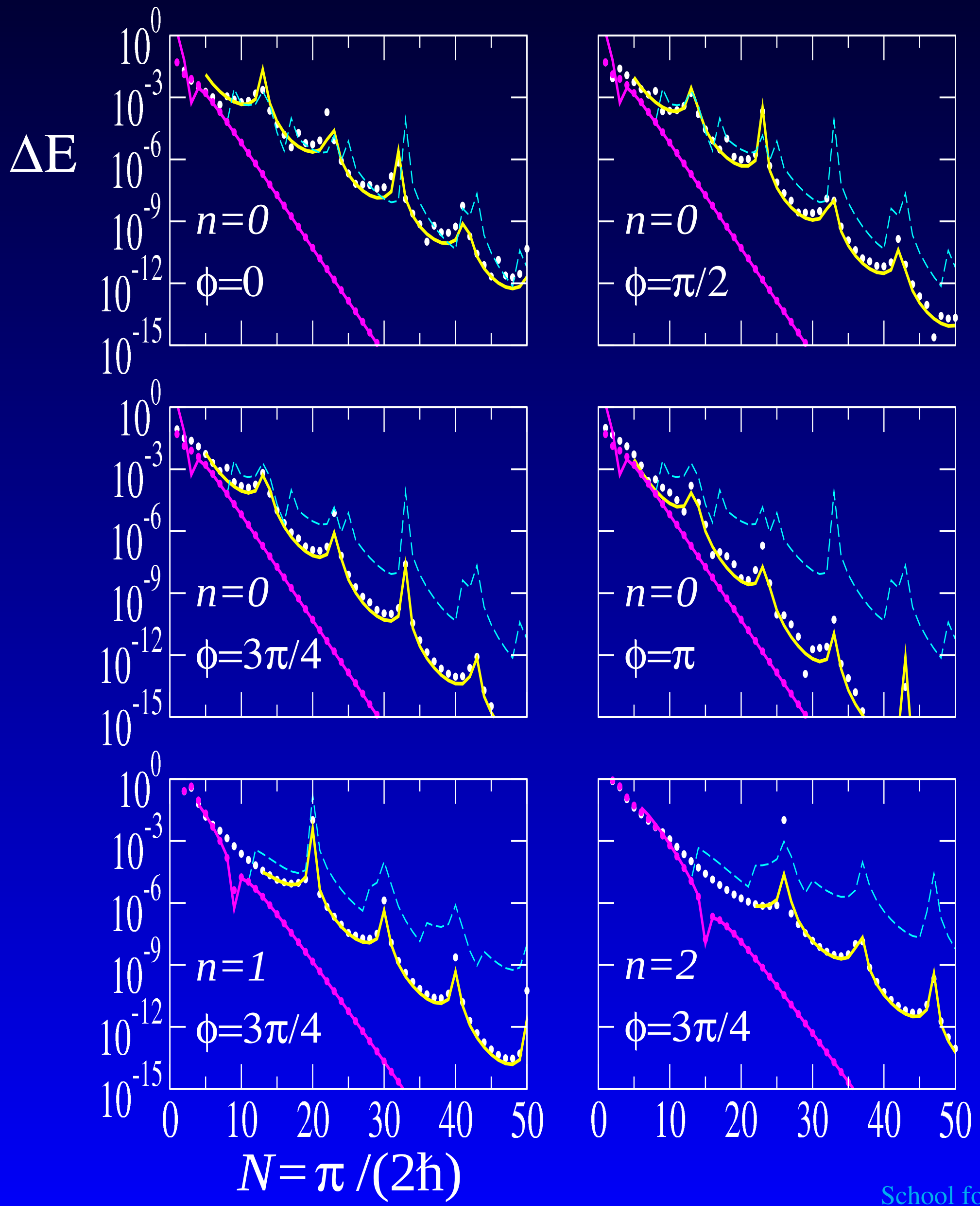

$a_1 = -.55$, $|b| = 0.05$

RAT vs semiclassical

$$A_n = \frac{e^{-S_{\mathfrak{c}}(E_n)/(2\hbar)}}{2\sin\left[\frac{\left(S_{\text{in}}(E_n)-S_{\text{out}}(E_n)\right)}{2\ell\hbar}\right]}$$

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Perspective: things to do and open questions

- ☞ Selection of the class of dominating complex orbits?

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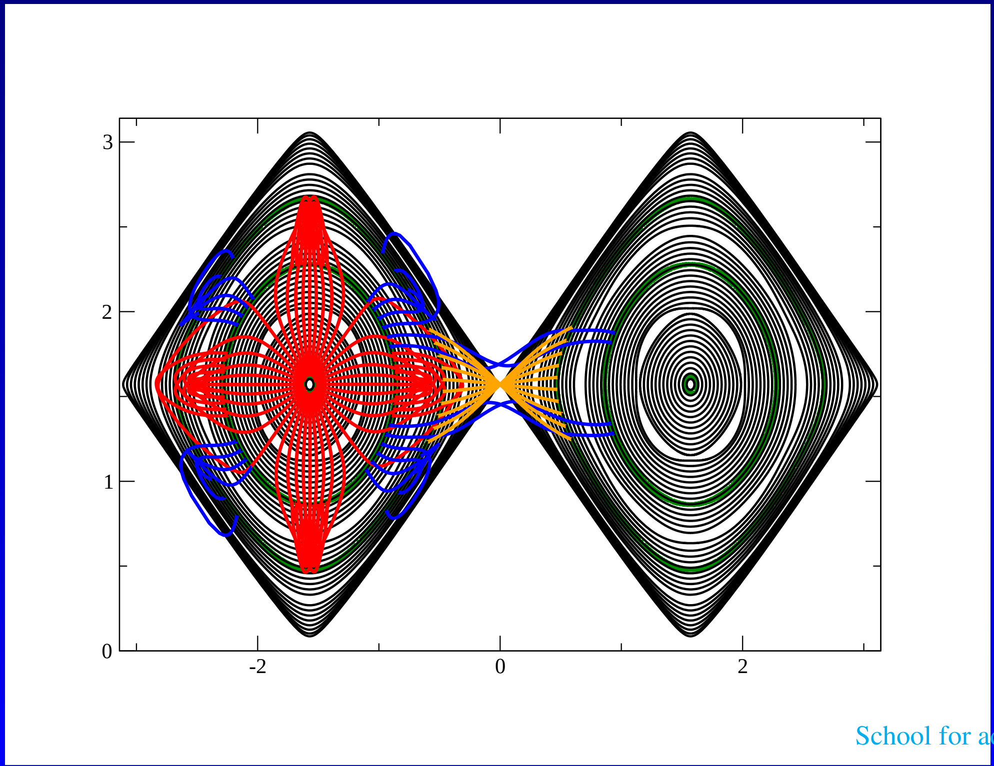
Two resonances chains can be taylored

$$v(I,\theta) = b_1(I - I_2)(2I)^{r_1/2}\cos(r_1\theta + \phi_1) + b_2(I - I_1)(2I)^{r_2/2}\cos(r_2\theta + \phi_2)$$



$r_1 = 4$

$r_2 = 6$

LE DEUNFF, MOUCHET, SCHLAGHECK work in progress.

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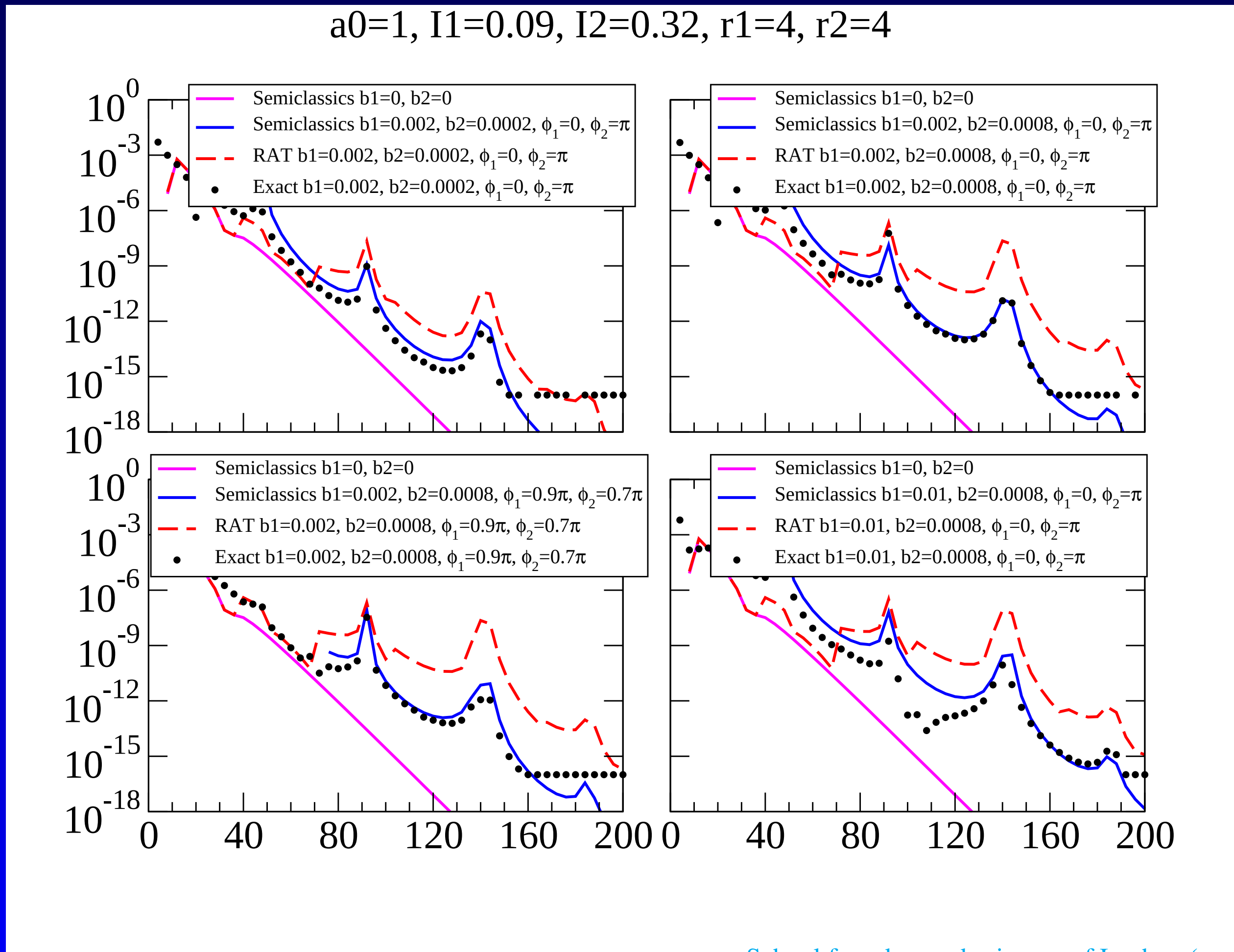

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- ☞ Connection with the work of the Japanese/Dresden school on complex structures (e.g. [MERTIG *et al.*, 2013])?
  $\longrightarrow$ see also AKIRA SHUDO, YASUTAKA HANADA and HIROMITSU HARADA's talks.

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- ☞ Recover multidimensionnal (non resonant) WILKINSON-CREAGH's formulae (1998)?

$$\Delta E \simeq \frac{\hbar^{3/2}}{\sqrt{\tau_R\{I_L, I_R\}\tau_L}} \mathrm{e}^{-K(I)/\hbar}$$

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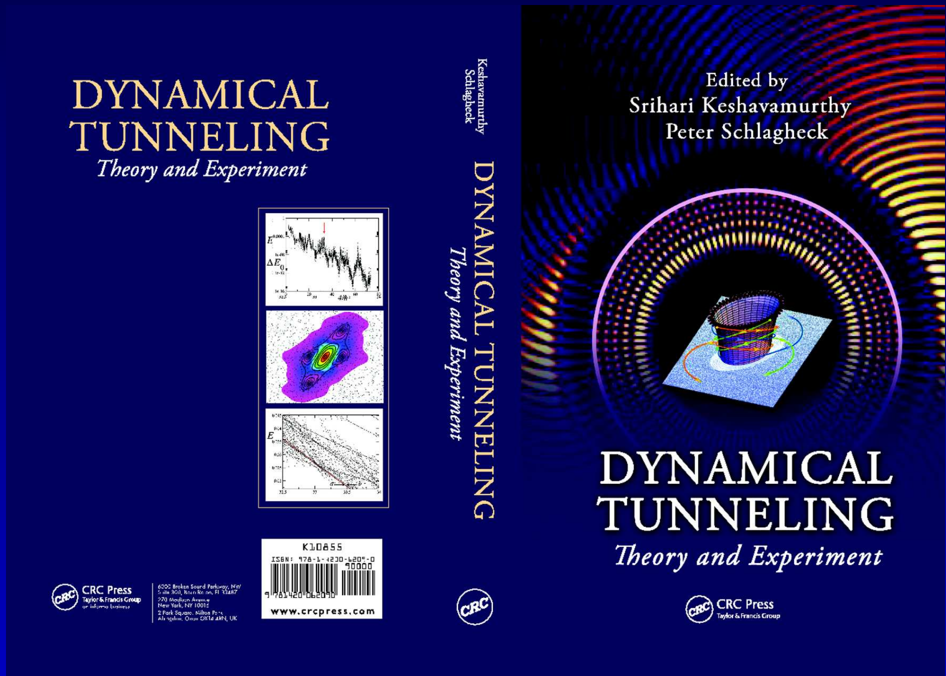
- Can $\mathrm{Im}\, T$ be interpreted as a dissipation which destroys tunnelling as in the CALDEIRA-LEGGETT (1981) model?

PROPAGANDA

☞ (2011)



☞ *Semiclassical description of resonance-assisted tunneling in one-dimensional integrable models* J. LE DEUNFF, A. MOUCHET & P. SCHLAGHECK *Phys. Rev. E* **88**, 04292 (2013)

☞ *Instantons revisited: dynamical tunnelling and resonant tunnelling* J. LE DEUNFF & A. MOUCHET *Phys. Rev. E* **81**, 046205 (2010)