# Normally hyperbolic trapping: from quantum dispersion to classical mixing

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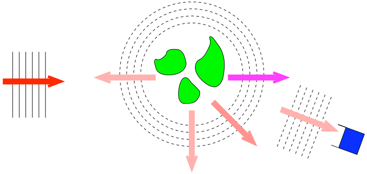
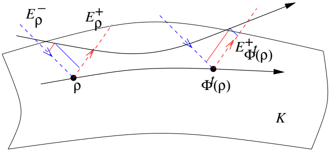
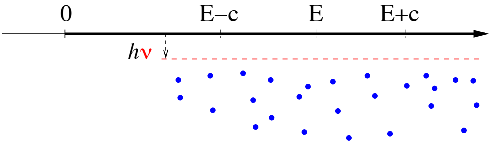


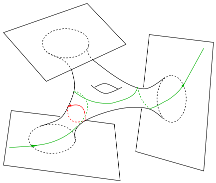


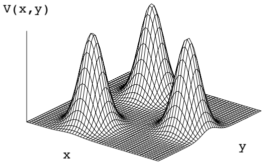



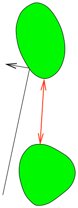
# Outline

- reminder on quantum (/wave) scattering, resonance spectrum.
- **semiclassical** distribution of long-living resonances near energy $E$ $\longleftrightarrow$ structure of set $K_E$ of *classical trapped trajectories*
- focus: $K_E$ **normally hyperbolic** symplectic submanifold
- Normal hyperbolicity $\Longrightarrow$ explicit resonance gap
- Application to classical chaos: *quantitative exponential mixing* for Anosov geodesic flows

# Classical and quantum scattering





$(X, g)$ of infinite volume, Euclidean outside of a bounded region
$\rightarrow$ *scattering* by geometry / potential / obstacles

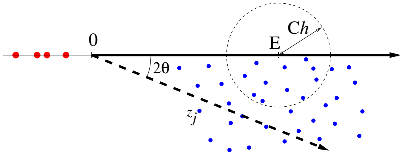
**Classical scattering:** particles follow the geodesic / Hamiltonian flow (with reflection on obstacles).

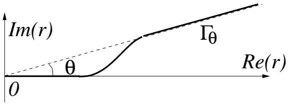
**Quantum scattering:** wave propagation. Two types of situations:

- Schrödinger equation: $i\hbar\partial_t\psi = H_\hbar\psi$, with the Hamiltonian operator $H_\hbar \stackrel{\text{def}}{=} -\frac{\hbar^2\Delta}{2} + V(x)$, or $H_\hbar = -\frac{\hbar^2\Delta_{Dir}}{2}$
- wave equation $(\partial_t^2 - \Delta)\psi(x,t) = 0$ ($\Leftrightarrow$ Schrödinger with $H_\hbar = \sqrt{-\hbar^2\Delta}$)

High frequency régime: fix $E > 0$, take the semiclassical limit $\hbar \to 0$.

# Quantum resonances



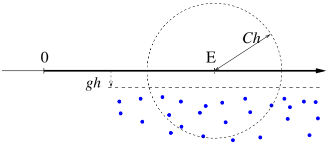




$X$ Euclidean near infinity $\Longrightarrow$ Spec($H_\hbar$) purely abs. continuous on $\mathbb{R}^+$.
$\Longrightarrow$ the resolvent $(H_\hbar - z)^{-1}$ diverges when $\operatorname{Im} z \to 0$.

- however, the Green's function $G(y, x; z) = \langle y|(H_\hbar - z)^{-1}|x\rangle$ can be meromorphically continued from $\{\operatorname{Im} z > 0\}$ to $\{\operatorname{Im} z < 0\}$.
  Poles (of finite multiplicity) = **resonances** $\{z_j(\hbar)\}$ (indep. of $x, y$)
- each $z_j \longleftrightarrow$ **lifetime** $\tau_j(\hbar) = \frac{\hbar}{2|\operatorname{Im} z_j|}$
  **Long-living** resonance: $|\operatorname{Im} z_j(\hbar)| \leq C\hbar$
- A way to uncover resonances: **complex deformation** of $H_\hbar$
  [Aguilar-Balslev-Combes,Simon,Helffer-Sjöstrand..]

  $H_\hbar$ on $\Gamma_\theta \Longleftrightarrow H_{\hbar,\theta}$ on $X$, $H_{\hbar,\theta} = -e^{-2i\theta}\hbar^2\frac{\Delta}{2}$ for $|x| > R$

## Questions in the semiclassical régime $h \ll 1$




- For $E > 0$ fixed, what is the semiclassical distribution of the **long-living** resonances $z_j(\hbar)$ near $E$? Resonance-free strip?
- bounds on $G(x, y; z)$ (or on the cutoff resolvent operator) for $z$ in the resonance free strip?
- Gap + good resolvent bound $\Longrightarrow$ fast decays as $t \to \infty$
  - Schrödinger "correlations" $\langle e^{-itH_\hbar/\hbar}\psi_1, \psi_2\rangle$
  - wave eq.: local energy $\mathcal{E}_\Omega(\psi(t)) \stackrel{\text{def}}{=} \frac{1}{2}\int_\Omega (|\partial_t \psi(t,x)|^2 + |\nabla\psi(t,x)|^2) dx$
  - correlations for Anosov geodesic flow $\int f(x) g(\varphi^t x)\, dx - \int f(x) dx \int g(x) dx$

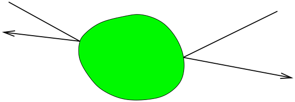
# Semiclassical distribution of resonances - Trapped set

**Main idea**: the distribution of long-living resonances near $E$ is guided by the set of **trapped classical trajectories** for the Hamiltonian flow $\Phi^t$,

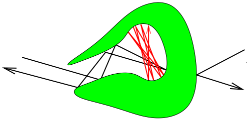
$$K_E = \{(x,p) \in T^*X,\ H(x,p) = E,\ \Phi^t(x,p) \not\to \infty,\ t \to \pm\infty\}$$

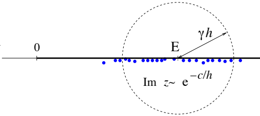
$K_E$ compact subset of $\{H(x,p) = E\}$, invariant through $\Phi^t$.

- $K_E = \emptyset \Longrightarrow \operatorname{Im} z_j \leq -C\hbar \log \hbar^{-1}$.
  No long-living resonances
  [LAX-PHILLIPS'69. . . MARTINEZ'02].



- $K_E$ contains a stable periodic orbit.
  Resonances $\operatorname{Im} z_j(\hbar) = \mathcal{O}(\hbar^\infty)$: very long lifetimes
  [POPOV, VODEV, TANG-ZWORSKI, STEFANOV]





# Normally hyperbolic trapped set

Focus on the case where $K = \cup_{|E'-E|\leq\delta} K_{E'}$ is a (smooth) $2d_{\parallel}$-dimensional symplectic submanifold of $T^*X$, and such that the transverse dynamics is hyperbolic. *Normally Hyperbolic Invariant Manifold* [WIGGINS'94...]

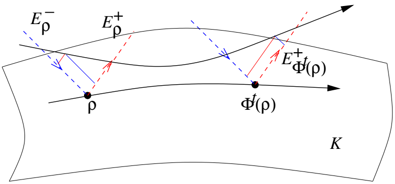
$$forall \rho \in K, \quad T_\rho(T^*X) = T_\rho K \oplus (T_\rho K)^\perp, \quad (T_\rho K)^\perp = E_\rho^- \oplus E_\rho^+, \quad \dim E_\rho^\pm = d - d_\parallel$$

$E_\rho^-$, $E_\rho^+$ are the transverse stable and unstable subspaces:

$$\forall \rho \in K, \quad \forall t > 0, \quad \|d\Phi^t \restriction_{E_\rho^-}\| \leq C\, e^{-\lambda t}, \quad \|d\Phi^{-t} \restriction_{E_\rho^+}\| \leq C\, e^{-\lambda t}$$

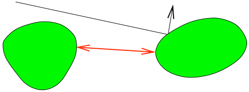
The subspaces $\{E_\rho^\mp,\ \rho \in K\}$ are $\Phi^t$-invariant, and assumed continuous w.r.t. $\rho$.

$E_\rho^\mp$ tangent to the stable/unstable manifolds $\Gamma^\mp$.



# 1st example: trapped set = 1 hyperbolic orbit

- $K_E$ = single hyperbolic periodic orbit ($d_\parallel = 1$)
[IKAWA'85,GÉRARD-SJÖSTRAND'87,GÉRARD'88. . . ]



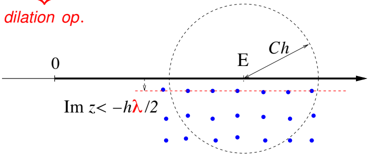
Construct a **Quantum Normal Form** for $H_\hbar$ near the orbit

Ex. (d=2): NF variables $(x_1, x_2) \in \mathbb{R} \times \mathbb{S}^1$, $K_E = \{x_1 = p_1 = p_2 = 0, x_2 \in \mathbb{S}^1\}$

$$\textbf{NF:} \quad H(x_1, p_1, x_2, p_2) = E + \lambda_E\, x_1 p_1 + \frac{p_2}{T_E} + \dots$$

$$\textbf{QNF:} \quad U_\hbar^* H_\hbar U_\hbar \equiv E + \lambda_E \underbrace{\frac{\hbar}{2i}(x_1\,\partial_{x_1} + \partial_{x_1} x_1)}_{\text{dilation op.}} + \frac{\hbar \partial_{x_2}}{i T_E} + \dots \text{ on } L^2(\mathbb{R} \times \mathbb{S}^1)$$

$\rightsquigarrow$ explicit resonances near $z = E$:
deformed half-lattice

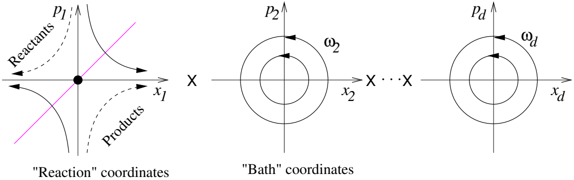



$$z_{\ell,k}(\hbar) = E(\hbar) - i\hbar\lambda_E(1/2 + \ell) + \frac{\hbar k}{T_E} + \mathcal{O}(\hbar^2), \quad \ell \in \mathbb{N}, \; k \in \mathbb{Z}$$

Hyperbolicity $\Longrightarrow$ resonance gap: **hyperbolic dispersion**

## Another example from quantum chemistry

Chemical reaction dynamics [GOUSSEV-SCHUBERT-WAALKENS-WIGGINS'10]:
Neighbourhood of a saddle-center-center fixed point ($d_\parallel = d - 1$)




Quadratic approximation near the fixed point:

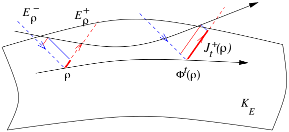
$$H(x,p) = E + \lambda\, x_1 p_1 + \sum_{k=2}^{d} \frac{\omega_k}{2}(x_k^2 + \xi_k^2) + \ldots, \quad K = \{x_1 = p_1 = 0\}$$

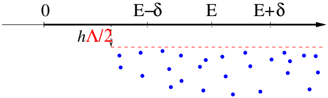
$$H_\hbar = E + \lambda \frac{\hbar}{2i}(x_1\,\partial_{x_1} + \partial_{x_1} x_1) + \sum_{k=2}^{d} \frac{\omega_k}{2}\big(-\hbar^2 \partial_{x_k}^2 + x_k^2\big) + \ldots$$

Nonresonance condition on the $\omega_2, \ldots, \omega_d \Longrightarrow$ **QNF**

Explicit resonances : $z_{\ell,\boldsymbol{n}} \approx E - i\hbar\lambda(1/2 + \ell) + \sum_{k=2}^{d} \hbar\omega_k(n_k + 1/2)$.

# Our main result: Normal hyperbolicity implies a resonance gap




If the dynamics on $K$ is not integrable, NO normal forms, NO expression for resonances. Still, one can prove a resonance gap.

Normal hyperbolicity $\rightarrow |\det d\Phi^t \restriction_{E_\rho^+}| \sim e^{\Lambda(\rho)t}$ for $t \gg 1$

$\rightsquigarrow$ minimal transverse expanding rate $\Lambda \stackrel{\text{def}}{=} \inf_{\rho\in K} \Lambda(\rho)$

## Theorem (N-Zworski'14)

*Assume the trapped set $K$ is a normally hyperbolic symplectic manifold. Then, for $\delta, \epsilon > 0$ and $\hbar > 0$ small enough, the strip $\{|E - \operatorname{Re} z| \leq \delta,\ 0 \geq \operatorname{Im} z \geq -\hbar\Lambda/2 + \epsilon\}$ is free of resonances.*
*(+ polynomial bound for the resolvent in the strip)*

Intuition: wavepackets localized on $K$ disperse exponentially fast along $\Gamma^+$, due to transverse hyperbolicity.

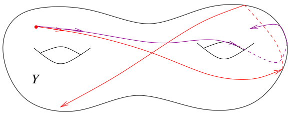
Consequences: exponential decay for wave dynamics

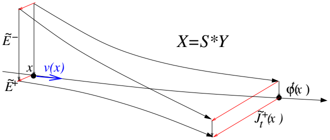
# A non-quantum application: exponential mixing for Anosov flows

$(Y, g)$ compact Riemannian manifold of negative curvature. $X = S^* Y$ (unit cotangent bundle) carries the geodesic flow $\varphi^t$, generated by $v(x) \in T_x X$

Negative curvature $\Longrightarrow$ the flow $\varphi^t$ is Anosov (uniformly hyperbolic):

$$T_x X = \mathbb{R} v(x) \oplus \tilde{E}_x^+ \oplus \tilde{E}_x^- , \qquad \| d\varphi^{\mp t} \restriction_{\tilde{E}_x^\pm} \| \leq C\, e^{-\nu t}, \quad t > 0.$$




$\Longrightarrow$ $\varphi^t$ ergodic and mixing w.r.t. Liouville measure: decay of correlations
$C_{fg}(t) \stackrel{\text{def}}{=} \int f(x) g(\varphi^t(x))\, dx - \int f(x) dx \int g(x)\, dx \xrightarrow{t \to \infty} 0$

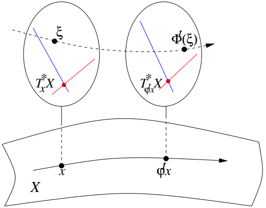
[DOGOPYAT'98,LIVERANI'04]: the mixing is exponential : $|C_{fg}(t)| \leq e^{-\gamma t}$

The decay is controlled by Ruelle–Pollicott resonances $\{Z_j\}$ ($\mathrm{Im}\, Z_j < 0$).

Question: how are the R-P resonances distributed?

# Anosov flow $\equiv$ scattering problem with $K$ Normal. Hyp.

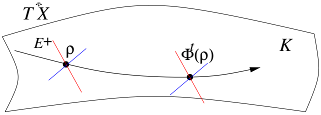
Original idea [FAURE-SJÖSTRAND'10]: analyze $\varphi^t : X \to X$ as a quantum scattering propagator



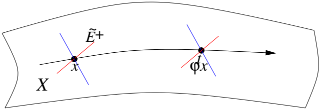

Fact: the transfer operator $\mathcal{L}^t f = f \circ \varphi^{-t}$ is identical to the quantum propagator $\mathcal{L}^t = e^{-itH_\hbar/\hbar}$, for the Hamiltonian $H_\hbar = \frac{\hbar}{i} v(x) \cdot \partial_x$

$\rightsquigarrow$ resonances of $H_\hbar \equiv$ R-P resonances : $z_j(\hbar) = \hbar Z_j$

The corresponding classical Hamiltonian $H(x,p) = v(x) \cdot p$ on $T^*X$ generates the Hamiltonian flow $\Phi^t : T^*X \to T^*X$, lift of $\varphi^t : X \to X$.




- $\forall E$, the energy shell $\{H(x,p) = E\}$ is unbounded in the momentum direction ($\simeq$scattering system)
- $\varphi^t$ preserves the Liouville 1-form $\alpha$ on X $\Longrightarrow$ trapped set $K_E = \{(x, p = E\alpha_x),\ x \in X\}$.
  $K = \cup_E K_E$ normally hyperb. smooth submanifold,
  $E^\pm_\rho$ = lift of $\tilde{E}^\pm_x$,
  $\Lambda = \tilde{\Lambda}$ minimal expanding rate along $\tilde{E}^+$



# Applying our gap result to the Ruelle-Pollicott resonances

## Theorem

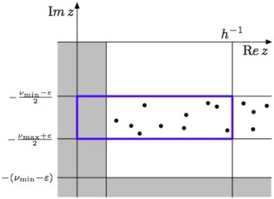
*Consider the geodesic flow on $(Y, g)$ compact of negative sectional curvature.*
*Then there can be at most finitely many Ruelle-Pollicott resonances $Z_j$ in the strip $\{0 \geq \operatorname{Im} Z_j \geq -\tilde{\Lambda}/2 + \epsilon\}$.*
*As a consequence, the correlations $C_{fg}(t)$ decay as*

$$\Longrightarrow C_{fg}(t) = \sum_{\operatorname{Im} Z_j > -\tilde{\Lambda}/2} e^{-iZ_j t} M_j(f, g) + \mathcal{O}(e^{-t\tilde{\Lambda}/2})$$

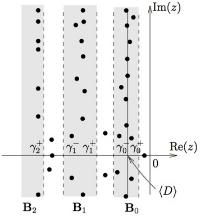
$(\tilde{\Lambda} = \inf_{x \in X} \liminf_{t \to \infty} \frac{1}{t} \log |\det d\varphi^t \restriction_{E^+(x)} |)$

Same result by [Tsujii'10,'12], by studying the action of $\mathcal{L}_t$ on anisotropic Sobolev spaces adapted to the dynamics.

## Beyond this resonance gap: resonances in strips




[DYATLOV'13]




[FAURE-TSUJII'13]

- wave propagation on Kerr(-de Sitter) metrics.
  Assuming pinching condition $\Lambda_{\text{max}} < 2\Lambda_{\text{min}}$, resonances in isolated strip $\{-\frac{\nu_{\text{max}}}{2} \leq \operatorname{Im} z/\hbar \leq -\frac{\nu_{\text{min}}}{2}\}$. Counting satisfies a Weyl's law. [DYATLOV'13]
- Anosov flow: same type of result for Ruelle-Pollicott resonances [FAURE-TSUJII'13].

Thank you for your attention, and good appetite!

# Applications to wave decay

- Schrödinger eq.: $\psi_1, \psi_2 \in L^2(B(0,R))$, $\chi \in C^\infty((E-\delta, E+\delta))$.
  Exponential decay of "correlations":

  $$\langle \psi_2, e^{-itH_\hbar/\hbar}\chi(H_\hbar)\psi_1\rangle \leq C_R \hbar^{-\beta} e^{-\Lambda t/2} + C_{R,N}\hbar^N, \text{ for all } t > 0.$$

- $X$ odd-dimensional. $(\partial_t^2 - \Delta_X)\psi = 0$, with $(\psi(0), \partial_t\psi(0)) \in C_c^\infty(X)$.
  For $\Omega \subset X$ bounded, exponential decay of the local energy:

  $$\mathcal{E}_\Omega(\psi(t)) \leq C_\epsilon\, e^{-\nu_\epsilon t}\left(\|\psi(0)\|^2_{H^{1+\epsilon}} + \|\partial_t\psi(0)\|^2_{H^\epsilon}\right).$$

- wave propagation in certain stationary Lorentzian metrics: perturbations of slowly-rotating Kerr (-de Sitter) metrics $\Longrightarrow$ $K$ normally hyperbolic.
  $\rightsquigarrow$ resonance gap [WUNSCH-ZWORSKI'10, DYATLOV'13,'14]
  (resonances = **Quasinormal modes**)
  $\rightsquigarrow$ local energy decay for $\psi_\lambda(0)$ concentrated near frequency $\lambda$:

  $$\mathcal{E}_\Omega(\psi_\lambda(t)) \leq C\,\lambda^{1/2} e^{-\Lambda t/2}\left(\|\psi_\lambda(0)\|^2_{H^1} + \|\partial_t\psi_\lambda(0)\|^2_{L^2}\right), \quad t \leq T\log\lambda\,.$$

# Normal hyperbolicity implies a resonance gap

**Theorem** (N-ZWORSKI'13)

*Assume $K$ is a normally hyperbolic smooth symplectic manifold, with $C^0$ invariant distributions.*
*Then, for any $\Lambda' < \Lambda$, the cutoff resolvent $\|R_\chi(z;\hbar)\| \leq C\,|\log\hbar|\,\hbar^{-1+c_0\,\mathrm{Im}\,z/h}$ in the strip $\{|E-\mathrm{Re}\,z| \leq \delta,\ 0 \geq \mathrm{Im}\,z \geq -\hbar\Lambda'/2\}$.*

[GÉRARD-SJÖSTRAND'88]: same gap for $P(x,hD)$ analytic differential op. on $\mathbb{R}^d$, weaker dynamical conditions: $K \subset \Sigma$ a $C^1$ symplectic submanifold, normally hyperbolic, $C^0$ invariant distributions.
Exponentially large resolvent estimate.

[WUNSCH-ZWORSKI'10]: $C^\infty$ setting, $K$ smooth symplectic, $\Gamma^\pm$ smooth of codimension 1 $\Longrightarrow$ (non-explicit) gap, resolvent estimates.

[DYATLOV'13,'14]: same assumptions as in [WUNSCH-ZWORSKI'10], + orientability of $\Gamma^\pm \Longrightarrow$ gap $\Lambda/2$ , sharper resolvent estimates.
(Much) simpler proof: no need for a refined escape function.

[TSUJII'12, FAURE-TSUJII'13], Anosov flow: explicitly use the transverse hyperbolic dispersion to compute the gap (and more..).

# Proof (1): making $H_\hbar$) absorbing away from $K$

1. Complex-deform $H_\hbar$ outside the "interaction region" $\Omega_{\rm int} \stackrel{\rm def}{=} \{|x| \leq R_0\}$, with angle $\theta = C\,\hbar|\log\hbar|$. $\rightsquigarrow$ nonselfadjoint op. $H_{\hbar,\theta}$.
In the energy shell $\mathcal{E}_E^\delta = \{|H(x,p) - E| \leq \delta\}$, its symbol $H_\theta(\rho)$ satisfies

$$\mathrm{Im}\, H_\theta(\rho) \leq -c\,\hbar\,|\log\hbar| \quad \text{for} \quad \rho \in \mathcal{E}_E^\delta \setminus \Omega_{\rm int}$$

$\Longrightarrow$ $H_{\hbar,\theta}$ is absorbing outside $\Omega_{\rm int}$:

$$\text{for any } \psi \text{ microlocalized in } \mathcal{E}_E^\delta \setminus \Omega_{\rm int}, \quad \|e^{-iH_{\hbar,\theta}/\hbar}\psi\| \leq e^{-c|\log\hbar|}\|\psi\|\,.$$

2. Extend absorption outside a thin neighbourhood $K(\hbar^{1/2})$.
Strategy: using normal hyperbolicity, construct an adapted escape function $g(x,pi;h)$:

$$\rho \in \mathcal{E}_E^\delta \setminus K(\hbar^{1/2}) \Longrightarrow \{H,g\}(\rho) \geq C > 0\,.$$

Take $G = \mathrm{Op}_\hbar(g) \Longrightarrow H_G \stackrel{\rm def}{=} e^{-G}\,H_{\hbar,\theta}\,e^{G} = H_{\hbar,\theta} - i\hbar\,\mathrm{Op}_\hbar(\{H,g\}) + \ldots$
absorbing outside $K(\hbar^{1/2})$:

$$\text{for any } \psi \text{ microlocalized in } \mathcal{E}_E^\delta \setminus K(\hbar^{1/2}), \quad \|e^{-iH_G/\hbar}\psi\| \leq e^{-C}\,\|\psi\|\,.$$

# Proof (2): transverse hyperbolic dispersion on $K(h^{1/2})$

3. Use local adapted Darboux coordinates $(x, x'; p, p')$ near $K = \{x = p = 0\}$: $K(\hbar^{1/2}) \equiv \{|x|^2 + |p|^2 \leq \hbar\}$,
Take $\chi(x, p; h)$ a transverse cutoff supported in $K(2\hbar^{1/2})$, $\chi = 1$ in $K(\hbar^{1/2})$.

Near $K$, write the propagator $e^{-itH_\hbar/\hbar}$ as the product of
– a unitary propagator on $L^2(\mathbb{R}_x^{d_\parallel})$, quantizing $\Phi^t\restriction_K: K \to K$)
– an operator $\mathrm{Op}_\hbar(M_t)$ on $L^2(\mathbb{R}_x^{d_\parallel})$, with symbol $M_t(x', p')$ taking values in the metaplectic operators on $L^2(\mathbb{R}_{x'}^{d_\perp})$. $M_t(x', p')$ quantizes the linearised (hyperbolic) transverse map $d\Phi^t\restriction_{(TK)^\perp}(x', p')$

$\Longrightarrow$ hyperbolic dispersion estimate from the linearized transverse dynamics:

$$\forall (x', p') \in K,\ \forall t > 0, \quad \|\,\mathrm{Op}_h(\chi) M_t(x', p')\,\mathrm{Op}_h(\chi)\|_{L^2_x \to L^2_x} \leq C\, J_t^+(\rho)^{-1/2}$$

$$\Longrightarrow \|\,\mathrm{Op}_h(\chi)\,\mathrm{Op}_h(M_t)\,\mathrm{Op}_h(\chi)\|_{L^2_{x,x'} \to L^2_{x,x'}} \leq C\, e^{-t\Lambda/2}$$

$$\Longrightarrow \|\,\mathrm{Op}_h(\chi) e^{-itP(h)/h}\,\mathrm{Op}_h(\chi)\|_{L^2_{x,x'} \to L^2_{x,x'}} \leq C\, e^{-t\Lambda/2}.$$

4. Combine the estimates near and away from $K \rightsquigarrow$ for any $\psi \in L^2$ microlocalized inside $\mathcal{E}_E^\delta$, in particular for $\psi$ an eigenstate of $H_G$ :

$$\|e^{-itH_G/\hbar}\psi\| \leq C\, e^{-t\Lambda/2}\, \|\psi\|, \quad t > 0 \quad \text{(indep. of } \hbar) \quad \square$$

## Proof (3): from propagator to resolvent estimate

5. Take $a \in C^\infty(T^*X)$ with $\operatorname{supp} a \subset \mathcal{E}_E^\delta$, $a \equiv 1$ in $\mathcal{E}_E^{\delta/2}$.:

$$\|e^{-itP_G/h}\operatorname{Op}_\hbar(a)\|_{L^2\to L^2} \le C\, e^{-t\Lambda/2}, \quad t>0 \quad \text{indep. of } h$$

- For $\operatorname{Im} z > -\Lambda'/2$, construct a parametrix for $(P_G - z)^{-1}$ on $\operatorname{supp} a$:
  Take $Q_a \stackrel{\text{def}}{=} \frac{i}{h}\int_0^T e^{-it(P_G-z)/h}\operatorname{Op}_\hbar(a) = \mathcal{O}(h^{-1})$.
  Then $(P_G - z)Q_a = \left(I - e^{-iT(P_G-z)/h}\right)\operatorname{Op}_\hbar(a) = \operatorname{Op}_\hbar(a) + \mathit{small}$ if $T \gg 1$
- $(P_G - z)$ semiclassically elliptic on $\operatorname{supp}(1-a)$
  $\rightsquigarrow$ construct $Q_{1-a} = \mathcal{O}(1)$ s.t. $(P_G - z)Q_{1-a} = (I - \operatorname{Op}_\hbar(a)) + \mathit{small}$

$$\Longrightarrow (P_G - z)(Q_a + Q_{1-a}) = Id + \mathit{small}$$
$$\Longrightarrow \|(P_G - z)^{-1}\| = \mathcal{O}(h^{-1}).$$

- by construction $\|e^{\pm G}\|_{L^2\to L^2} = \mathcal{O}(h^{-M})$

$$\Longrightarrow \|(P_\theta - z)^{-1}\| = \mathcal{O}(h^{-1-2M}) \quad \text{in the strip}$$
$$\Longrightarrow \|\chi(P-z)^{-1}\chi\| = \|\chi(P_\theta - z)^{-1}\chi\| = \mathcal{O}(h^{-1-2M}). \quad \square$$