Ergodicity-nonergodicity transitions in driven many-body systems

Tomaž Prosen

Department of Physics, FMF, University of Ljubljana, SLOVENIA

Superbagneres de Luchon, 20 March 2015

subtitle:

MANY-BODY QUANTUM CHAOS **WITHOUT** $\hbar$

Outline

- Quantum ergodicity, decay of correlations and fidelity decay
- Kicked Ising chain
  Integrability breaking ergodicity/non-ergodicity transition
- Heisenberg *XXZ* chain
  Integrable ergodicity/non-ergodicity transition
- Ergodicity/non-ergodicity transition in a completely integrable classical-mechanical model (Lattice-Laudau-Lifshitz)
- Kicked Ising spin system on a 2D lattice – dynamical phase transitions

Prosen PTPS 2000, Prosen PRE 2002

$$H(t) = \sum_{j=0}^{L-1} \left\{ J\sigma_j^{\mathrm{z}}\sigma_{j+1}^{\mathrm{z}} + (h_{\mathrm{x}}\sigma_j^{\mathrm{x}} + h_{\mathrm{z}}\sigma_j^{\mathrm{z}}) \sum_{m\in\mathbb{Z}} \delta(t-m) \right\}$$

$$U_{\mathrm{Floquet}} = T \exp\left(-\mathrm{i}\int_{0+}^{1+} \mathrm{d}t' H(t')\right) = \prod_j \exp\left(-\mathrm{i}(h_{\mathrm{x}}\sigma_j^{\mathrm{x}} + h_{\mathrm{z}}\sigma_j^{\mathrm{z}})\right) \exp\left(-\mathrm{i}J\sigma_j^{\mathrm{z}}\sigma_{j+1}^{\mathrm{z}}\right)$$

where $[\sigma_j^\alpha, \sigma_k^\beta] = 2\mathrm{i}\varepsilon_{\alpha\beta\gamma}\sigma_j^\gamma\delta_{jk}$.

Prosen PTPS 2000, Prosen PRE 2002

$$H(t) = \sum_{j=0}^{L-1} \left\{ J\sigma^{\mathrm{z}}_j \sigma^{\mathrm{z}}_{j+1} + (h_{\mathrm{x}}\sigma^{\mathrm{x}}_j + h_{\mathrm{z}}\sigma^{\mathrm{z}}_j) \sum_{m\in\mathbb{Z}} \delta(t-m) \right\}$$

$$U_{\mathrm{Floquet}} = T\exp\left(-\mathrm{i}\int_{0+}^{1+} \mathrm{d}t' H(t')\right) = \prod_j \exp\left(-\mathrm{i}(h_{\mathrm{x}}\sigma^{\mathrm{x}}_j + h_{\mathrm{z}}\sigma^{\mathrm{z}}_j)\right) \exp\left(-\mathrm{i}J\sigma^{\mathrm{z}}_j\sigma^{\mathrm{z}}_{j+1}\right)$$

where $[\sigma^\alpha_j, \sigma^\beta_k] = 2\mathrm{i}\varepsilon_{\alpha\beta\gamma}\sigma^\gamma_j\delta_{jk}$.
The model is *completely integrable* in terms of Jordan-Wigner transformation if

- $h_{\mathrm{x}} = 0$ (longitudinal field)
- $h_{\mathrm{z}} = 0$ (transverse field)

My favourite toy model of many-body quantum chaos: Kicked Ising Chain

Prosen PTPS 2000, Prosen PRE 2002

$$H(t) = \sum_{j=0}^{L-1} \left\{ J\sigma^{\mathrm{z}}_j \sigma^{\mathrm{z}}_{j+1} + (h_{\mathrm{x}}\sigma^{\mathrm{x}}_j + h_{\mathrm{z}}\sigma^{\mathrm{z}}_j) \sum_{m\in\mathbb{Z}} \delta(t-m) \right\}$$

$$U_{\mathrm{Floquet}} = T\exp\left(-\mathrm{i}\int_{0+}^{1+} \mathrm{d}t' H(t')\right) = \prod_j \exp\left(-\mathrm{i}(h_{\mathrm{x}}\sigma^{\mathrm{x}}_j + h_{\mathrm{z}}\sigma^{\mathrm{z}}_j)\right) \exp\left(-\mathrm{i}J\sigma^{\mathrm{z}}_j\sigma^{\mathrm{z}}_{j+1}\right)$$

where $[\sigma^{\alpha}_j, \sigma^{\beta}_k] = 2\mathrm{i}\varepsilon_{\alpha\beta\gamma}\sigma^{\gamma}_j \delta_{jk}$.
The model is *completely integrable* in terms of Jordan-Wigner transformation if

- $h_{\mathrm{x}} = 0$ (longitudinal field)
- $h_{\mathrm{z}} = 0$ (transverse field)

My favourite toy model of many-body quantum chaos: Kicked Ising Chain

Prosen PTPS 2000, Prosen PRE 2002

$$H(t) = \sum_{j=0}^{L-1} \left\{ J\sigma^{\mathrm{z}}_j \sigma^{\mathrm{z}}_{j+1} + (h_{\mathrm{x}}\sigma^{\mathrm{x}}_j + h_{\mathrm{z}}\sigma^{\mathrm{z}}_j) \sum_{m\in\mathbb{Z}} \delta(t-m) \right\}$$

$$U_{\mathrm{Floquet}} = T\exp\left(-\mathrm{i}\int_{0+}^{1+} \mathrm{d}t' H(t')\right) = \prod_j \exp\left(-\mathrm{i}(h_{\mathrm{x}}\sigma^{\mathrm{x}}_j + h_{\mathrm{z}}\sigma^{\mathrm{z}}_j)\right) \exp\left(-\mathrm{i}J\sigma^{\mathrm{z}}_j \sigma^{\mathrm{z}}_{j+1}\right)$$

where $[\sigma^{\alpha}_j, \sigma^{\beta}_k] = 2\mathrm{i}\varepsilon_{\alpha\beta\gamma}\sigma^{\gamma}_j \delta_{jk}$.
The model is *completely integrable* in terms of Jordan-Wigner transformation if

- $h_{\mathrm{x}} = 0$ (longitudinal field)
- $h_{\mathrm{z}} = 0$ (transverse field)

My favourite toy model of many-body quantum chaos: Kicked Ising Chain

Prosen PTPS 2000, Prosen PRE 2002

$$H(t) = \sum_{j=0}^{L-1} \left\{ J\sigma_j^{\mathrm{z}}\sigma_{j+1}^{\mathrm{z}} + (h_{\mathrm{x}}\sigma_j^{\mathrm{x}} + h_{\mathrm{z}}\sigma_j^{\mathrm{z}}) \sum_{m\in\mathbb{Z}} \delta(t-m) \right\}$$

$$U_{\mathrm{Floquet}} = T\exp\left(-\mathrm{i}\int_{0+}^{1+} \mathrm{d}t' H(t')\right) = \prod_j \exp\left(-\mathrm{i}(h_{\mathrm{x}}\sigma_j^{\mathrm{x}} + h_{\mathrm{z}}\sigma_j^{\mathrm{z}})\right) \exp\left(-\mathrm{i}J\sigma_j^{\mathrm{z}}\sigma_{j+1}^{\mathrm{z}}\right)$$

where $[\sigma_j^\alpha, \sigma_k^\beta] = 2\mathrm{i}\varepsilon_{\alpha\beta\gamma}\sigma_j^\gamma\delta_{jk}$.
The model is *completely integrable* in terms of Jordan-Wigner transformation if

- $h_{\mathrm{x}} = 0$ (longitudinal field)
- $h_{\mathrm{z}} = 0$ (transverse field)

Time-evolution of local observables is quasi-exact, e.g. for computing

$$U_{\mathrm{Floquet}}^{-t}\sigma_j^\alpha U_{\mathrm{Floquet}}^{t}$$

only $2t+1$ sites in the range $[j-t, j+t]$ are needed!.
Quantum cellular automaton in the sense of Schumacher and Werner (2004).

Quasi-energy level statistics of KI [C. Pineda, TP, PRE 2007]

Fix $J = 0.7, h_\mathrm{x} = 0.9, h_\mathrm{z} = 0.9$, s.t. KI is (strongly) non-integrable.

Diagonalize $U_\mathrm{Floquet}|n\rangle = \exp(-\mathrm{i}\varphi_n)|n\rangle$. For each conserved total momentum K quantum number, we find $\mathcal{N} \sim 2^L/L$ levels, normalized to mean level spacing as $s_n = (\mathcal{N}/2\pi)\varphi_n$.

Fix $J = 0.7, h_{\mathrm{x}} = 0.9, h_{\mathrm{z}} = 0.9$, s.t. KI is (strongly) non-integrable.

Diagonalize $U_{\mathrm{Floquet}}|n\rangle = \exp(-\mathrm{i}\varphi_n)|n\rangle$. For each conserved total momentum K quantum number, we find $\mathcal{N} \sim 2^L/L$ levels, normalized to mean level spacing as $s_n = (\mathcal{N}/2\pi)\varphi_n$.

$$N(s) = \#\{s_n < s\} = N_{\mathrm{smooth}}(s) + N_{\mathrm{fluct}}(s)$$

Fix $J = 0.7$, $h_{\mathrm{x}} = 0.9$, $h_{\mathrm{z}} = 0.9$, s.t. KI is (strongly) non-integrable.

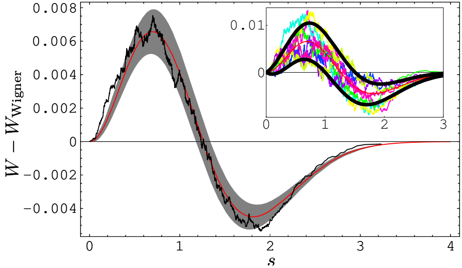
Diagonalize $U_{\mathrm{Floquet}}|n\rangle = \exp(-\mathrm{i}\varphi_n)|n\rangle$. For each conserved total momentum K quantum number, we find $\mathcal{N} \sim 2^L/L$ levels, normalized to mean level spacing as $s_n = (\mathcal{N}/2\pi)\varphi_n$.

$$N(s) = \#\{s_n < s\} = N_{\mathrm{smooth}}(s) + N_{\mathrm{fluct}}(s)$$

For kicked quantum quantum systems spectra are expected to be statistically uniformly dense

$$\mathcal{N}_{\mathrm{smooth}}(s) = s$$

We plot cumulative level spacing distribution
$W(s) = \int_0^s \mathrm{d}sP(s) = Prob\{s_{n+1} - s_n < s\}$.



The noisy curve shows the difference between the numerical data for 18 qubits, averaged over the different momentum sectors, and the Wigner RMT surmise. The smooth (red) curve is the difference between infinitely dimensional COE solution and the Wigner surmise. In the inset we present a similar figure with the results for each of quasi-moemtnum sector K.

Spectral form factor $K_2(\tau)$ is for *nonzero* integer t defined as

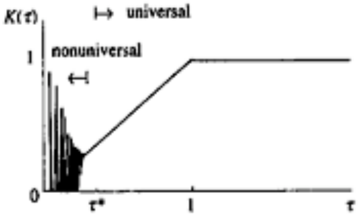
$$K_2(t/\mathcal{N}) = \frac{1}{\mathcal{N}} \left|\mathrm{tr}\, U^t\right|^2 = \frac{1}{\mathcal{N}} \left|\sum_n e^{-\mathrm{i}\varphi_n t}\right|^2.$$

Spectral form factor $K_2(\tau)$ is for *nonzero* integer t defined as

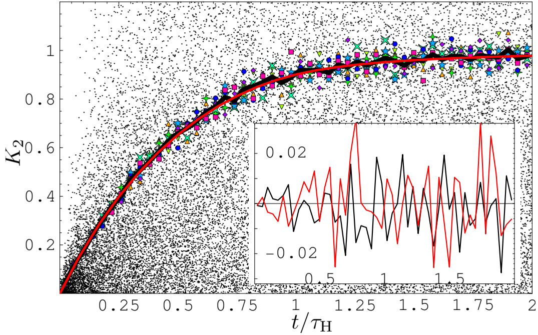
$$K_2(t/\mathcal{N}) = \frac{1}{\mathcal{N}} \left|\mathrm{tr}\, U^t\right|^2 = \frac{1}{\mathcal{N}} \left|\sum_n e^{-\mathrm{i}\varphi_n t}\right|^2 .$$

In non-integrable systems with a chaotic classical lomit, form factor has two regimes:

- universal described by RMT,
- non-universal described by short classical periodic orbits.



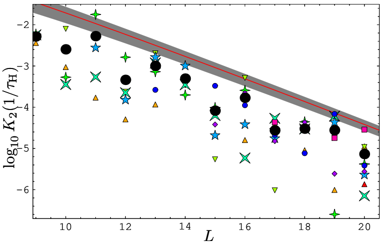
Note that for kicked systems, Heisenberg integer time $\tau_{\mathrm{H}} = \mathcal{N}$

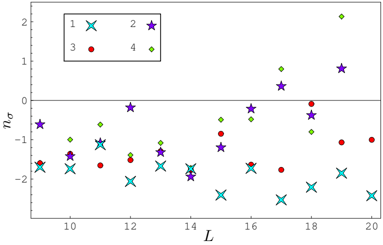


We show the behavior of the form factor for $L = 18$ qubits. We perform averaging over short ranges of time ($\tau_{\mathrm{H}}/25$). The results for each of the K-spaces are shown in colors. The average over the different spaces as well as the theoretical COE(N) curve is plotted as a black and red curve, respectively.

Surprise!? Deviaton from universality at short times

Similarly as for semi-classical systems, we find notable statistically significant deviations from universal COE/GOE predictions for short times of *few kicks*.





But there is no underlying classical structure! Dynamical explanation of this phenomenon needed!

Quantum ergodicity and its "order parameter"

- Temporal correlation of an extensive traceless observable A $(\mathrm{tr}\, A = 0, \mathrm{tr}\, A^2 \propto L)$:

$$C_A(t) = \lim_{L\to\infty} \frac{1}{L2^L} \mathrm{tr}\, A U^{-t} A U^t$$

- Average correlator

$$D_A = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} C_A(t)$$

signals quantum ergodicity if $D_A = 0$.

- Quantum chaos regime in KI chain seems compatible with exponential decay of correlations. For integrable, and weakly non-integrable cases, though, we find saturation of temporal correlations $D \neq 0$.

- Temporal correlation of an extensive traceless observable A $(\mathrm{tr}\, A = 0, \mathrm{tr}\, A^2 \propto L)$:

$$C_A(t) = \lim_{L\to\infty} \frac{1}{L2^L} \mathrm{tr}\, AU^{-t}AU^t$$

- Average correlator

$$D_A = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} C_A(t)$$

signals quantum ergodicity if $D_A = 0$.

- Quantum chaos regime in KI chain seems compatible with exponential decay of correlations. For integrable, and weakly non-integrable cases, though, we find saturation of temporal correlations $D \neq 0$.

Quantum ergodicity and its "order parameter"

- Temporal correlation of an extensive traceless observable A $(\mathrm{tr}\,A = 0, \mathrm{tr}\,A^2 \propto L)$:

$$C_A(t) = \lim_{L\to\infty} \frac{1}{L2^L} \mathrm{tr}\, AU^{-t}AU^t$$

- Average correlator

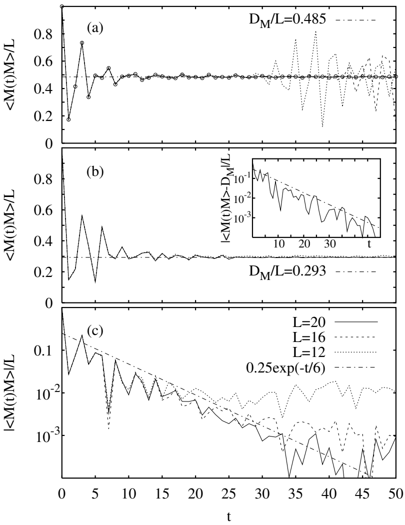
$$D_A = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} C_A(t)$$

  signals quantum ergodicity if $D_A = 0$.

- Quantum chaos regime in KI chain seems compatible with exponential decay of correlations. For integrable, and weakly non-integrable cases, though, we find saturation of temporal correlations $D \neq 0$.

Three typical cases of parameters:

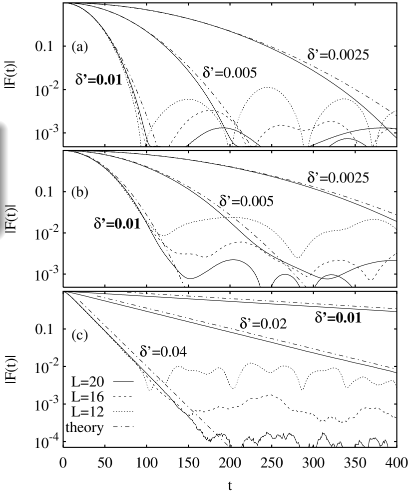
- (a) $J = 1, h_x = 1.4, h_z = 0.0$ (completely integrable).
- (b) $J = 1, h_x = 1.4, h_z = 0.4$ (intermediate).
- (c) $J = 1, h_x = 1.4, h_z = 1.4$ ("quantum chaotic").



Decay of correlations is closely related to fidelity decay $F(t) = \langle U^{-t} U_\delta^t(t) \rangle$ due to perturbed evolution $U_\delta = U \exp(-\mathrm{i}\delta A)$ (Prosen PRE 2002) e.g. in a linear response approximation:

$$F(t) = 1 - \frac{\delta^2}{2} \sum_{t', t''=1}^{t} C(t' - t'')$$

- (a) $J = 1, h_x = 1.4, h_z = 0.0$ (completely integrable).
- (b) $J = 1, h_x = 1.4, h_z = 0.4$ (intermediate).
- (c) $J = 1, h_x = 1.4, h_z = 1.4$ ("quantum chaotic").



PHYSICAL REVIEW E, VOLUME 65, 036208

General relation between quantum ergodicity and fidelity of quantum dynamics

Tomaž Prosen
Physics Department, Faculty of Mathematics and Physics, University of Ljubljana, Ljubljana, Slovenia
(Received 26 June 2001; published 11 February 2002)

A general relation is derived, which expresses the fidelity of quantum dynamics, measuring the stability of time evolution to small static variation in the Hamiltonian, in terms of ergodicity of an observable generating the perturbation as defined by its time correlation function. Fidelity for *ergodic* dynamics is predicted to decay *exponentially* on time scale $\propto \delta^{-2}$, $\delta \sim$ strength of perturbation, whereas faster, typically *Gaussian* decay on shorter time scale $\propto \delta^{-1}$ is predicted for *integrable*, or generally *nonergodic* dynamics. This result needs the perturbation δ to be sufficiently small such that the fidelity decay time scale is larger than any (quantum) relaxation time, e.g., mixing time for mixing dynamics, or averaging time for nonergodic dynamics (or Ehrenfest time for wave packets in systems with chaotic classical limit). Our surprising predictions are demonstrated in a quantum Ising spin-(1/2) chain periodically *kicked* with a tilted magnetic field where we find finite parameter-space regions of nonergodic and nonintegrable motion in the thermodynamic limit.

$$F(t)=1+\sum_{m=1}^{\infty}\frac{i^m\delta^m}{m!}\hat{\mathcal{T}}\sum_{t_1,t_2\ldots t_m=0}^{t-1}\langle A_{t_1}A_{t_2}\cdots A_{t_m}\rangle. \quad (3)$$

(I) Ergodicity and fast mixing. Here we assume that $C_A(t)\to 0$ sufficiently fast that the total sum converges, $S_A := (1/2)\Sigma_{t=-\infty}^{\infty}C_A(t), |S_A|<\infty$. For times t much larger than the so-called *mixing time scale* $t \gg t_{\rm mix}$, which effectively characterizes the correlation decay, e.g., $t_{\rm mix} = \Sigma_t |tC_A(t)|/\Sigma_t|C_A(t)|$, it follows that the fidelity drops linearly in time $F_{\rm e}(t)=1-t/\tau_{\rm e}+\mathcal{O}(\delta^3)$ on a scale

$$\tau_{\rm e}=S_A^{-1}\delta^{-2}. \quad (5)$$

$$F_{\rm e}(t)=\sum_{k=0}^{\infty}\frac{(-1)^k(2k-1)!!2^k\delta^{2k}S_A^k}{(2k)!}=\exp(-t/\tau_{\rm e}). \quad (6)$$

(II) Nonergodicity. Here we assume that the autocorrelation function of the perturbation does not decay asymptotically but has a nonvanishing time average, $D_A := \lim_{t\to\infty}(1/t)\Sigma_{t'=0}^{t-1}C_A(t')$, though the first moment is vanishing $\langle A\rangle=0$. For times t larger than the *averaging time* $t_{\rm ave}$ in which a finite time average effectively relaxes into the stationary value D_A, we can write fidelity to second order, which decays quadratically in time, $F_{\rm ne}(t)=1-(1/2)\times(t/\tau_{\rm ne})^2+\mathcal{O}(\delta^3)$, on a scale

$$\tau_{\rm ne}=D_A^{-1/2}\delta^{-1}. \quad (7)$$

More general result can be formulated in terms of a time-averaged operator $\bar{A} := \lim_{t\to\infty}(1/t)\Sigma_{t'=0}^{t-1}A_{t'}$, namely, for $t \gg t_{\rm ave}$ Eq. (3) can be rewritten as

$$F_{\rm ne}(t)=1+\sum_{m=2}^{\infty}\frac{i^m\delta^m t^m}{m!}\langle\bar{A}^m\rangle=\langle\exp(i\bar{A}\delta t)\rangle. \quad (8)$$

TP, J. Phys. A **35**, L737 (2002)
Transfer matrix approach to exponential decay of correlation:
Truncated quantum Perron-Frobenius map and Ruelle resonances.

TP, J. Phys. A **35**, L737 (2002)
Transfer matrix approach to exponential decay of correlation:
Truncated quantum Perron-Frobenius map and Ruelle resonances.

We construct a matrix representation of the following dynamical Heisenberg map

$$\hat{T}A = [U^\dagger A U]_r$$

truncated with respect to the following basis of translationally invariant extensive observables

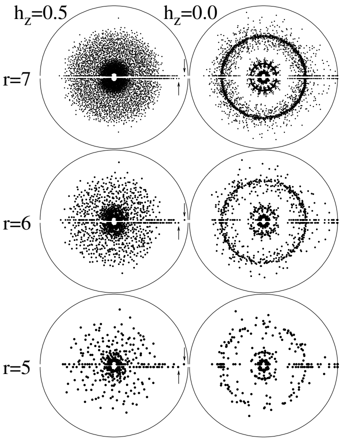
$$Z_{(s_0 s_1 \ldots s_{r-1})} = \sum_{j=-\infty}^{\infty} \sigma_j^{s_0} \sigma_{j+1}^{s_1} \cdots \sigma_{j+r-1}^{s_{r-1}}$$

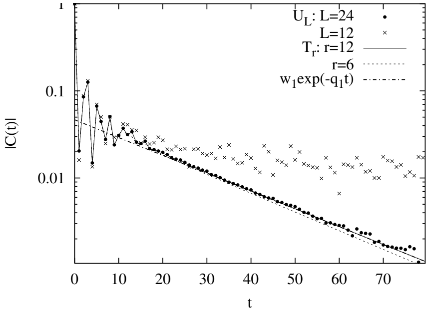
and inner product

$$(A|B) = \lim_{L\to\infty} \frac{1}{L2^L} \operatorname{tr} A^\dagger B, \quad A = \sum_s a_s Z_s \Rightarrow (A|A) = \sum_s |a_s|^2 < \infty.$$

Quantum Ruelle resonances

$J = 0.7, h_x = 1.1$





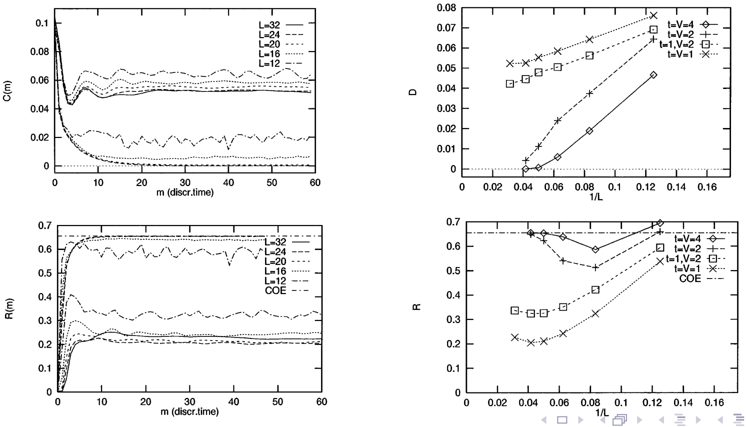
VOLUME 80, NUMBER 9 PHYSICAL REVIEW LETTERS 2 MARCH 1998

Time Evolution of a Quantum Many-Body System: Transition from Integrability to Ergodicity in the Thermodynamic Limit

Tomaž Prosen
Physics Department, Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1111 Ljubljana, Slovenia
(Received 17 July 1997)

Numerical evidence is given for nonergodic (nonmixing) behavior, exhibiting ideal transport, of a simple nonintegrable many-body quantum system in the thermodynamic limit, namely, the kicked t-V model of spinless fermions on a ring. However, for sufficiently large kick parameters t and V we recover quantum ergodicity, and normal transport, which can be described by random matrix theory.

$$H(\tau) = \sum_{j=0}^{L-1} [-\tfrac{1}{2} t(c_j^\dagger c_{j+1} + \text{H.c.}) + \delta_p(\tau) V n_j n_{j+1}],$$



Considerable followup up activity on periodically driven quantum spin chains only after cca. 2011, reviewed recently in:

`Marin Bukov, Luca D'Alessio, and Anatoli Polkovnikov, arXiv:1407.4803, to appear in Adv. Phys.`

Green-Kubo formulae express the conductivities in terms of current autocorrelaion functions

$$\kappa(\omega) = \lim_{t\to\infty} \lim_{L\to\infty} \frac{\beta}{L} \int_0^t \mathrm{d}t' e^{\mathrm{i}\omega t} \langle J(t') J(0) \rangle_\beta$$

When d.c. conductivity diverges, one defines a Drude weight D

$$\kappa(\omega) = 2\pi D \delta(\omega) + \kappa_{\mathrm{reg}}(\omega)$$

which in linear response expresses as

$$D = \lim_{t\to\infty} \lim_{L\to\infty} \frac{\beta}{2tL} \int_0^t \mathrm{d}t' \langle J(t') J(0) \rangle_\beta = \frac{\beta}{2L} \langle J \bar{J} \rangle_\beta = \frac{\beta}{2L} \langle \bar{J}^2 \rangle_\beta$$

Green-Kubo formulae express the conductivities in terms of current autocorrelaion functions

$$\kappa(\omega) = \lim_{t\to\infty} \lim_{L\to\infty} \frac{\beta}{L} \int_0^t \mathrm{d}t' e^{\mathrm{i}\omega t} \langle J(t')J(0)\rangle_\beta$$

When d.c. conductivity diverges, one defines a Drude weight D

$$\kappa(\omega) = 2\pi D\delta(\omega) + \kappa_{\mathrm{reg}}(\omega)$$

which in linear response expresses as

$$D = \lim_{t\to\infty} \lim_{L\to\infty} \frac{\beta}{2tL} \int_0^t \mathrm{d}t' \langle J(t')J(0)\rangle_\beta = \frac{\beta}{2L}\langle J\bar{J}\rangle_\beta = \frac{\beta}{2L}\langle \bar{J}^2\rangle_\beta$$

For integrable quantum systems, Zotos et al. (1997) suggested to use Mazur's (1969) and Suzuki's (1971) bound, estimating Drude weight in terms of local conserved operators F_j, $[H, F_j] = 0$:

$$D \geq \lim_{L\to\infty} \frac{\beta}{2L} \sum_j \frac{\langle JF_j\rangle_\beta^2}{\langle F_j^2\rangle_\beta}$$

where operators F_j are chosen mutually orthogonal $\langle F_j F_k\rangle = 0$ for $j \neq k$.

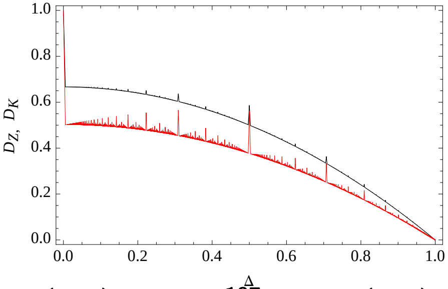
XXZ spin 1/2 chain

$$H = \sum_{j=1}^{n-1} (\sigma_j^{\rm x}\sigma_{j+1}^{\rm x} + \sigma_j^{\rm y}\sigma_{j+1}^{\rm y} + \Delta\sigma_j^{\rm z}\sigma_{j+1}^{\rm z}).$$

Fractal Drude weight bound (at high temperature $\beta \to 0$)

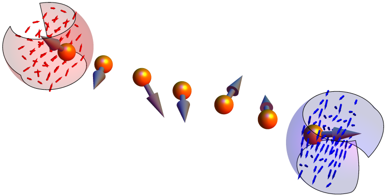
$$\frac{D}{\beta} \geq D_Z := \frac{\sin^2(\pi l/m)}{\sin^2(\pi/m)}\left(1 - \frac{m}{2\pi}\sin\left(\frac{2\pi}{m}\right)\right), \quad \Delta = \cos\left(\frac{\pi l}{m}\right)$$

and $\frac{D}{\beta} = 0$ for $|\Delta| > 1$.



TP, PRL **106**, 217206 (2011); TP, PRL **107**, 137201 (2011); TP, Ilievski, PRL **111**, 057203 (2013); TP, NPB **886**, 1177 (2014)

The new quasi-local conservation law Z, satisfying $[H, Z] = \sigma^{\rm z}_1 - \sigma^{\rm z}_n$, comes from studying the far from equilibrium problem:

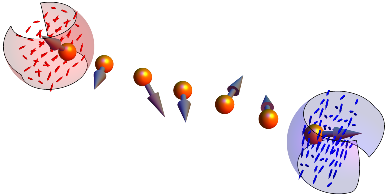


Canonical markovian master equation for the many-body density matrix:

The Lindblad (L-GKS) equation:

$$\frac{{\rm d}\rho}{{\rm d}t} = \hat{\mathcal{L}}\rho := -{\rm i}[H,\rho] + \sum_{\mu}\left(2L_\mu \rho L_\mu^\dagger - \{L_\mu^\dagger L_\mu, \rho\}\right).$$

The new quasi-local conservation law Z, satisfying $[H, Z] = \sigma_1^z - \sigma_n^z$, comes from studying the far from equilibrium problem:



Canonical markovian master equation for the many-body density matrix:

The Lindblad (L-GKS) equation:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \hat{\mathcal{L}}\rho := -\mathrm{i}[H, \rho] + \sum_{\mu} \left(2L_\mu \rho L_\mu^\dagger - \{L_\mu^\dagger L_\mu, \rho\} \right).$$

- *Bulk*: Fully **coherent**, local interactions,e.g. $H = \sum_{x=1}^{n-1} h_{x,x+1}$.
- *Boundaries:* Fully **incoherent**, ultra-local dissipation, jump operators L_μ supported near boundaries $x = 1$ or $x = n$.

Steady state Lindblad equation $\hat{\mathcal{L}}\rho_\infty = 0$:

$$\mathrm{i}[H, \rho_\infty] = \sum_\mu \left(2L_\mu \rho_\infty L_\mu^\dagger - \{L_\mu^\dagger L_\mu, \rho_\infty\}\right)$$

The XXZ Hamiltonian:

$$H = \sum_{x=1}^{n-1} (2\sigma_x^+ \sigma_{x+1}^- + 2\sigma_x^- \sigma_{x+1}^+ + \Delta \sigma_x^{\mathrm{z}} \sigma_{x+1}^{\mathrm{z}})$$

and symmetric boundary (ultra local) Lindblad jump operators:

$$\begin{aligned} L_1^{\mathrm{L}} &= \sqrt{\frac{1}{2}(1-\mu)\varepsilon}\ \sigma_1^+, \quad L_1^{\mathrm{R}} = \sqrt{\frac{1}{2}(1+\mu)\varepsilon}\ \sigma_n^+, \\ L_2^{\mathrm{L}} &= \sqrt{\frac{1}{2}(1+\mu)\varepsilon}\ \sigma_1^-, \quad L_2^{\mathrm{R}} = \sqrt{\frac{1}{2}(1-\mu)\varepsilon}\ \sigma_n^-. \end{aligned}$$

Two key boundary parameters:

- ε System-bath coupling strength
- μ Non-equilibrium driving strength (bias)

TP, PRL**106**(2011); PRL**107**(2011); Karevski, Popkov, Schütz, PRL**111**(2013)

$$\rho_\infty = (\mathrm{tr}\, R)^{-1} R, \quad R = SS^\dagger$$

$$S = \sum_{(s_1,\ldots,s_n)\in\{+,-,0\}^n} \langle 0|\mathbf{A}_{s_1}\mathbf{A}_{s_2}\cdots\mathbf{A}_{s_n}|0\rangle \sigma^{s_1}\otimes\sigma^{s_2}\cdots\otimes\sigma^{s_n} = \langle 0| \begin{pmatrix} \mathbf{A}_0 & \mathbf{A}_+ \\ \mathbf{A}_- & \mathbf{A}_0 \end{pmatrix}^{\otimes n} |0\rangle$$

TP, PRL**106**(2011); PRL**107**(2011); Karevski, Popkov, Schütz, PRL**111**(2013)

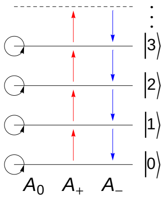
$$\rho_\infty = (\mathrm{tr}\, R)^{-1} R, \quad R = SS^\dagger$$

$$S = \sum_{(s_1,\ldots,s_n)\in\{+,-,0\}^n} \langle 0|\mathbf{A}_{s_1}\mathbf{A}_{s_2}\cdots\mathbf{A}_{s_n}|0\rangle \sigma^{s_1}\otimes\sigma^{s_2}\cdots\otimes\sigma^{s_n} = \langle 0| \begin{pmatrix} \mathbf{A}_0 & \mathbf{A}_+ \\ \mathbf{A}_- & \mathbf{A}_0 \end{pmatrix}^{\otimes n} |0\rangle$$

$$\mathbf{A}_0 = \sum_{k=0}^{\infty} a_k^0 |k\rangle\langle k|,$$

$$\mathbf{A}_+ = \sum_{k=0}^{\infty} a_k^+ |k\rangle\langle k+1|,$$

$$\mathbf{A}_- = \sum_{k=0}^{\infty} a_k^- |k+1\rangle\langle r|,$$



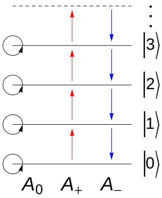
Cholesky decomposition of NESS and Matrix Product Ansatz (for $\mu = 1$)

TP, PRL**106**(2011); PRL**107**(2011); Karevski, Popkov, Schütz, PRL**111**(2013)

$$\rho_\infty = (\mathrm{tr}\, R)^{-1} R, \quad R = SS^\dagger$$

$$S = \sum_{(s_1,\ldots,s_n)\in\{+,-,0\}^n} \langle 0|\mathbf{A}_{s_1}\mathbf{A}_{s_2}\cdots\mathbf{A}_{s_n}|0\rangle \sigma^{s_1}\otimes\sigma^{s_2}\cdots\otimes\sigma^{s_n} = \langle 0| \begin{pmatrix} \mathbf{A}_0 & \mathbf{A}_+ \\ \mathbf{A}_- & \mathbf{A}_0 \end{pmatrix}^{\otimes n} |0\rangle$$

$$\begin{aligned} \mathbf{A}_0 &= \sum_{k=0}^{\infty} a_k^0 |k\rangle\langle k|, \\ \mathbf{A}_+ &= \sum_{k=0}^{\infty} a_k^+ |k\rangle\langle k+1|, \\ \mathbf{A}_- &= \sum_{k=0}^{\infty} a_k^- |k+1\rangle\langle r|, \end{aligned}$$

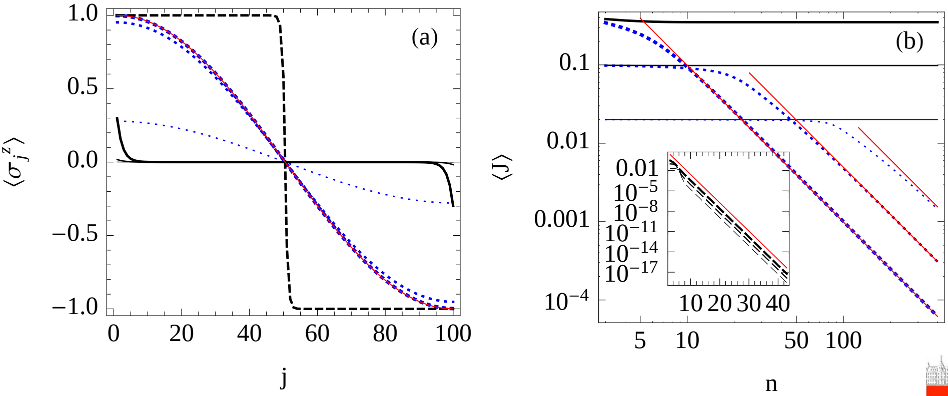


$$\begin{aligned} a_k^0 &= \cos((s-k)\eta) \qquad \cos\eta := \Delta, \\ a_k^+ &= \sin((k+1)\eta) \qquad \tan(\eta s) := \frac{\varepsilon}{2\mathrm{i}\sin\eta} \\ a_k^- &= \cos((2s-k)\eta) \end{aligned}$$

s is a q$-$deformed complex spin $q = e^{\mathrm{i}\eta}$

Observables in NESS: From insulating to ballistic transport

- For $|\Delta| < 1$, $\langle J \rangle \sim n^0$ (ballistic)
- For $|\Delta| > 1$, $\langle J \rangle \sim \exp(-\mathrm{const}\, n)$ (insulating)
- For $|\Delta| = 1$, $\langle J \rangle \sim n^{-2}$ (anomalous)



There is an example of ergodicity/non-ergodicity transition even in a classical mechanical completely integrable many body system!

PRL **111,** 040602 (2013) PHYSICAL REVIEW LETTERS week ending 26 JULY 2013

Macroscopic Diffusive Transport in a Microscopically Integrable Hamiltonian System

Tomaž Prosen

Physics Department, Faculty of Mathematics and Physics, University of Ljubljana, 1000 Ljubljana, Slovenia

Bojan Žunkovič

Departamento de Física, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Casilla 487-3, Santiago, Chile

(Received 26 April 2013; published 26 July 2013)

We demonstrate that a completely integrable classical mechanical model, namely the lattice Landau-Lifshitz classical spin chain, supports diffusive spin transport with a finite diffusion constant in the easy-axis regime, while in the easy-plane regime, it displays ballistic transport in the absence of any known relevant local or quasilocal constant of motion in the symmetry sector of the spin current. This surprising finding should open the way towards analytical computation of diffusion constants for integrable interacting systems and hints on the existence of new quasilocal classical conservation laws beyond the standard soliton theory.

Transition from **ballistic** to **diffusive** transport in **integrable** classical chain

Transition from **ballistic** to **diffusive** transport in **integrable** classical chain

Locally interacting spin chain Hamiltonian

$$H = \sum_{x=1}^{n} h(\vec{S}_x, \vec{S}_{x+1}),$$

where for Lattice-Landau-Lifshitz model, the energy density reads

$$\begin{aligned} h(\vec{S}, \vec{S}') &= \log \Big| \cosh(\rho S_3)\cosh(\rho S_3') + \coth^2(\rho R)\sinh(\rho S_3)\sinh(\rho S_3') \\ &+ \sinh^{-2}(\rho R) F(S_3) F(S_3')(S_1 S_1' + S_2 S_2') \Big| \end{aligned}$$

and $F(S) \equiv \sqrt{(\sinh^2(\rho R) - \sinh^2(\rho S))/(R^2 - S^2)}$.

Transition from **ballistic** to **diffusive** transport in **integrable** classical chain

Locally interacting spin chain Hamiltonian

$$H = \sum_{x=1}^{n} h(\vec{S}_x, \vec{S}_{x+1}),$$

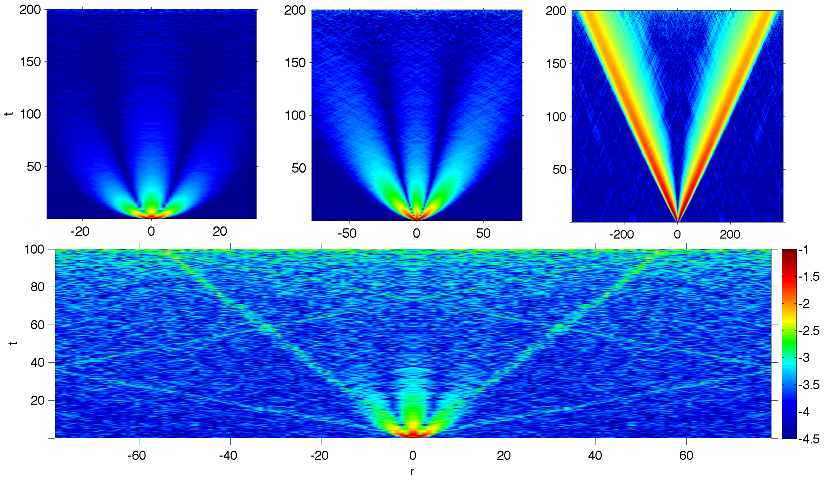
where for Lattice-Landau-Lifshitz model, the energy density reads

$$\begin{aligned} h(\vec{S}, \vec{S}') \quad = \quad & \log\left|\cosh(\rho S_3)\cosh(\rho S_3') + \coth^2(\rho R)\sinh(\rho S_3)\sinh(\rho S_3')\right. \\ & \left. + \sinh^{-2}(\rho R) F(S_3) F(S_3')(S_1 S_1' + S_2 S_2')\right| \end{aligned}$$

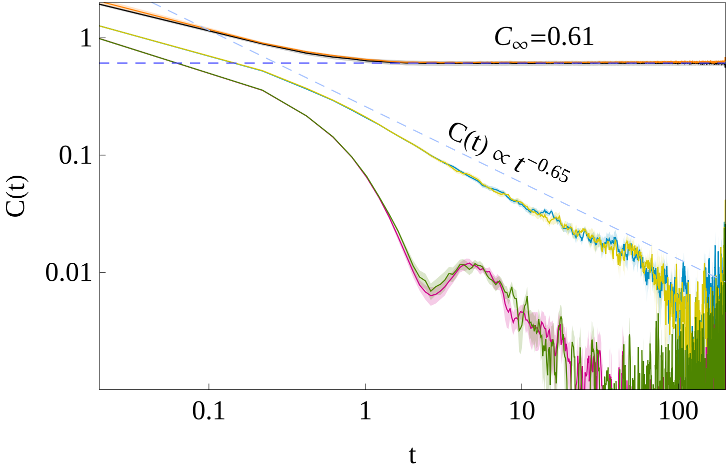
and $F(S) \equiv \sqrt{(\sinh^2(\rho R) - \sinh^2(\rho S))/(R^2 - S^2)}$.

Writing **anisotropy parameter** $\delta = \rho^2$ we study three cases:

- $\delta > 0$, easy axis regime (Ising-like) **diffusive!!!**
- $\delta < 0$, easy plane regime (XY-like) **ballistic!!!**
- $\delta = 0$, isotropic regime (where $h(\vec{S}, \vec{S}') = \log\left(1 + \frac{\vec{S}\cdot\vec{S}'}{R^2}\right)$) **anomalous!!!**



Spatio-temporal current-current c.f. shown in log-scale with color scale ranging from $10^{-4.5}$ to 10^{-1} indicated in the bottom-right. In the upper panels we show data averaged over ensembles of $N \approx 10^3$ initial conditions in easy-axis (left; $n = 5120$), isotropic (center; $n = 5120$) and easy-plane (right; $n = 2560$) regimes. Bottom: smaller $n = 160$, $N = 600$ where **scars of solitons** emerging from local thermal fluctuations are still clearly visible.



$C(t)$ in log-log scale for easy-plane regime (top curves, orange: $n = 160$, black: $n = 2560$), isotropic regime (middle curves, yellow: $n = 2560$, blue: $n = 5120$) and easy-axis regime (bottom curves, violet: $n = 2560$, green: $n = 5120$). Shaded regions denote the estimated statistical error for ensemble averages over $N \approx 10^3$ initial conditions. Dashed lines denote asymptotic behavior for large time in the easy-plane regime (dark-blue) and isotropic regime (light-blue).

So much for 1D quantum (and classical) lattice systems.

However, situation gets even more puzzling for 2D systems..

C. Pineda, TP and E. Villasenor, NJP **16**, 123044 (2014).

Taking an Ising Hamiltonian on a rectangular lattice

$$H_1 = JH_{\rm I}, \quad H_{\rm I} = \sum_{m=0}^{L_x-1}\sum_{n=0}^{L_y-1}(\sigma^z_{m,n}\sigma^z_{m+1,n} + \sigma^z_{m,n}\sigma^z_{m,n+1}),$$

with periodic boundary conditions $\sigma^\alpha_{m,L_y} \equiv \sigma^\alpha_{m,0}$, $\sigma^\alpha_{L_x,n} \equiv \sigma^\alpha_{0,n}$. and a Zeeman Hamiltonian for a spatially homogeneous magnetic field $\vec{b}$

$$H_0 = \sum_{m=0}^{L_x-1}\sum_{n=0}^{L_y-1}\vec{b}\cdot\vec{\sigma}_{m,n} = \vec{b}\cdot\vec{S}, \quad \vec{S} =: \sum_{m=0}^{L_x-1}\sum_{n=0}^{L_y-1}\vec{\sigma}_{m,n}.$$

we consider the kicked Hamiltonian

$$H(t) = H_1 + H_0\sum_{j\in\mathbb{Z}}\delta(t - j\tau).$$

C. Pineda, TP and E. Villasenor, NJP **16**, 123044 (2014).

Taking an Ising Hamiltonian on a rectangular lattice

$$H_1 = JH_{\mathrm{I}}, \quad H_{\mathrm{I}} = \sum_{m=0}^{L_x-1}\sum_{n=0}^{L_y-1}(\sigma^z_{m,n}\sigma^z_{m+1,n} + \sigma^z_{m,n}\sigma^z_{m,n+1}),$$

with periodic boundary conditions $\sigma^\alpha_{m,L_y} \equiv \sigma^\alpha_{m,0}$, $\sigma^\alpha_{L_x,n} \equiv \sigma^\alpha_{0,n}$. and a Zeeman Hamiltonian for a spatially homogeneous magnetic field $\vec{b}$

$$H_0 = \sum_{m=0}^{L_x-1}\sum_{n=0}^{L_y-1}\vec{b}\cdot\vec{\sigma}_{m,n} = \vec{b}\cdot\vec{S}, \quad \vec{S} =: \sum_{m=0}^{L_x-1}\sum_{n=0}^{L_y-1}\vec{\sigma}_{m,n}.$$

we consider the kicked Hamiltonian

$$H(t) = H_1 + H_0\sum_{j\in\mathbb{Z}}\delta(t - j\tau).$$

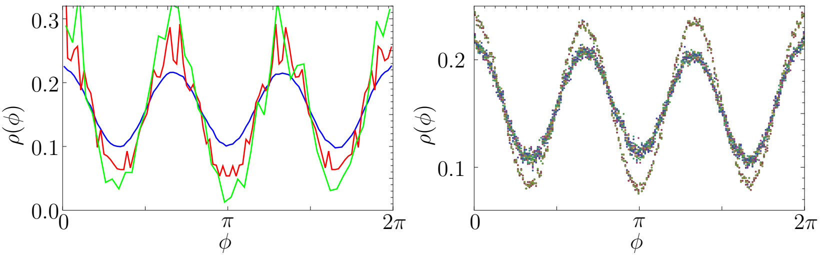
We observe three *unrelated* transitions as we vary the parameters $J, \vec{b}$...

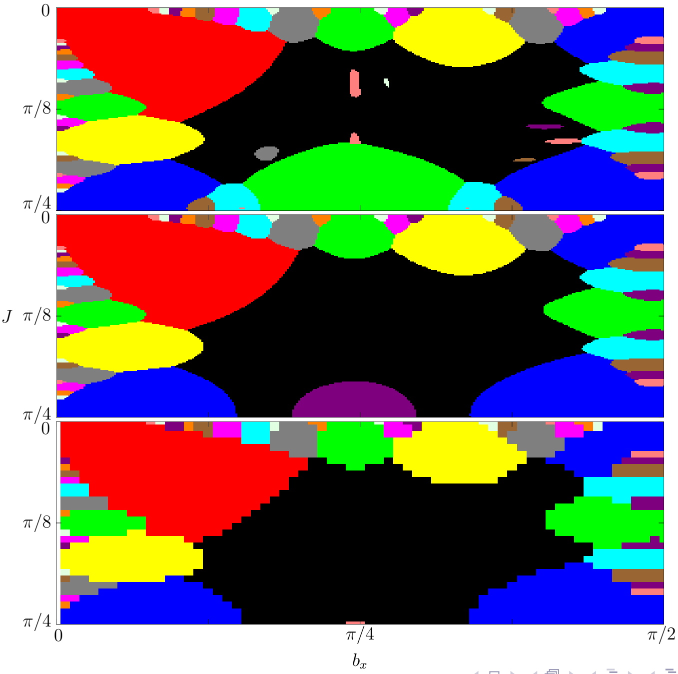
Floquet map spectrum:

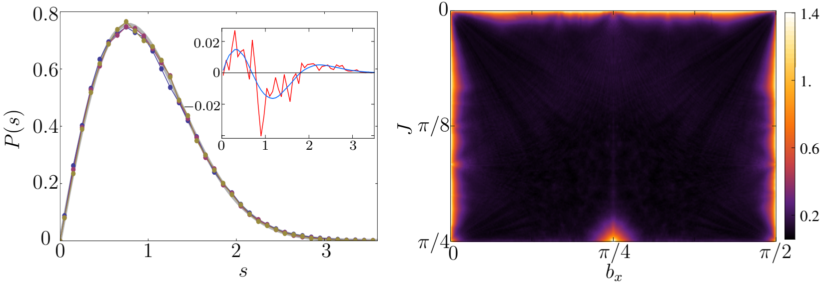
$$U_{\rm KI}|\psi_n\rangle = e^{-{\rm i}\phi_n}|\psi_n\rangle, \quad U_{\rm KI} = e^{-{\rm i}H_1}e^{-{\rm i}H_0}$$

Spectral density ($\mathcal{N} = 2^{L_x L_y}$):

$$\rho(\phi) = \frac{1}{\mathcal{N}}\sum_{n=1}^{\mathcal{N}}\delta(\phi-\phi_n) = \frac{1}{2\pi}\left(1+\sum_{k=1}^{\infty}\cos(k\phi)\frac{2}{\mathcal{N}}{\rm tr}\,U^k\right).$$

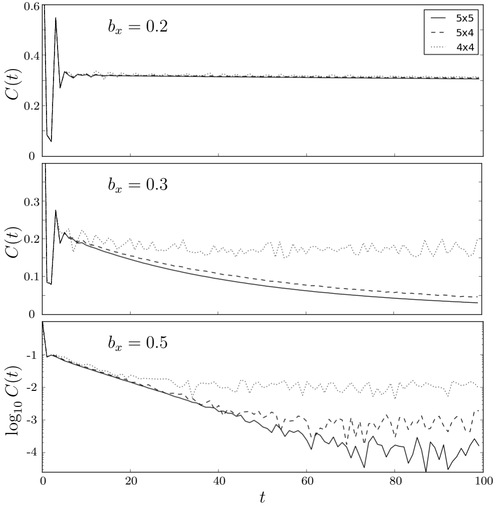



0
π/8
π/4
0
J π/8
π/4
0
π/8
π/4
0
π/4
π/2
b_x



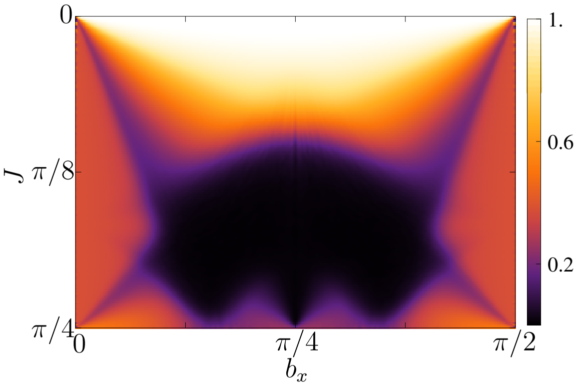
Analysis of the distribution of the nearest neighbour spacing $P(s)$. On the left panel, we observe the nearest neighbour spacing distribution for three different transverse fields, $b_x = 0.2$, 0.3 and 0.5 in red, green and yellow respectively, $J = 0.5$, and we consider a 5×4 lattice. In all cases, we are considering $s_x = \pm 1$, $k_x \in \{1, 2\}$ and $k_y = 1$. The thick black curve correspond to the Wigner surmise. In the inset, we show the average of these three curves, minus the Wigner surmise, together with the theoretical prediction. On the right panel, we consider the Kolmogorov distance between the unfolded $P(s)$, and the Wigner surmise, for all the parameters of the model, and a 4×3 lattice. Very good agreement with the RMT prediction is observed except when J or b_x are zero, or $J = b_x = \pi/4$.

Ergodicity breaking transition

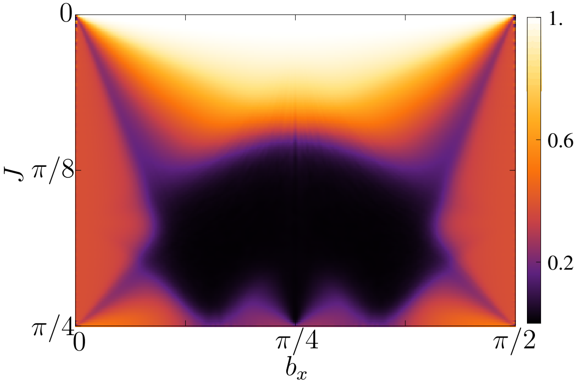


Correlation decay (of transverse magnetisation) for the transverse field KI model, varying b_x, for different dimensions and fixed $J = 0.5$. The calculation is done using a single random state.

Phase diagram of ergodicity



Phase diagram of time averaged correlator for the Ising model, for $M = S^x$, as a function of b_x and J, with $M = S_x$ and $b_z = 0$.



Phase diagram of time averaged correlator for the Ising model, for $M = S^x$, as a function of b_x and J, with $M = S_x$ and $b_z = 0$.

The phase diagram has no resemblance to phase diagram of level density, whereas spectral statistics is Wigner-Dyson-like almost everywhere!