# Quantum tunneling in nonintegrable systems: beyond the leading order semiclassical description

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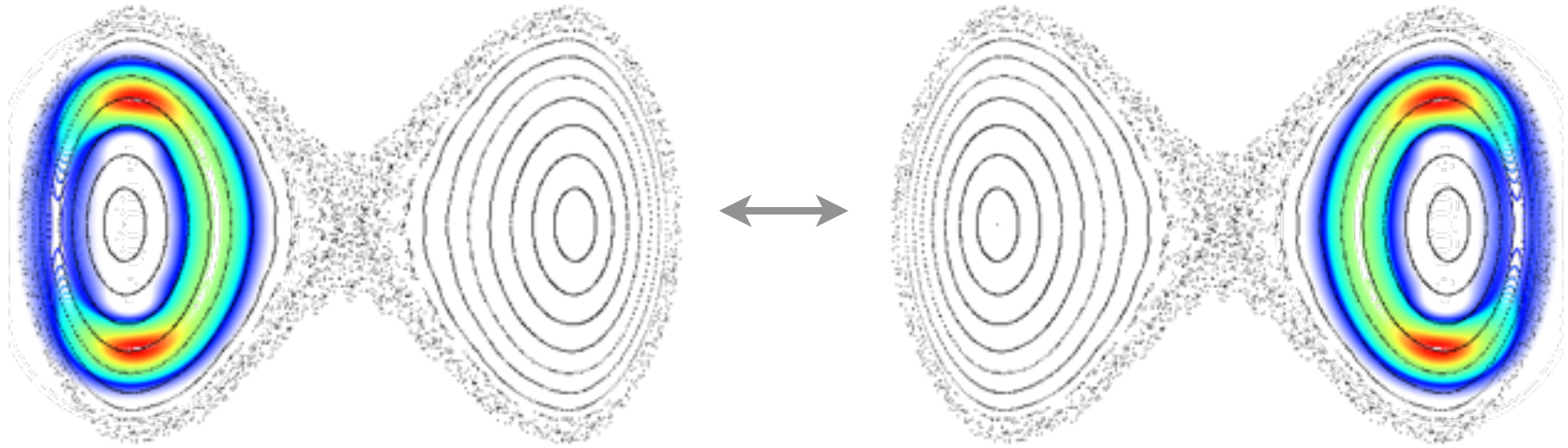
collaboration with Y Hanada and K S Ikeda

# Introduction

## Dynamical tunneling



- **Energy domain approach based on trance formula**
  No semiclassical formula for mixed systems
  (cf. hyperbolic : Gutzwiller, completely integrable : Berry-Tabor)

- **Time domain approach based on Van-Vleck Gutzwiller**
  works well within the leading order semiclassical approximation
  (cf. recent advances in theory of complex dynamical systems by Bedford and Smillie)
  but depends on initial and final states, or representations

- **Here, not long-time, but just a single step semiclassical analysis**
  as close as possible to the energy domain by adjusting initial and final states

# Completely integrable model

$$L : \begin{pmatrix} q' \\ p' \end{pmatrix} \mapsto \begin{pmatrix} q + \omega \\ p + K \cos q \end{pmatrix}$$

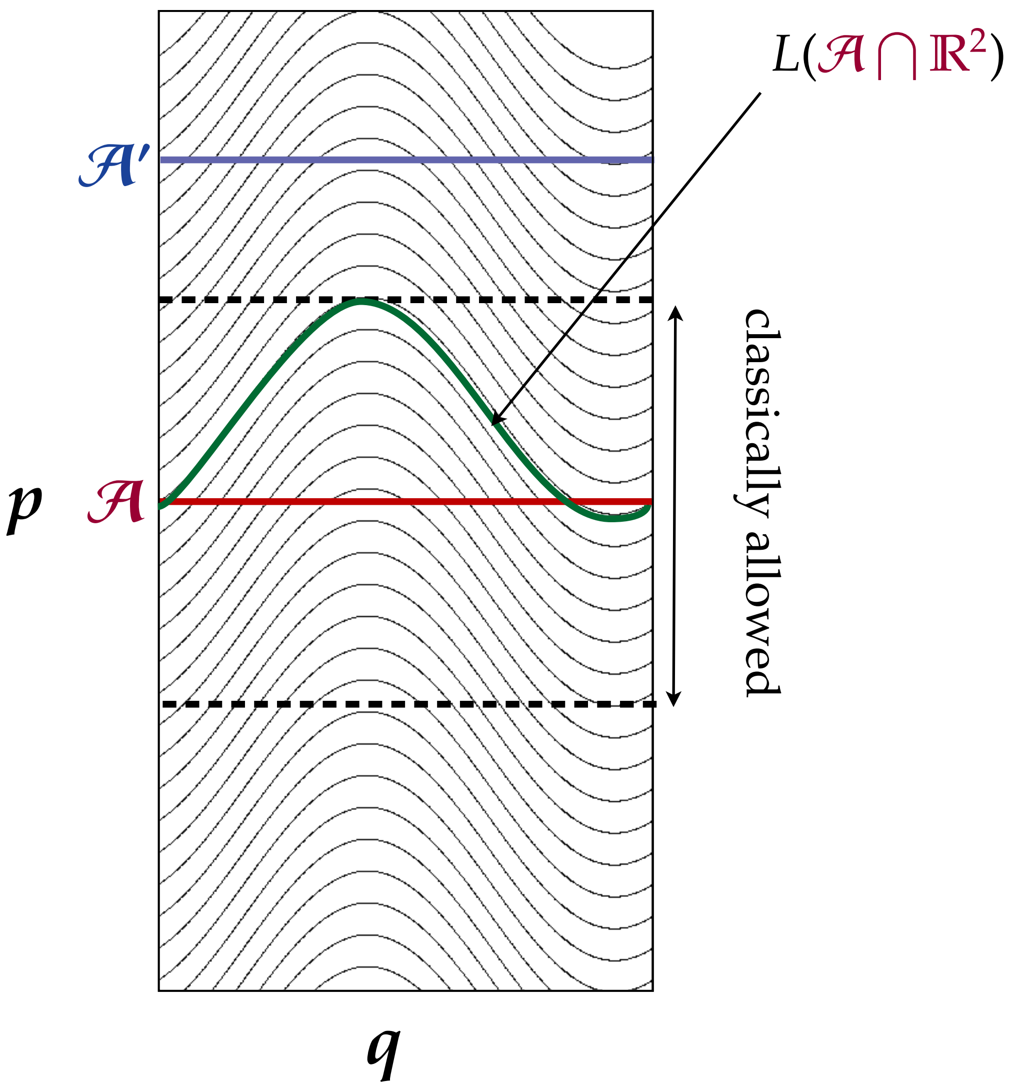
**In the real plane**

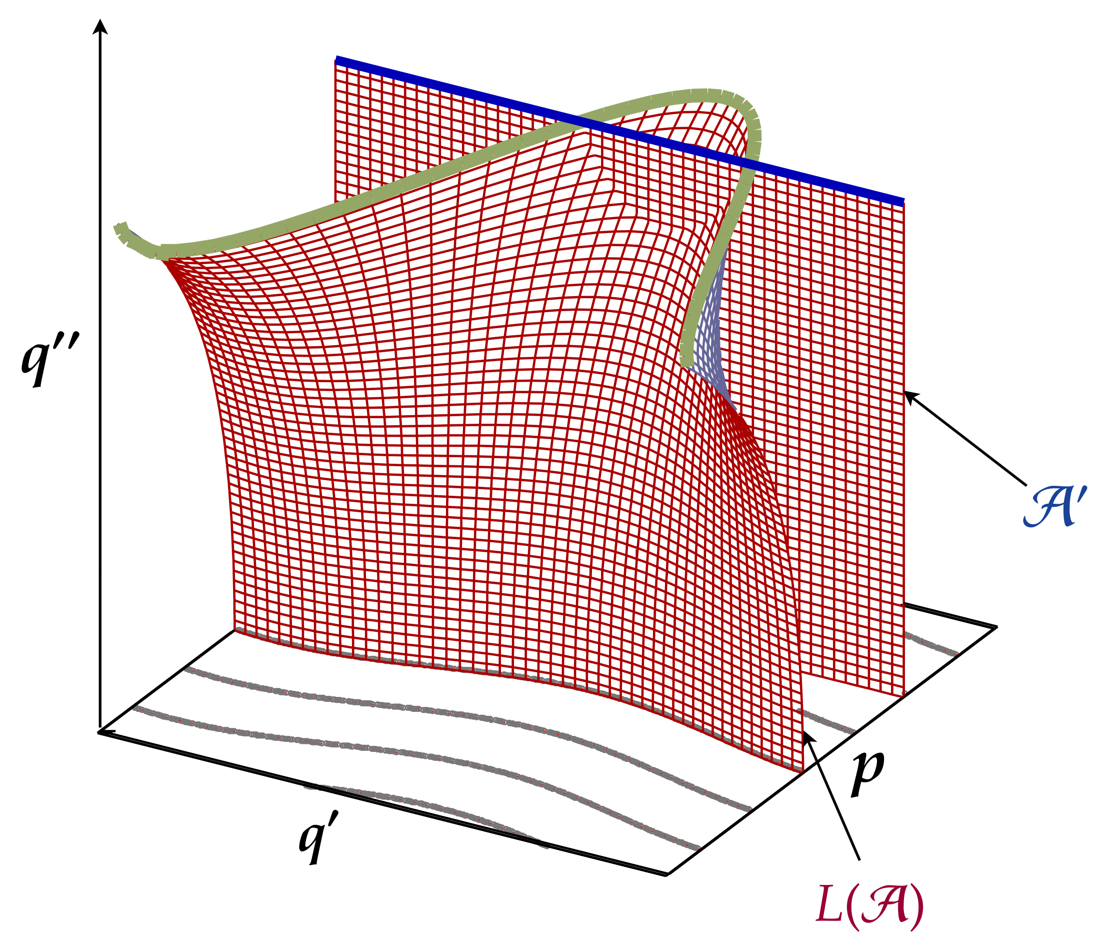
$L(\mathcal{A}) \bigcap \mathcal{A}' = \emptyset$

**if $\mathcal{A}'$ is outside the classically allowed region.**

**In the complex plane**

**$L(\mathcal{A}) \bigcap \mathcal{A}' \neq \emptyset$ for any $\mathcal{A}$ and $\mathcal{A}'$.**



# Completely integrable model

**1-step propagator**

$$\langle p' | \hat{U} | p \rangle = \int_{-\infty}^{\infty} dq \exp\left[-\frac{i}{\hbar}\left\{F(q; p', p)\right\}\right]$$

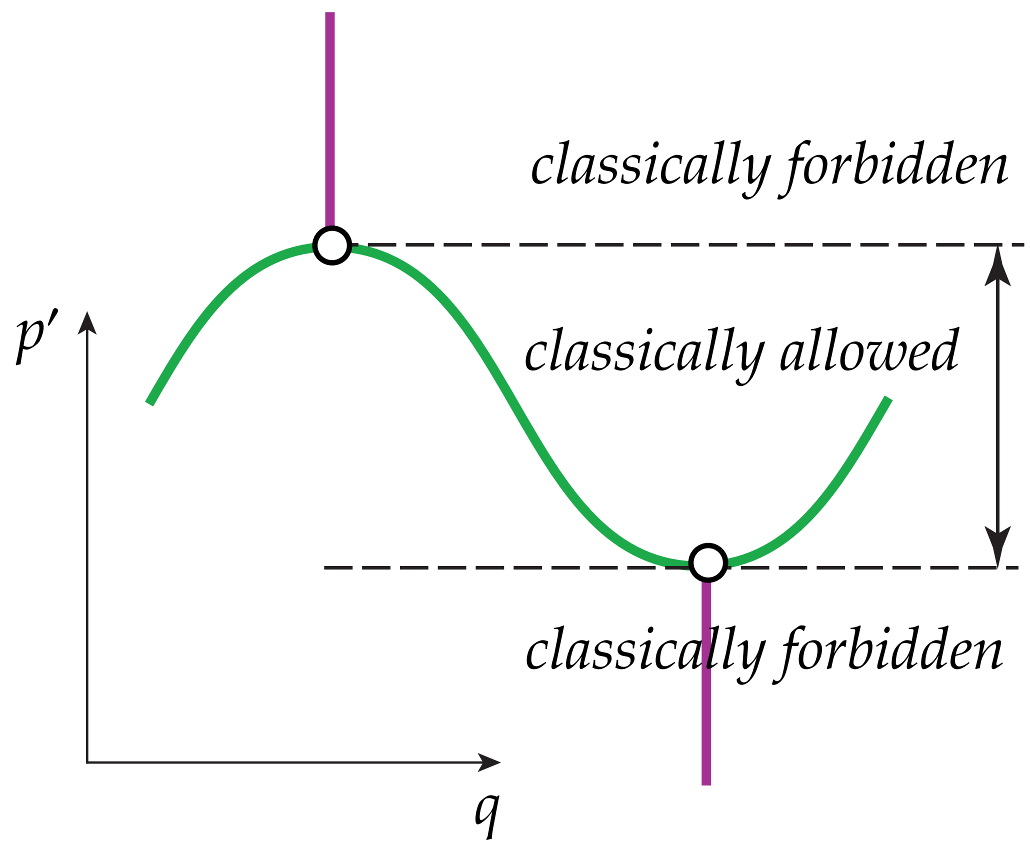
**where**

$$F(q; p', p) := T(p) + V(q) + q(p' - p)$$

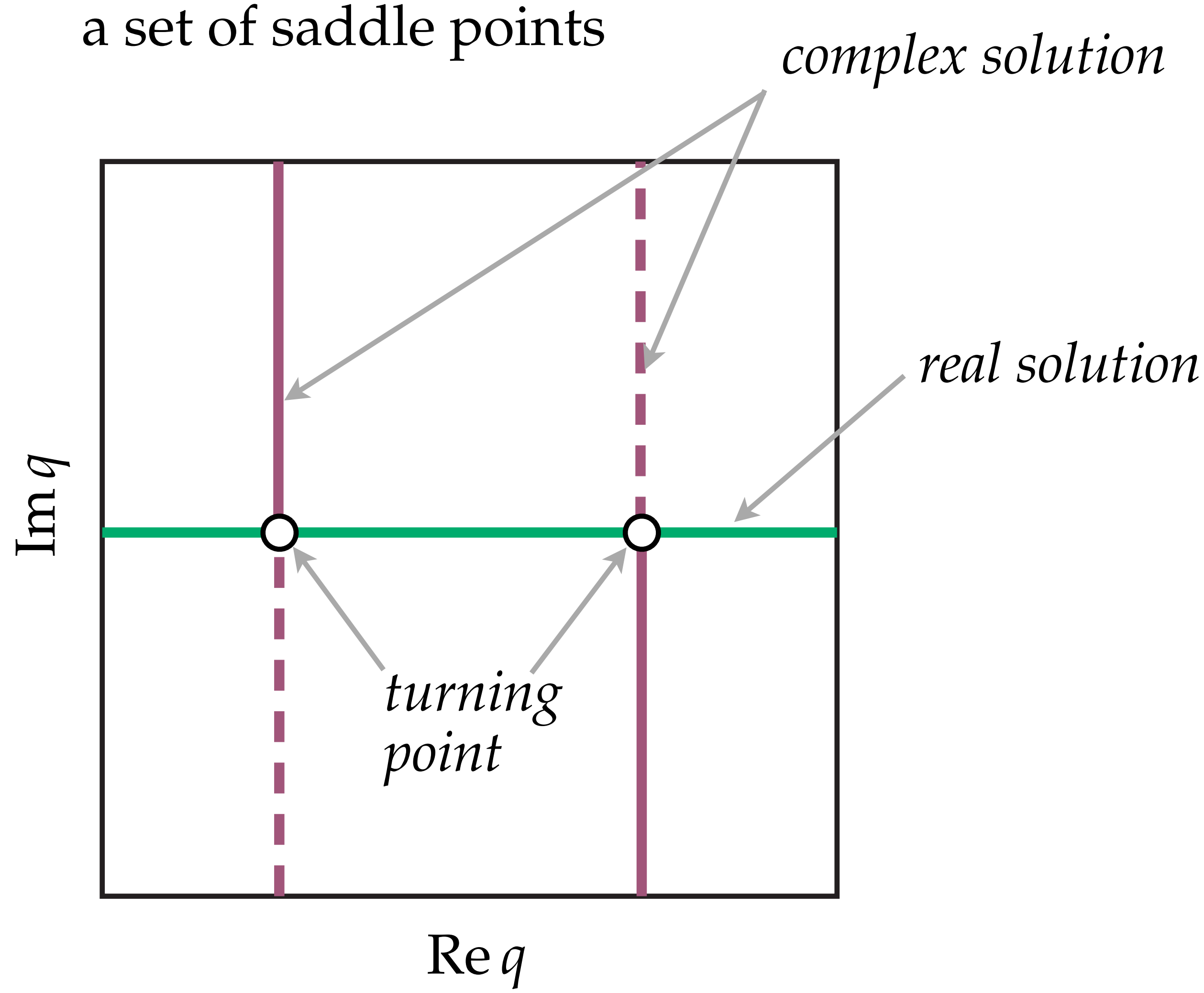
**Saddle point condition**

$$\frac{\partial F(q; p', p)}{\partial q} = 0$$

Langrangian manifold




a set of saddle points



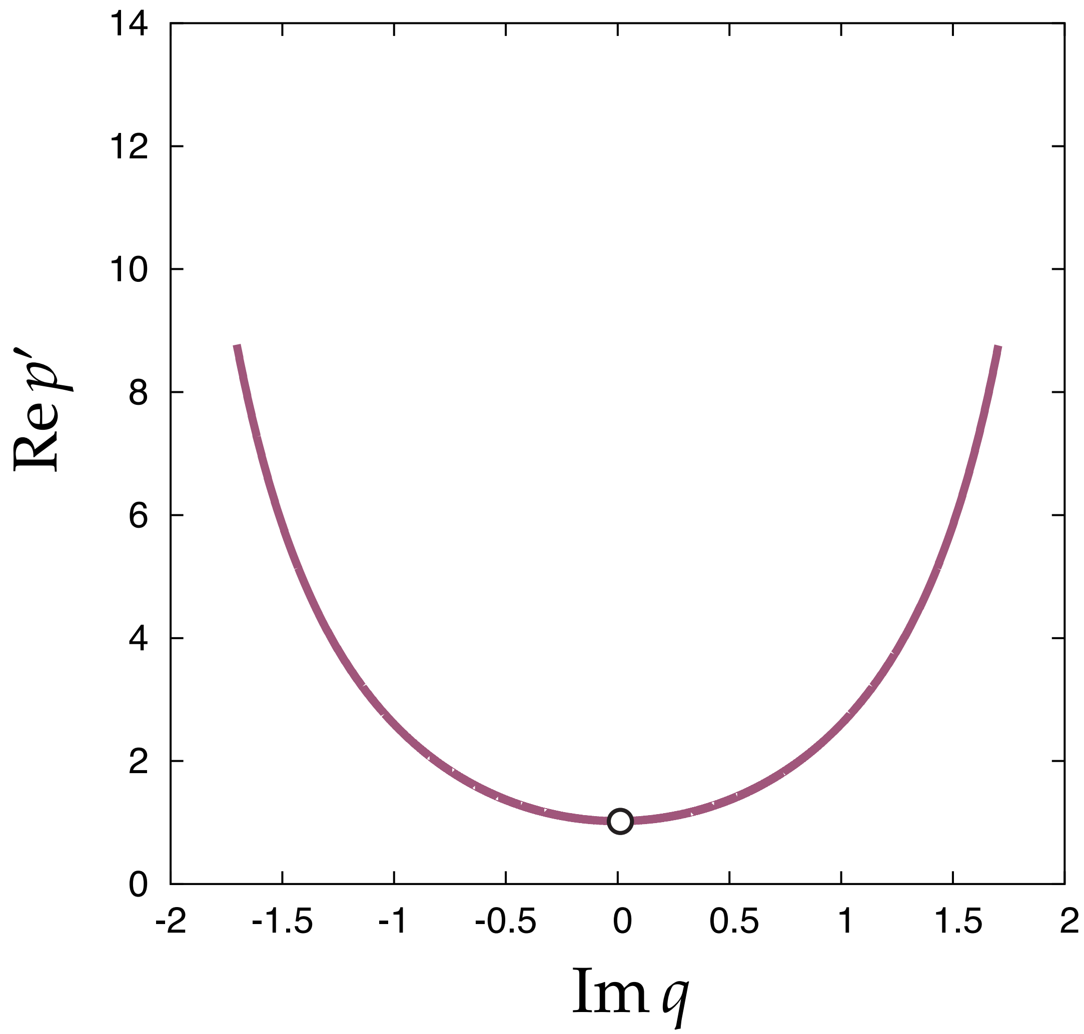
# Completely integrable model

**Manifold around the turning point**

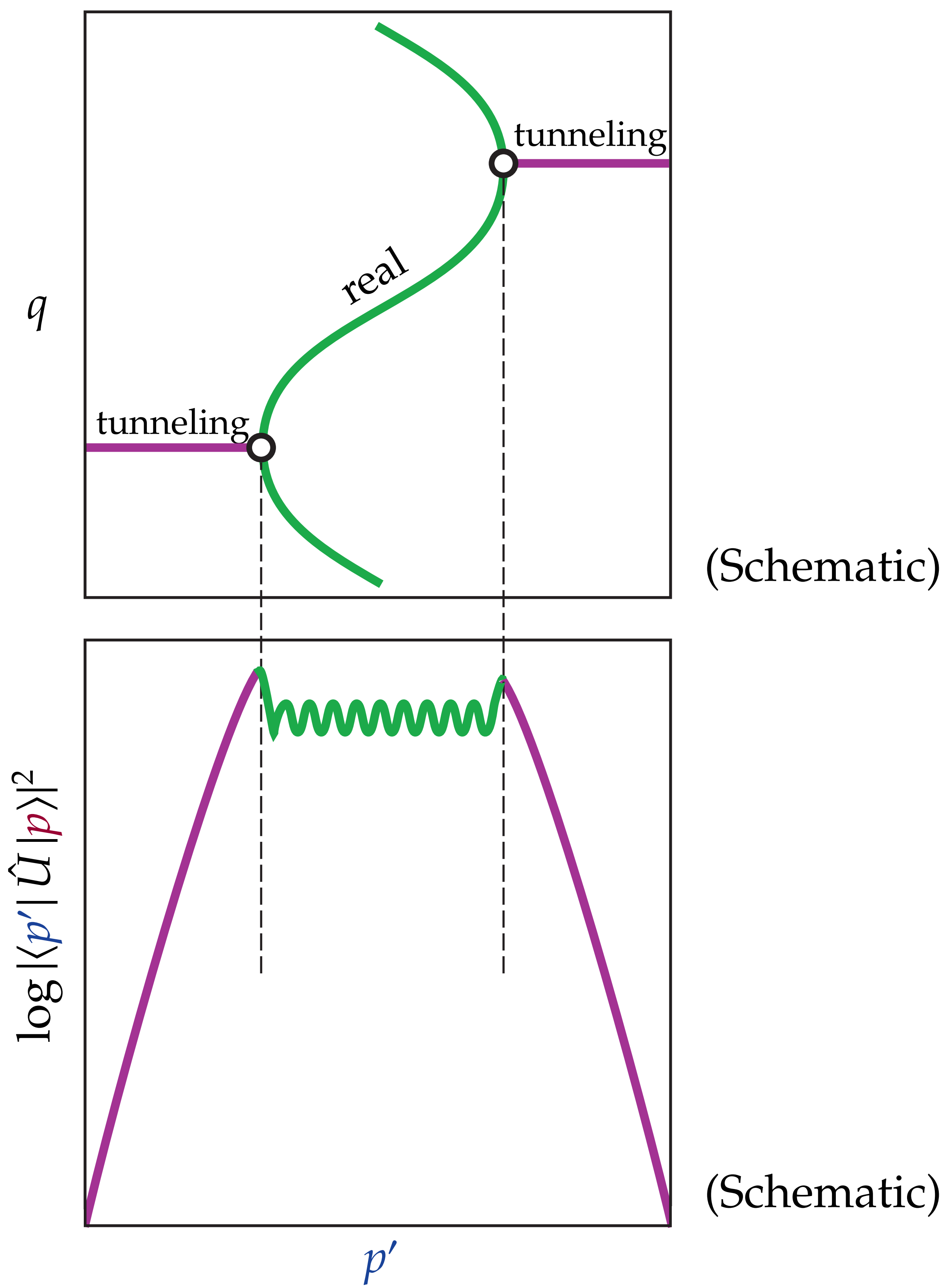
Locally, the behavior around the turning point is described by

$$\Psi_K(\mathbf{p}) = \int_{-\infty}^{\infty} \exp(\mathrm{i}\,\Phi_K(t;\mathbf{p}))\,dt, \qquad \text{where} \quad \Phi_K(t;\mathbf{p}) = t^{K+2} + \sum_{m=1}^{K} x_m t^m$$

with $K = 1$, that is the *Airy function*.



# Completely integrable model

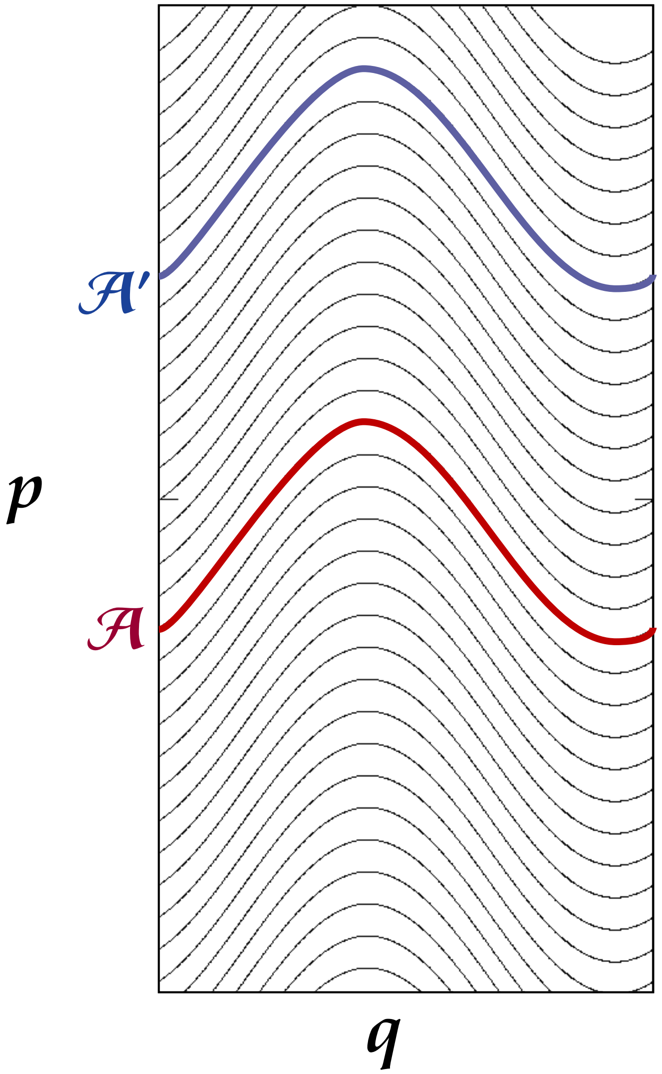


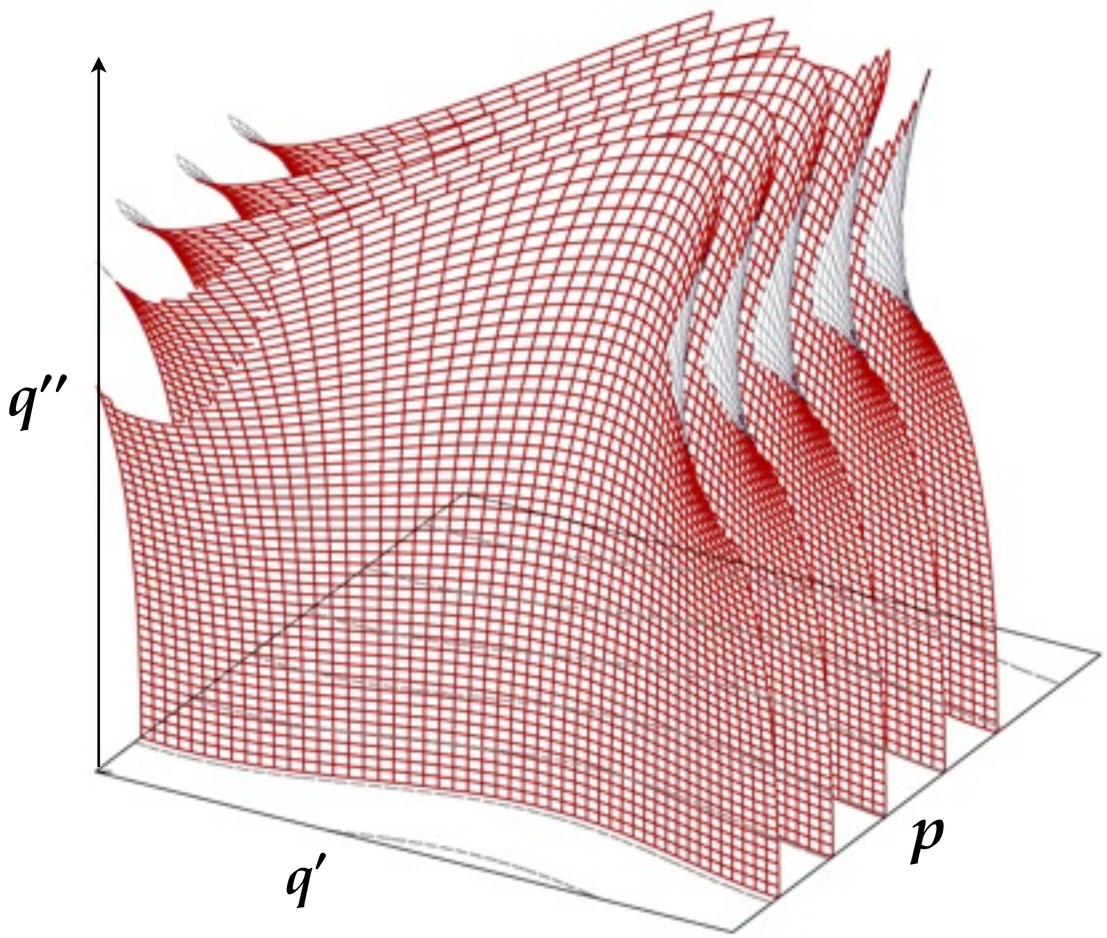
# Completely integrable model

$$L : \begin{pmatrix} q' \\ p' \end{pmatrix} \mapsto \begin{pmatrix} q + \omega \\ p + K \cos q \end{pmatrix}$$

$$\mathcal{A} = \{ (q,p) \in \mathbb{C}^2 \mid I(q,p) = I_a \in \mathbb{R} \}$$
$$\mathcal{A}' = \{ (q,p) \in \mathbb{C}^2 \mid I(q,p) = I_b \in \mathbb{R} \}$$

$L(\mathcal{A}) \bigcap \mathcal{A}' = \emptyset$ for any $\mathcal{A}$ and $\mathcal{A}'$.



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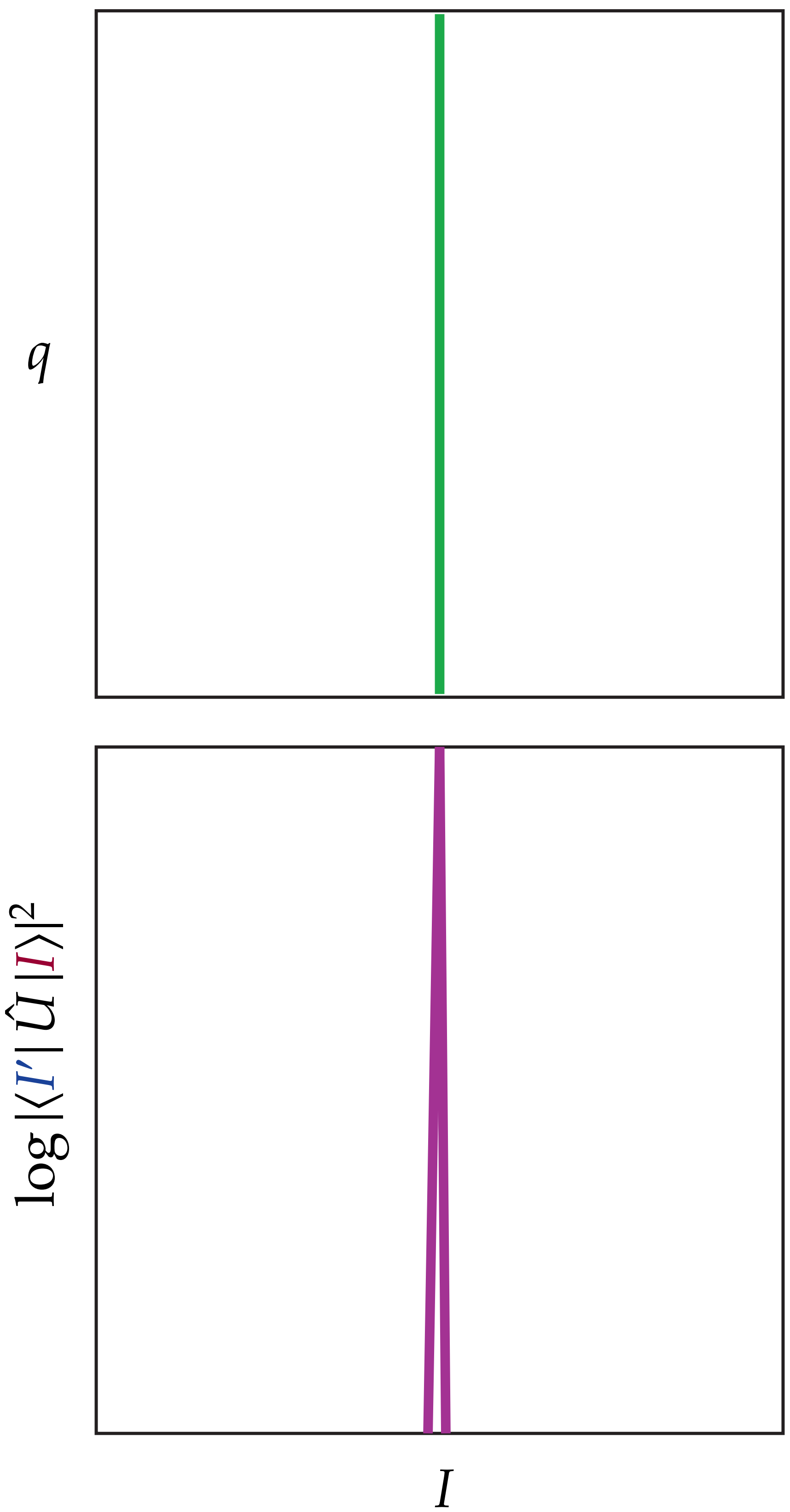


# Map with discontinuity

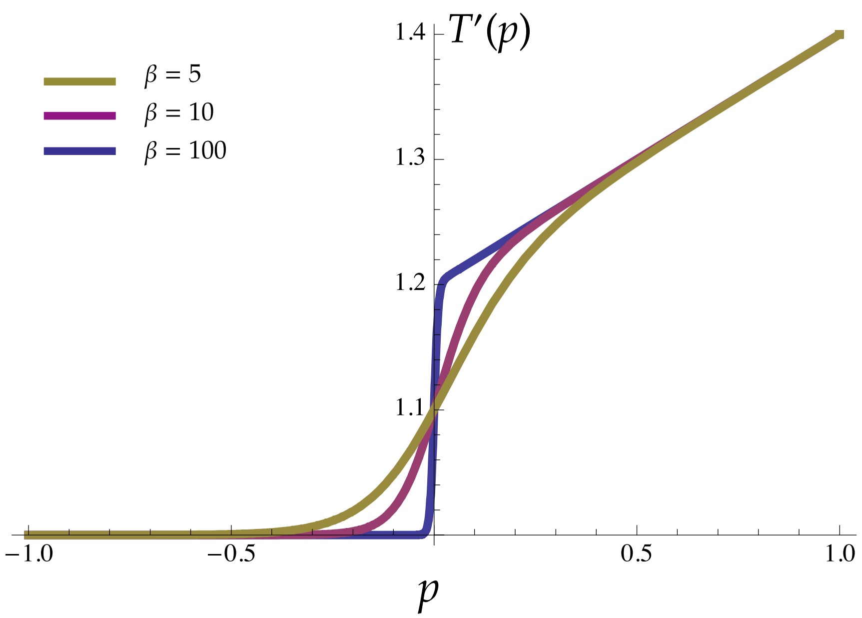
**Map:**

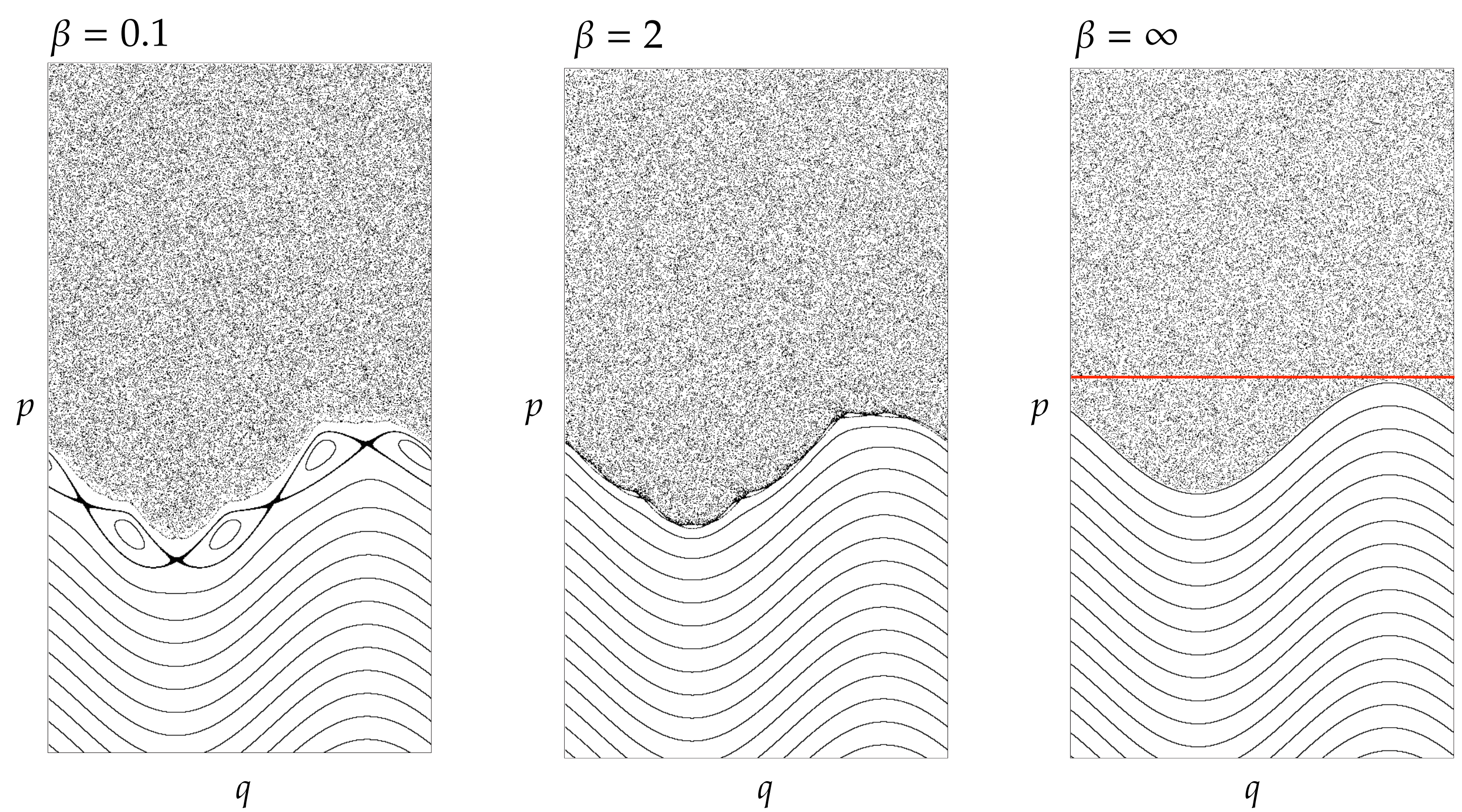
$$S_1 : \begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q + \tau T'(p) \\ p - \tau V'(q + \tau T'(p)) \end{pmatrix}$$

**where**

$$T(p) = \left[\frac{s}{2}(p-d)^2 + \omega(p-d)\right]\theta_\beta(p-d) \qquad \left( \theta_\beta(p) \equiv \frac{1}{2}\left[\tanh(\beta p) + 1\right] \right)$$

$$V(q) = K\cos(2\pi q)$$



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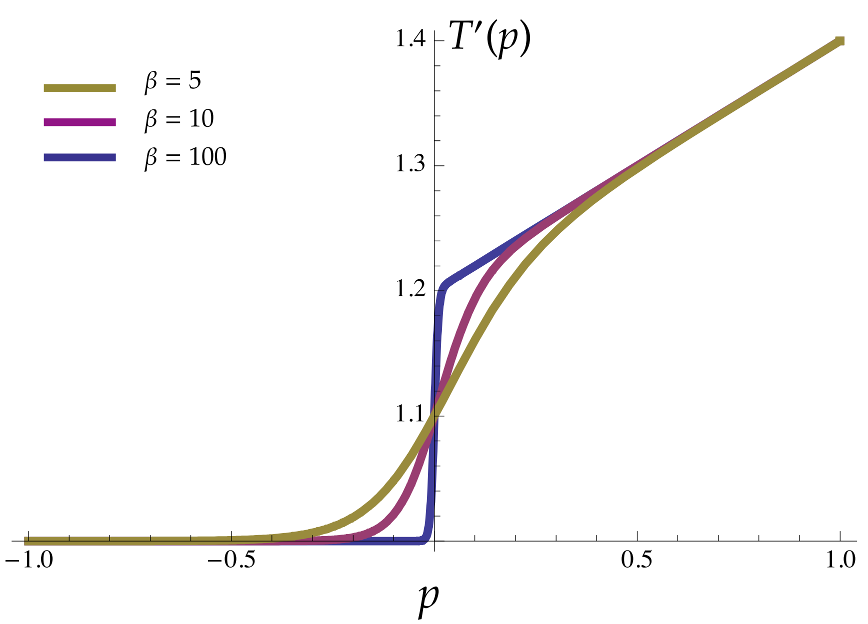
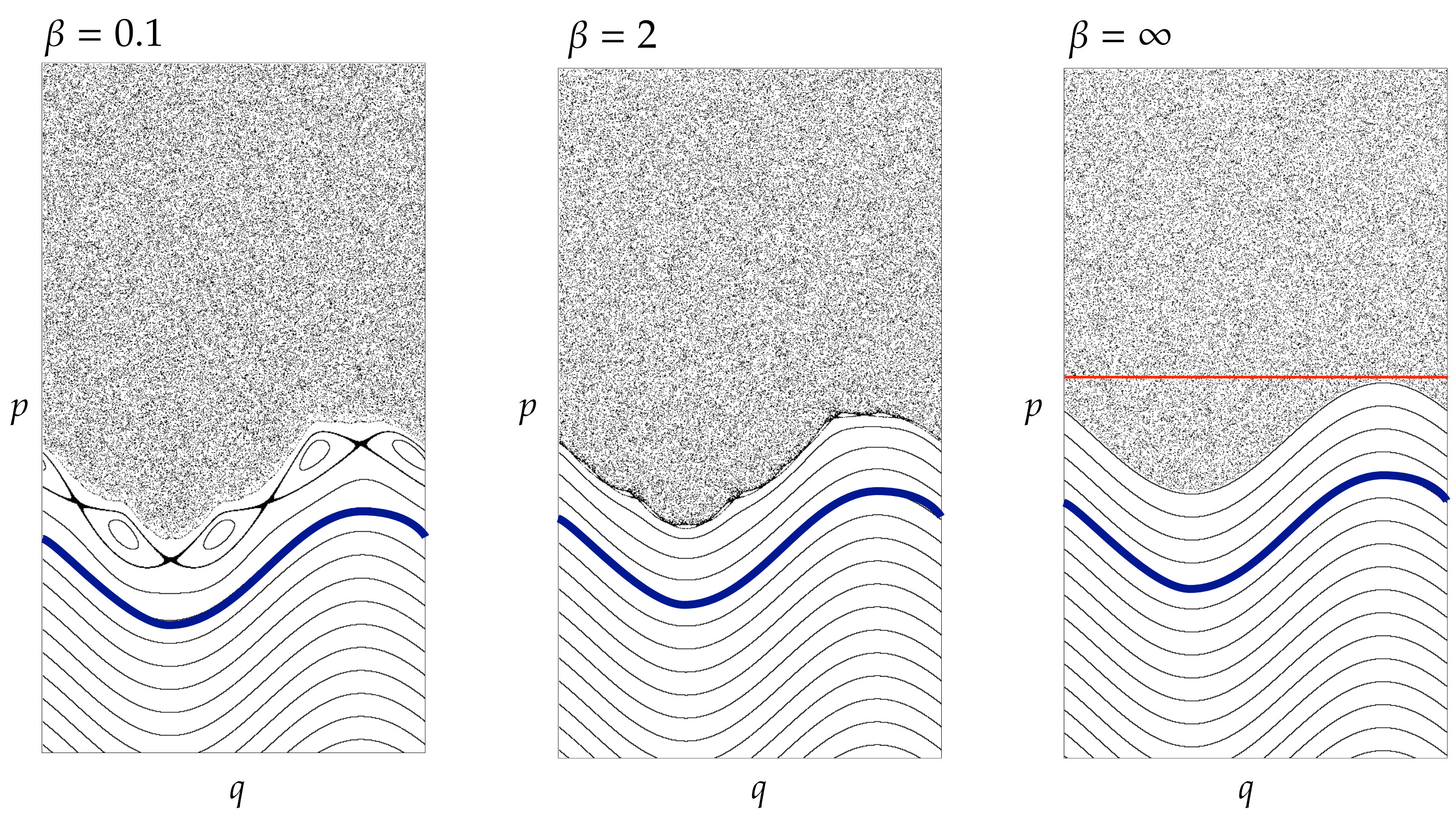


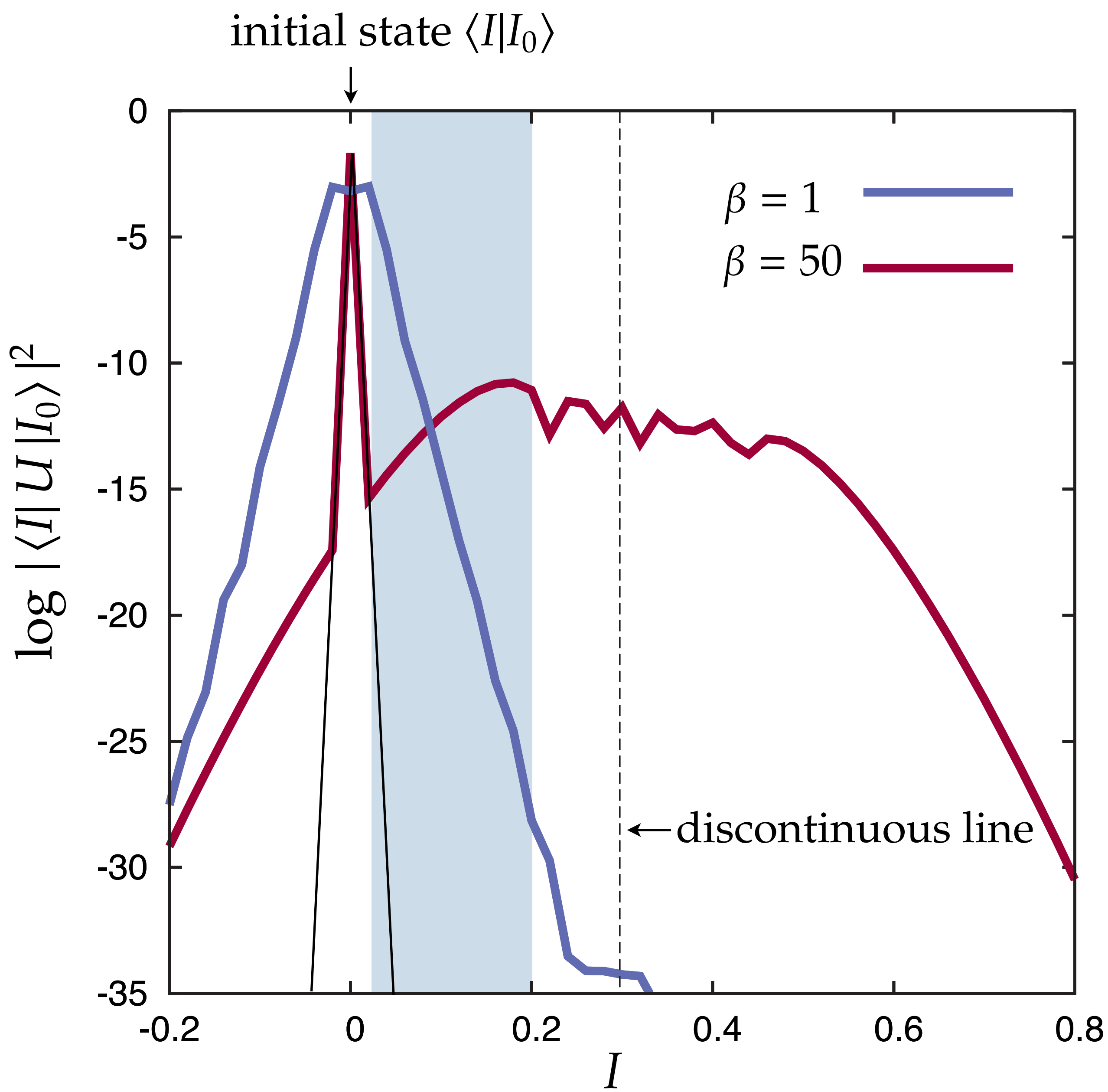



# Anomalous tail in the action representation

**1-step time evolution: $\langle I|U|I_0\rangle$ where $U = e^{-\frac{i}{\hbar}T(p)}e^{-\frac{i}{\hbar}V(q)}$**

Here $|I\rangle$ denotes the eigenfunction of the integrable map $L$:

$$U_0|I\rangle = e^{-\frac{i}{\hbar}E}|I\rangle \quad \text{where} \quad U_0 = e^{-\frac{i}{\hbar}\omega p}e^{-\frac{i}{\hbar}K\sin q}$$



# Semiclassical analysis for discontinuous limit ($\beta = \infty$)

**1-step propagator in the action representation**

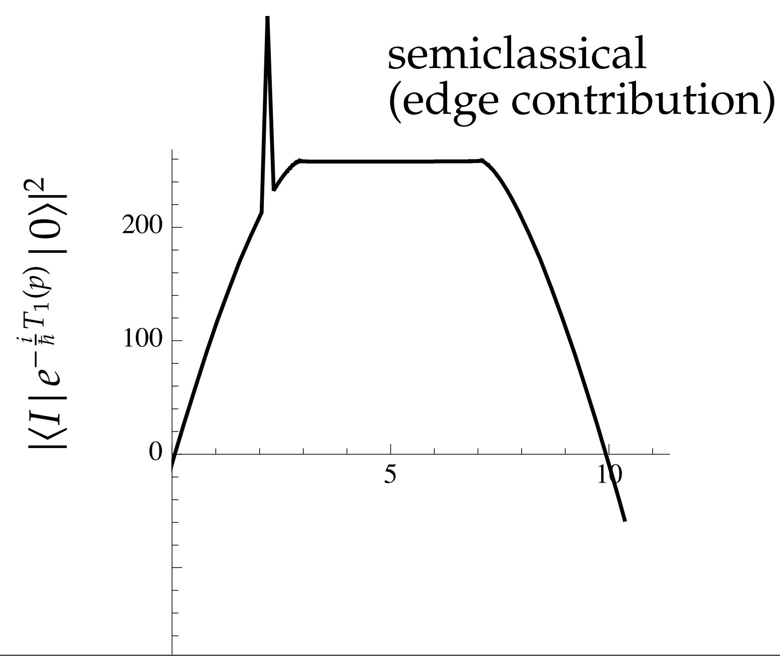
$$\langle I' | \hat{U} | I \rangle = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dq' \int_{-\infty}^{\infty} dp \exp\left[ -\frac{i}{\hbar} \left\{ F(q', p, q; I, I') \right\} \right]$$

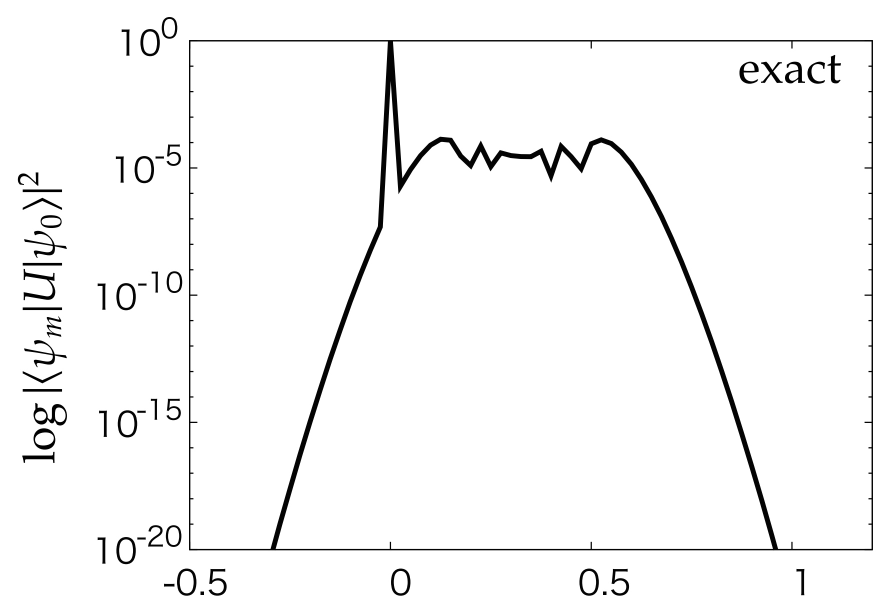
**where**

$$F(q', p, q; I, I') := S(I', q') - S(I, q) - p(q' - q) + T(p) + V(q)$$

**Since $T(p)$ has a discontinuity at $p = d$,**

$$\int dq' \int dp \int dq = \int dq' \left\{ \int_{-\infty}^{d} dp + \int_{d}^{+\infty} dp \right\} \int dq$$



# Semiclassical analysis for large but finite $\beta$

**1-step propagator in the action representation**

$$\langle I'|\,\hat{U}\,|I\rangle = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dq' \int_{-\infty}^{\infty} dp \exp\left[-\frac{i}{\hbar}\left\{F(q',p,q;I,I')\right\}\right]$$

**where**

$$F(q',p,q;I,I') := S(I',q') - S(I,q) - p(q'-q) + T(p) + V(q)$$

**for the present map**

$$T(p) = \left[\frac{s}{2}(p-d)^2 + \omega(p-d)\right]\theta_\beta(p-d)$$

$$V(q) = K\cos(2\pi q)$$

$$S(I,q) = Iq + K\sin q$$

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$$T(p) = \left[\frac{s}{2}(p-d)^2 + \omega(p-d)\right]\theta_\beta(p-d)$$

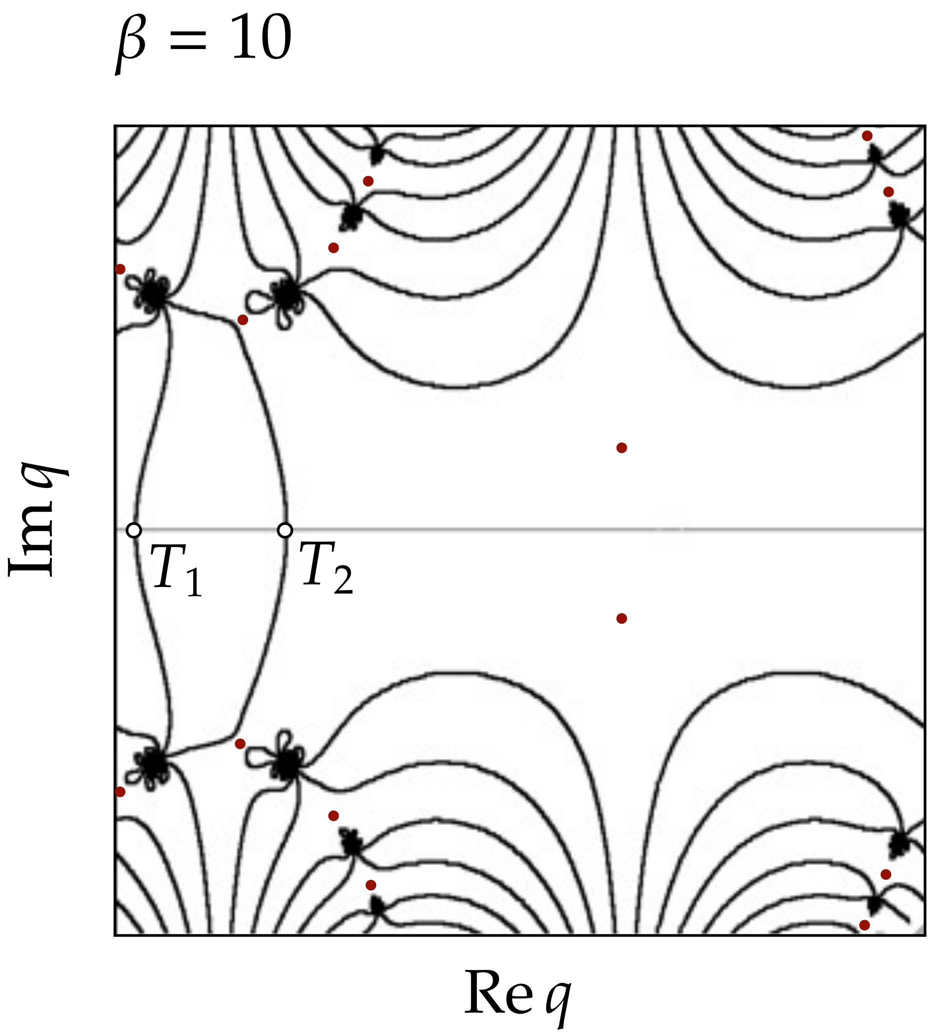
$$V(q) = K\cos(2\pi q)$$

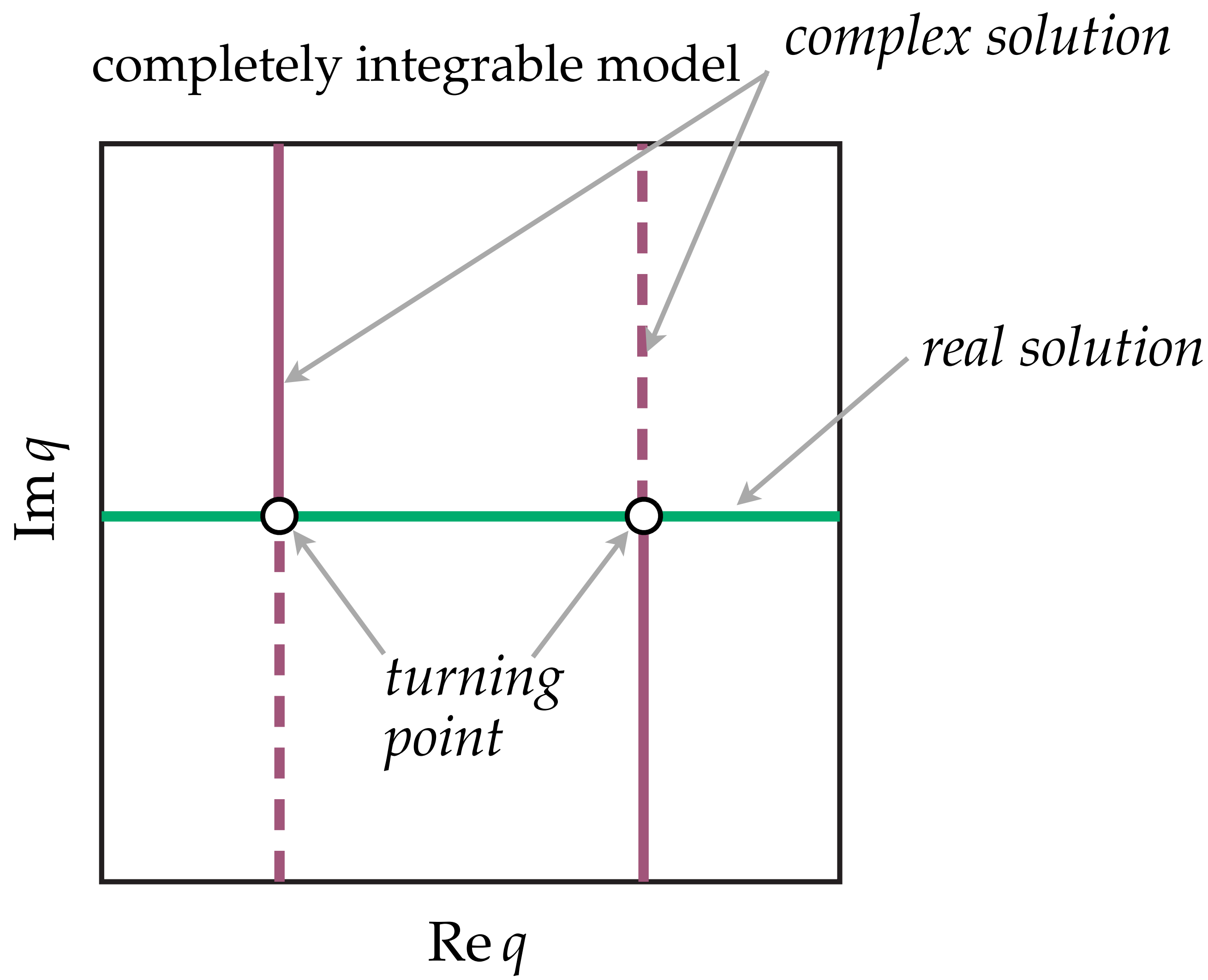
$$S(I,q) = Iq + K\sin q$$

**Saddle point condition:**

$$\frac{\partial F}{\partial q'} = 0, \quad \frac{\partial F}{\partial p} = 0, \quad \frac{\partial F}{\partial q} = 0 \iff \begin{pmatrix} I \\ q \end{pmatrix} \overset{S}{\longmapsto} \begin{pmatrix} q \\ p \end{pmatrix} \overset{S_2}{\longmapsto} \begin{pmatrix} q' \\ p' \end{pmatrix} \overset{S}{\longmapsto} \begin{pmatrix} I' \\ q' \end{pmatrix}$$

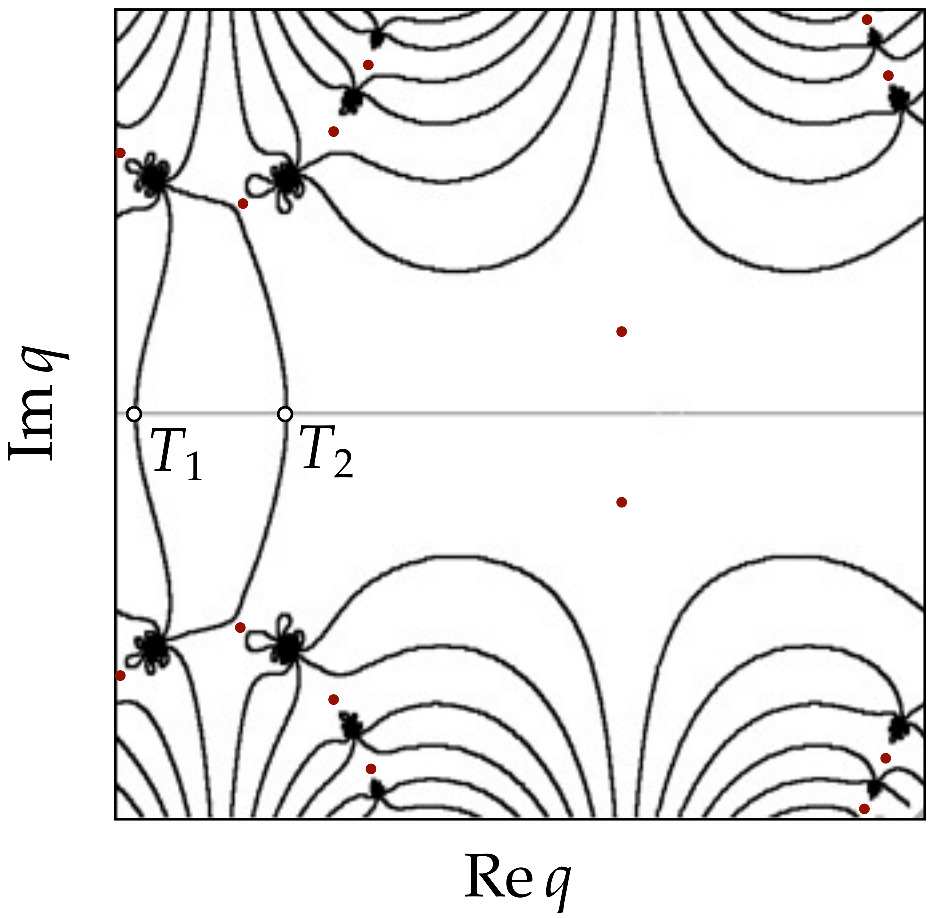
# A set of saddle points



# Two types of turning points

$\beta = 10$




- ∘ turning points on the real manifold
- • turning points in the complex plane

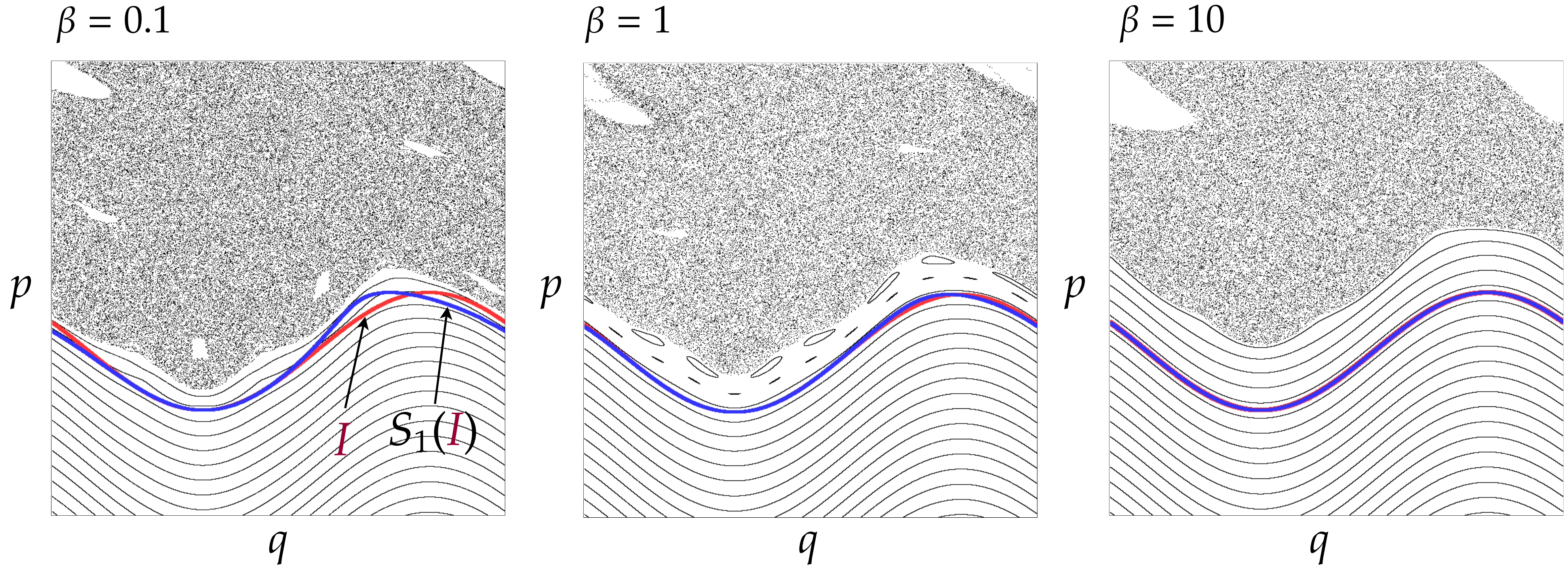
1. **Turning points on the real manifold**
   locally highly degenerated, reflecting tangency between $\boldsymbol{I}$ and $\boldsymbol{S_1(I)}$

2. **Turning points in the complex plane**
   increase as $\boldsymbol{\beta}$ gets large, reflecting the increase of singularities, and possibly the existence of *natural boundaries*

# 1-step time evolution of the real manifold $I$

**$I$ : An invariant curve of the integrable map**

**$S_1(I)$ : 1-step time evolution of $I$**



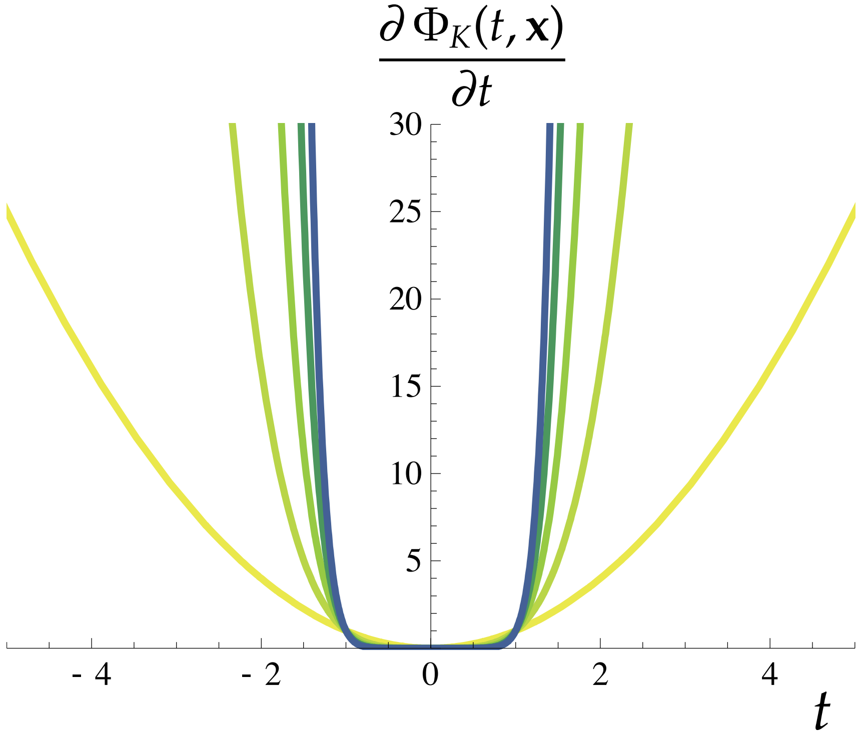

With increase in $\beta$, the initial manifold $I$ comes closer to KAM curves, and moves very slightly within a single step.

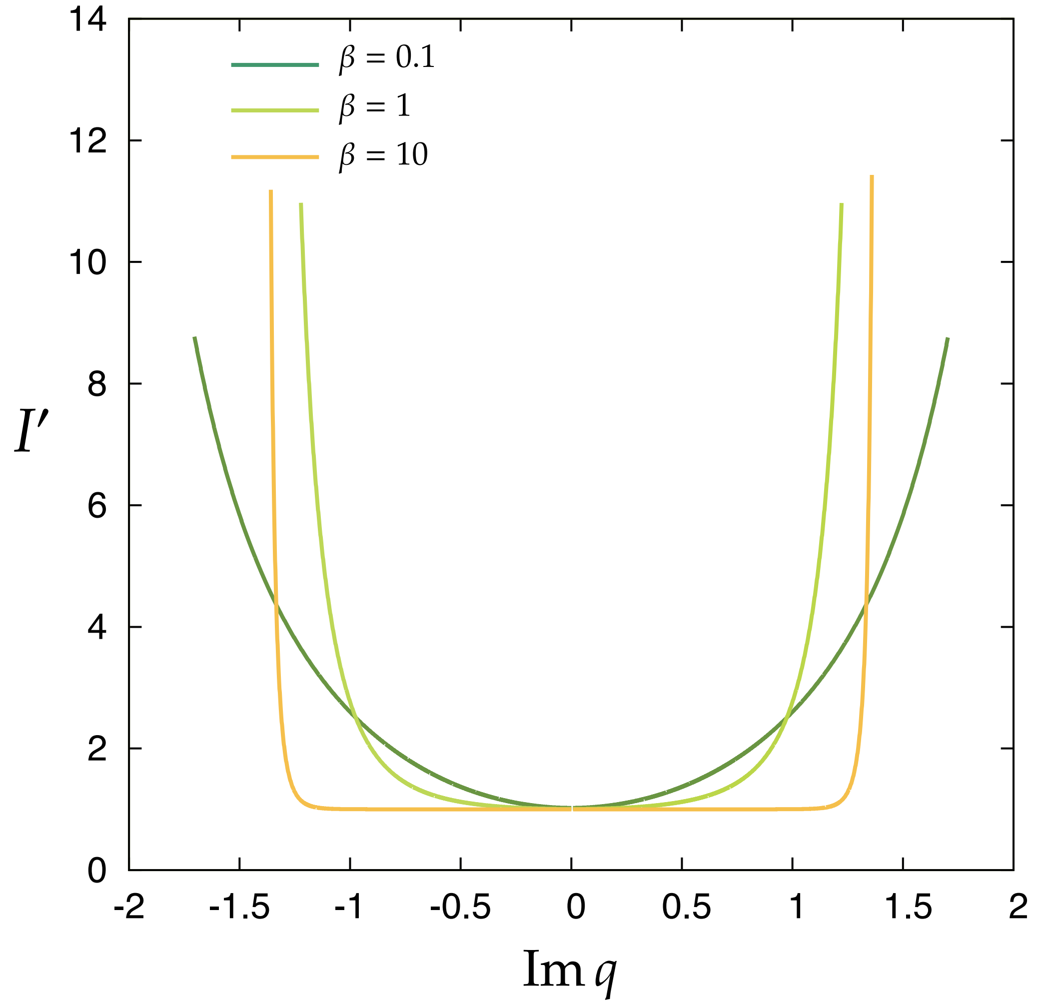
# Diffraction integrals with coalescing saddles

**Integrals with coalescing saddles**

$$\Psi_K(\mathbf{x}) = \int_{-\infty}^{\infty} \exp(\mathrm{i}\,\Phi_K(t;\mathbf{x}))\,dt, \qquad \text{where} \quad \Phi_K(t;\mathbf{x}) = t^{K+2} + \sum_{m=1}^{K} x_m t^m$$

$$\mathbf{x} = (x_1, 0, \cdots, 0)$$



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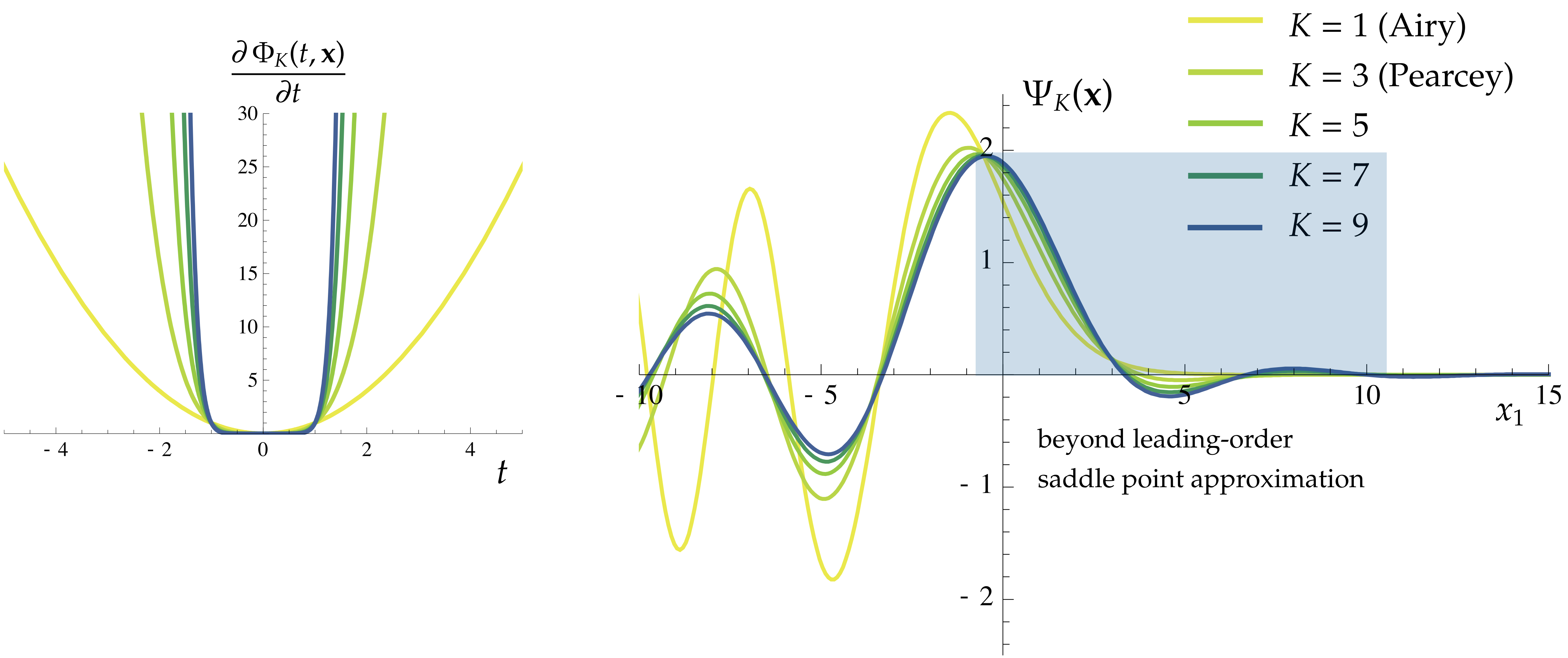
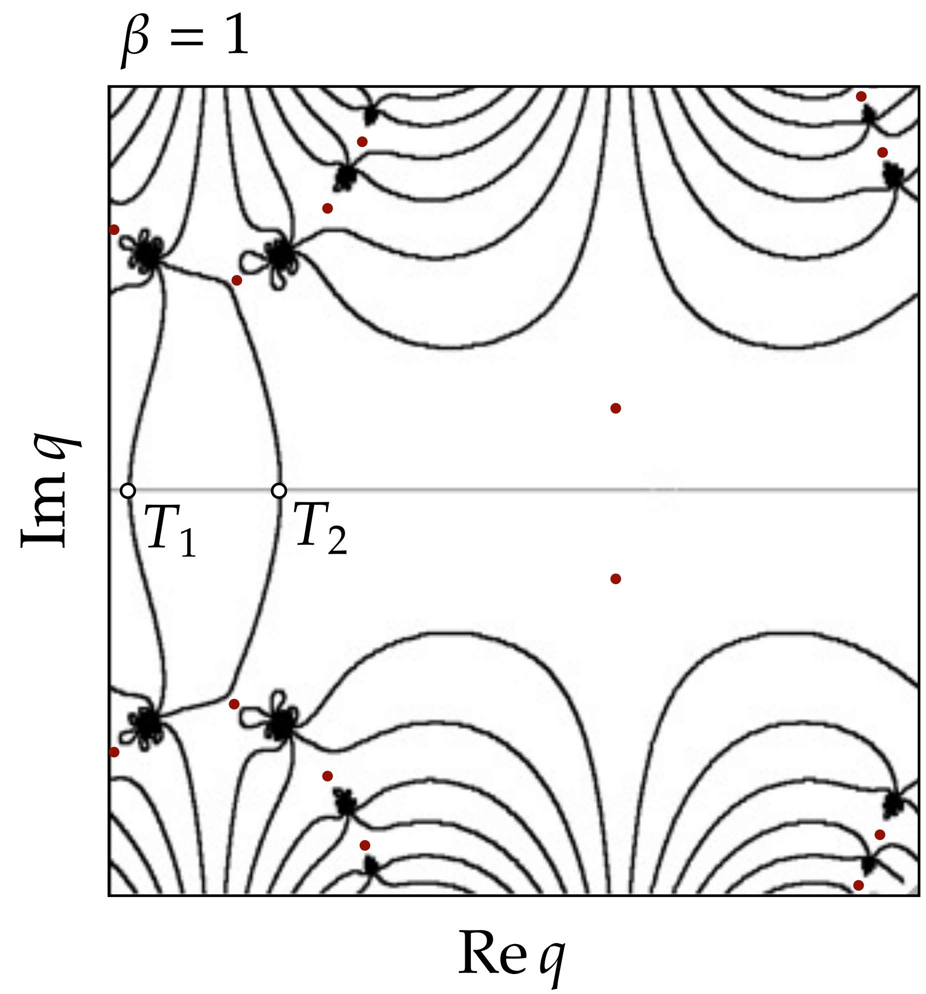


# Two types of turning points

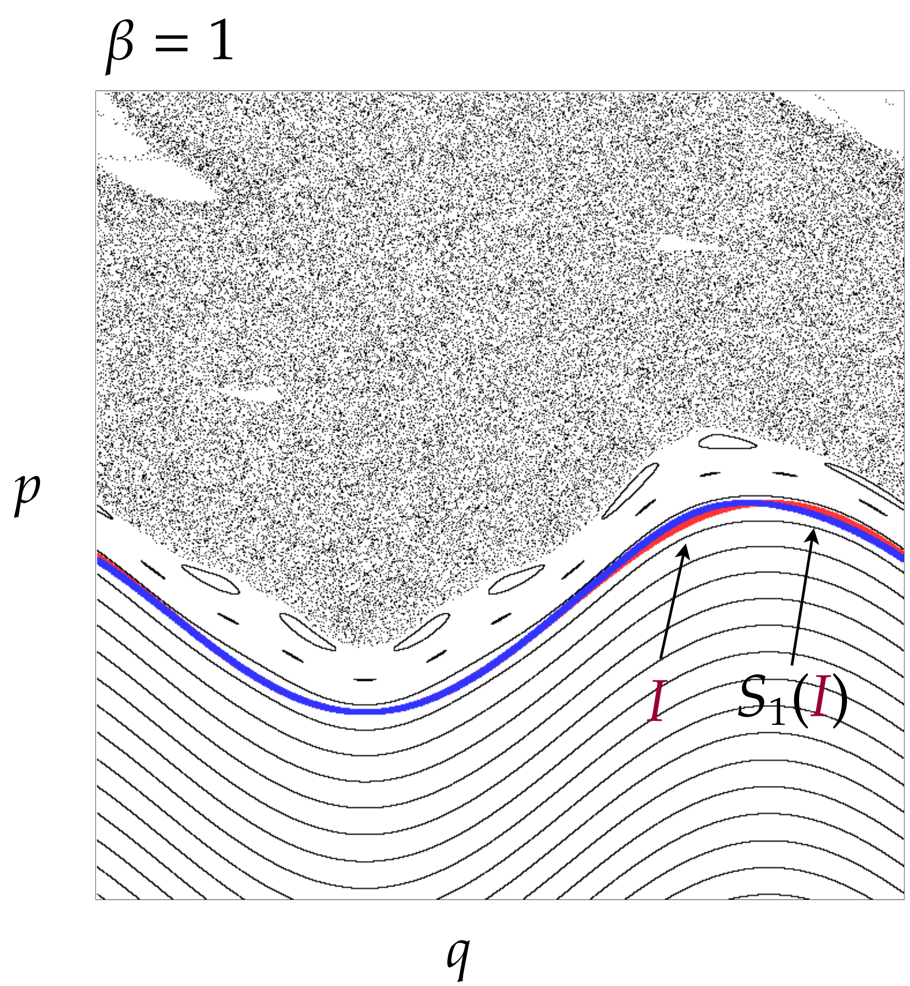
1. **Turning points on the real manifold**
   locally highly degenerated, reflecting tangency between ***I*** and $S_1(\boldsymbol{I})$

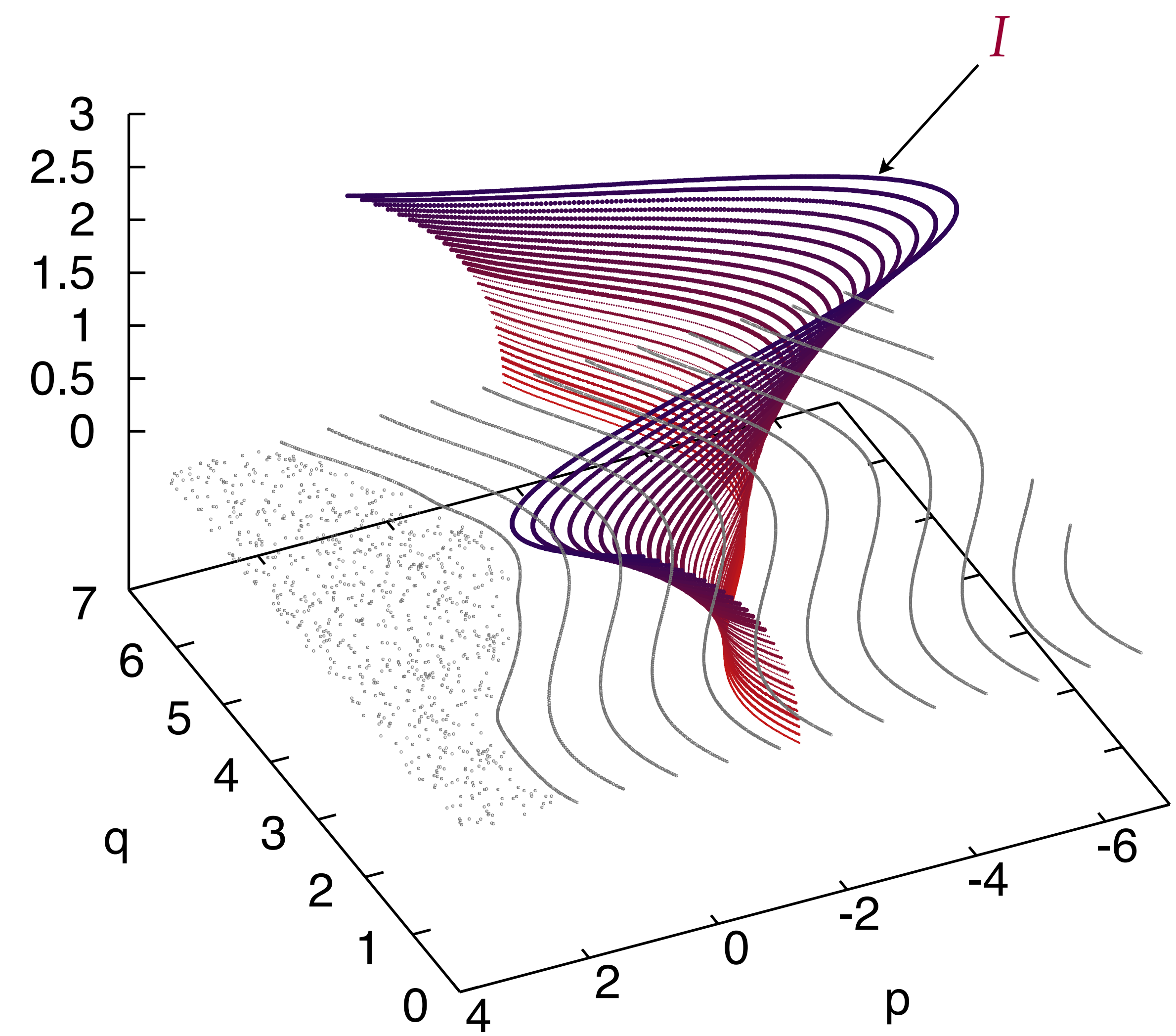
2. **Turning points in the complex plane**
   increase as $\boldsymbol{\beta}$ gets large, reflecting the increase of singularities, and possibly the existence of *natural boundaries*



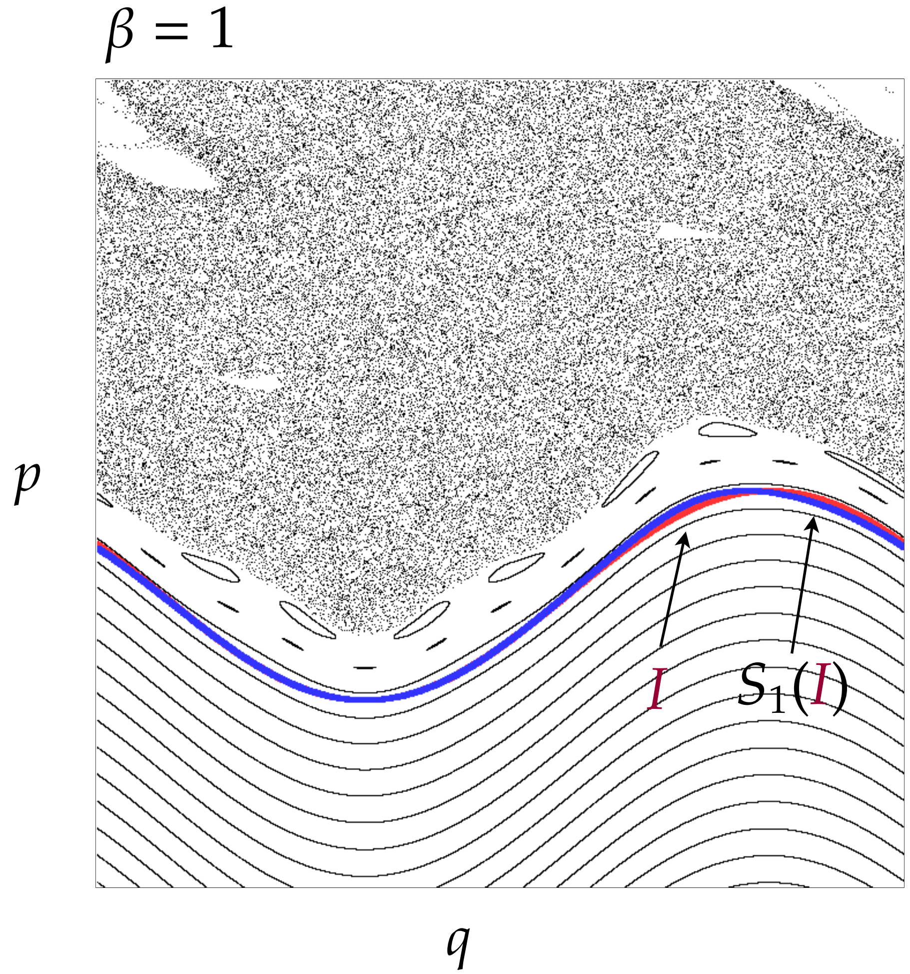

∘ turning points on the real manifold
• turning points in the complex plane

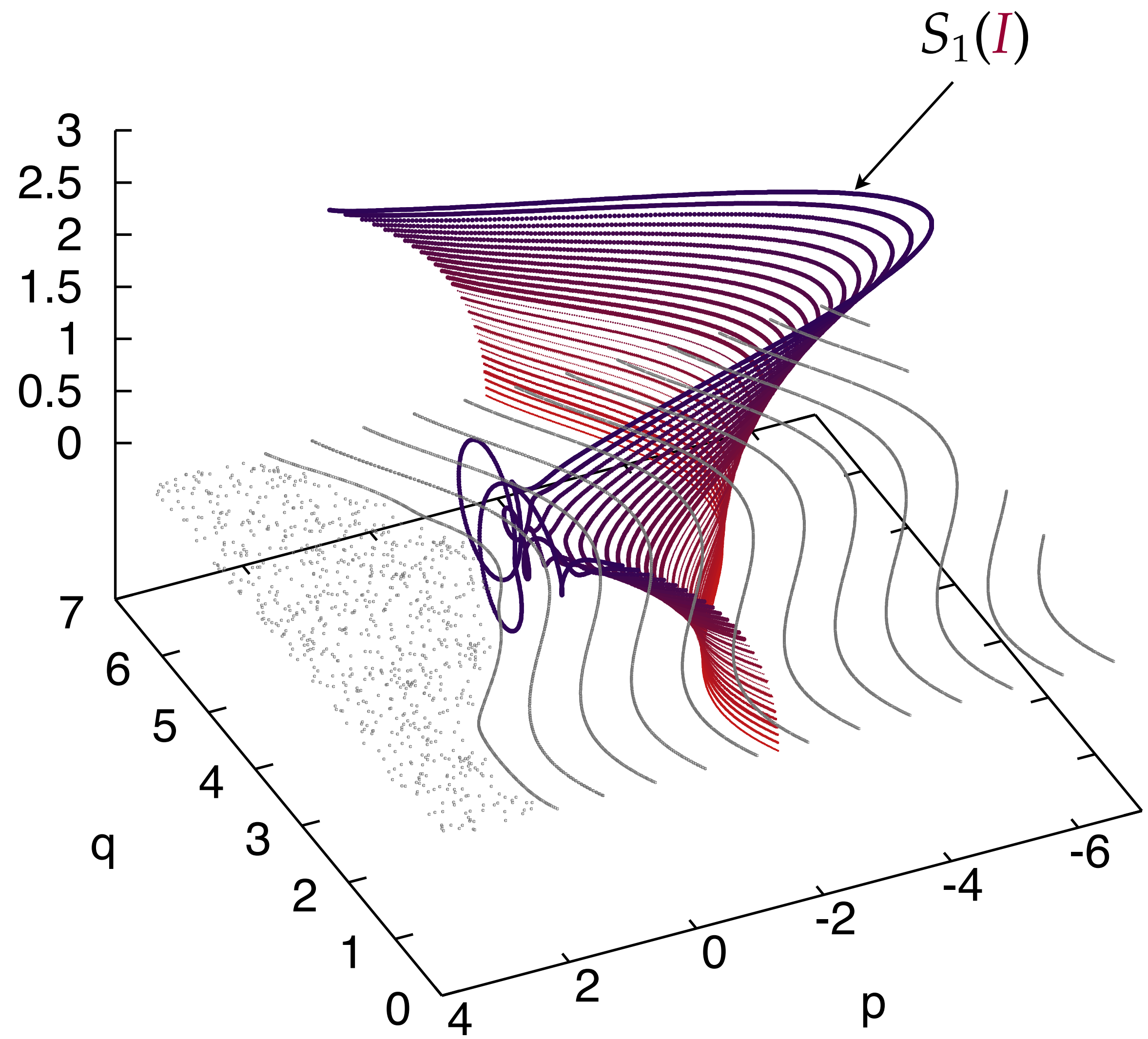
# Large deformation in the complex plane



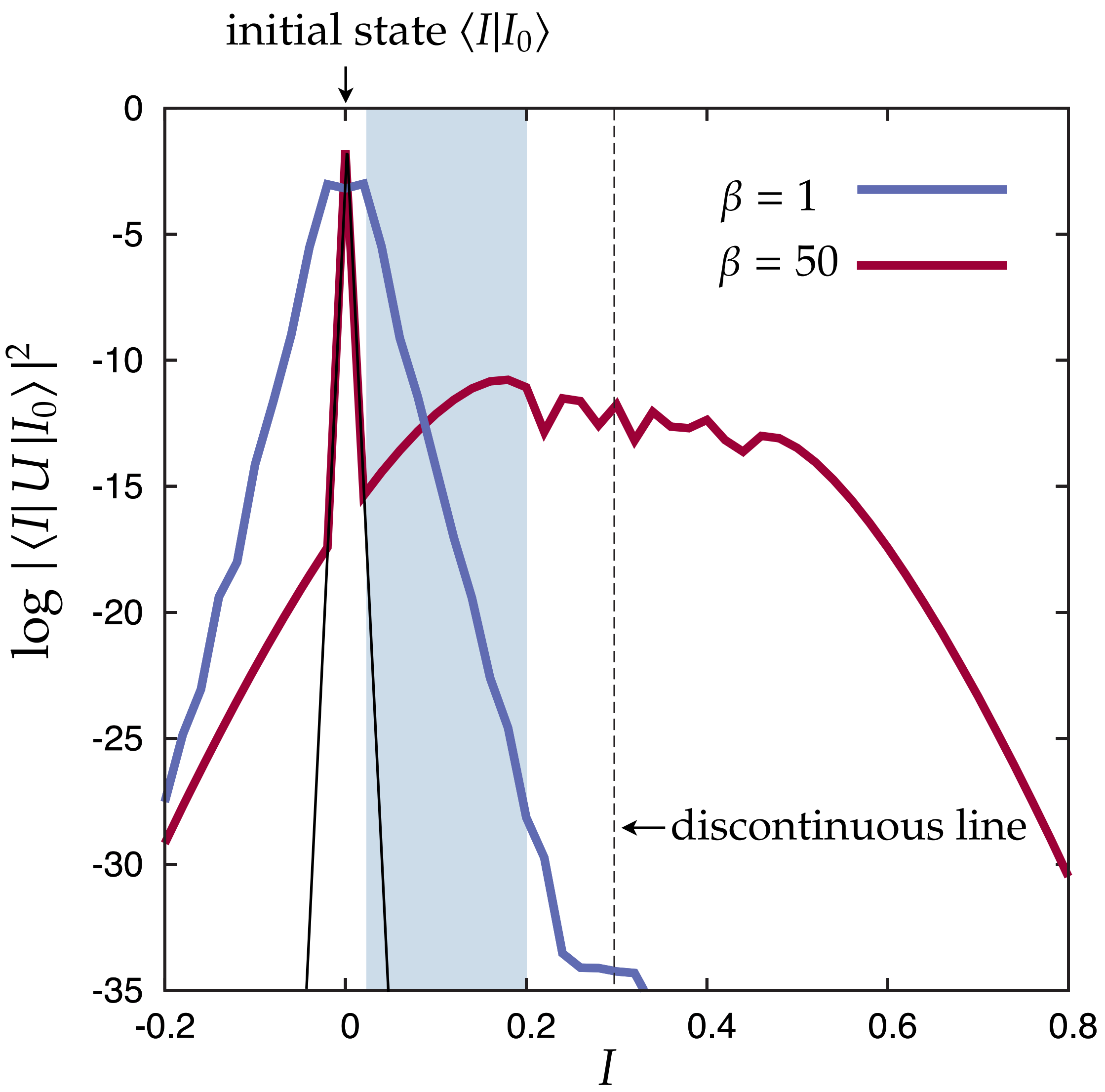



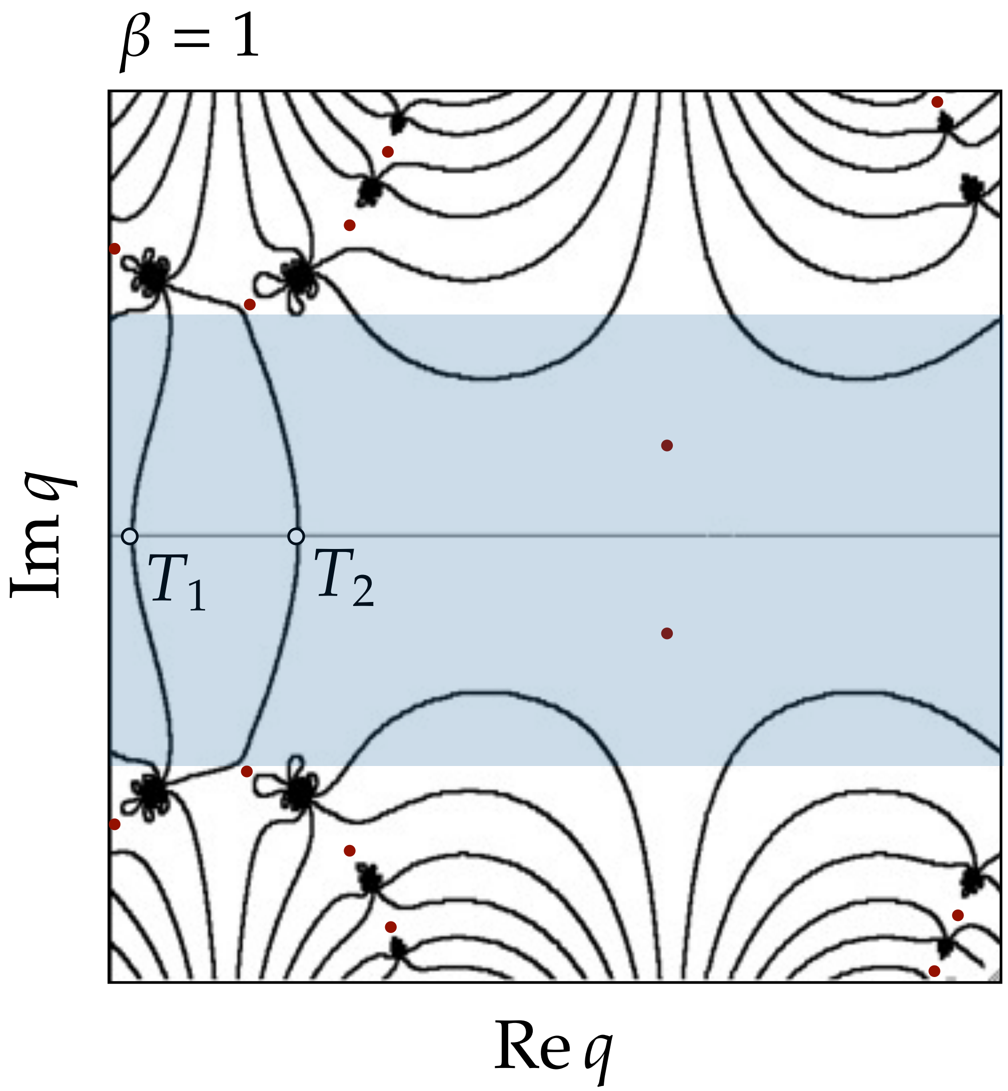
# Large deformation in the complex plane



# Range beyond the semiclassical approximation



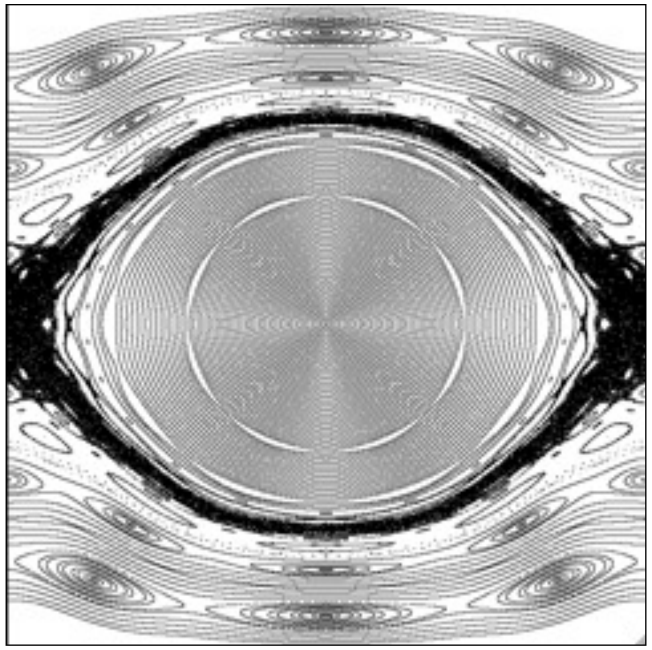

- ∘ turning points on the real manifold
- • turning points in the complex plane

# Standard map and integrable approximation

**Quantum unitary operator**

$$\hat{U} = \exp\left[-\frac{\mathrm{i}}{\hbar}\tau\frac{V(\hat{q})}{2}\right]\exp\left[-\frac{\mathrm{i}}{\hbar}\tau T(\hat{p})\right]\exp\left[-\frac{\mathrm{i}}{\hbar}\tau\frac{V(\hat{q})}{2}\right]$$

$$T(p) = \frac{p^2}{2}, \quad V(q) = K\sin q$$




**Integrable approximation of $\hat{U}$**

$$\hat{U}^{(M)} := \exp\left[-\frac{\mathrm{i}}{\hbar}\tau\hat{H}_{\mathrm{eff}}^{(M)}(\hat{q},\hat{p})\right]$$

**where**

$$\hat{H}_{\mathrm{eff}}^{(M)}(\hat{q},\hat{p}) = \hat{H}_1(\hat{q},\hat{p}) + \sum_{j=3}^{M}\left(\frac{\mathrm{i}\tau}{\hbar}\right)^{j-1}\hat{H}_j(\hat{q},\hat{p})$$

**$\hat{H}_j$: the $j$-th order term in the Baker-Campbell-Hausdorff (BCH) series.**

**Classical Hamiltonian**

$$H_{\mathrm{eff}}^{(M)}(q,p) = H_1(q,p) + \sum_{\substack{j=3 \\ (j\in\text{odd int.})}}^{M}\left(\frac{\mathrm{i}\tau}{\hbar}\right)^{j-1}H_j(q,p).$$

**$H_j(q,p)$: obtained by replacing commutators in the BCH series by Poisson brackets.**

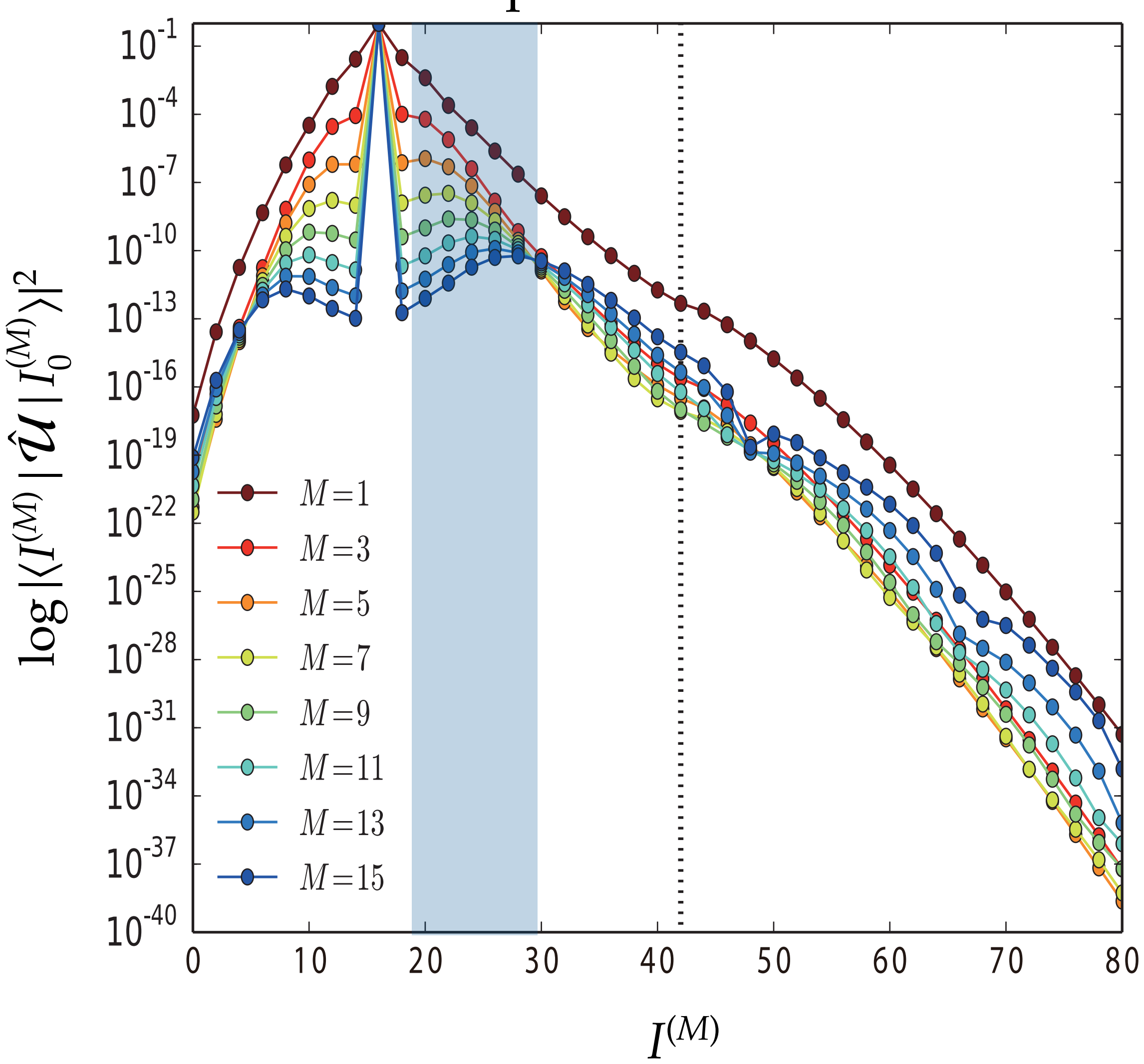
# 1-step time evolution in the action representation

**1-step time evolution: $\langle I'^{(M)}|\,\hat{U}\,|I^{(M)}\rangle$ where $\hat{U} = e^{-\frac{i}{\hbar}T(p)}e^{-\frac{i}{\hbar}V(q)}$**

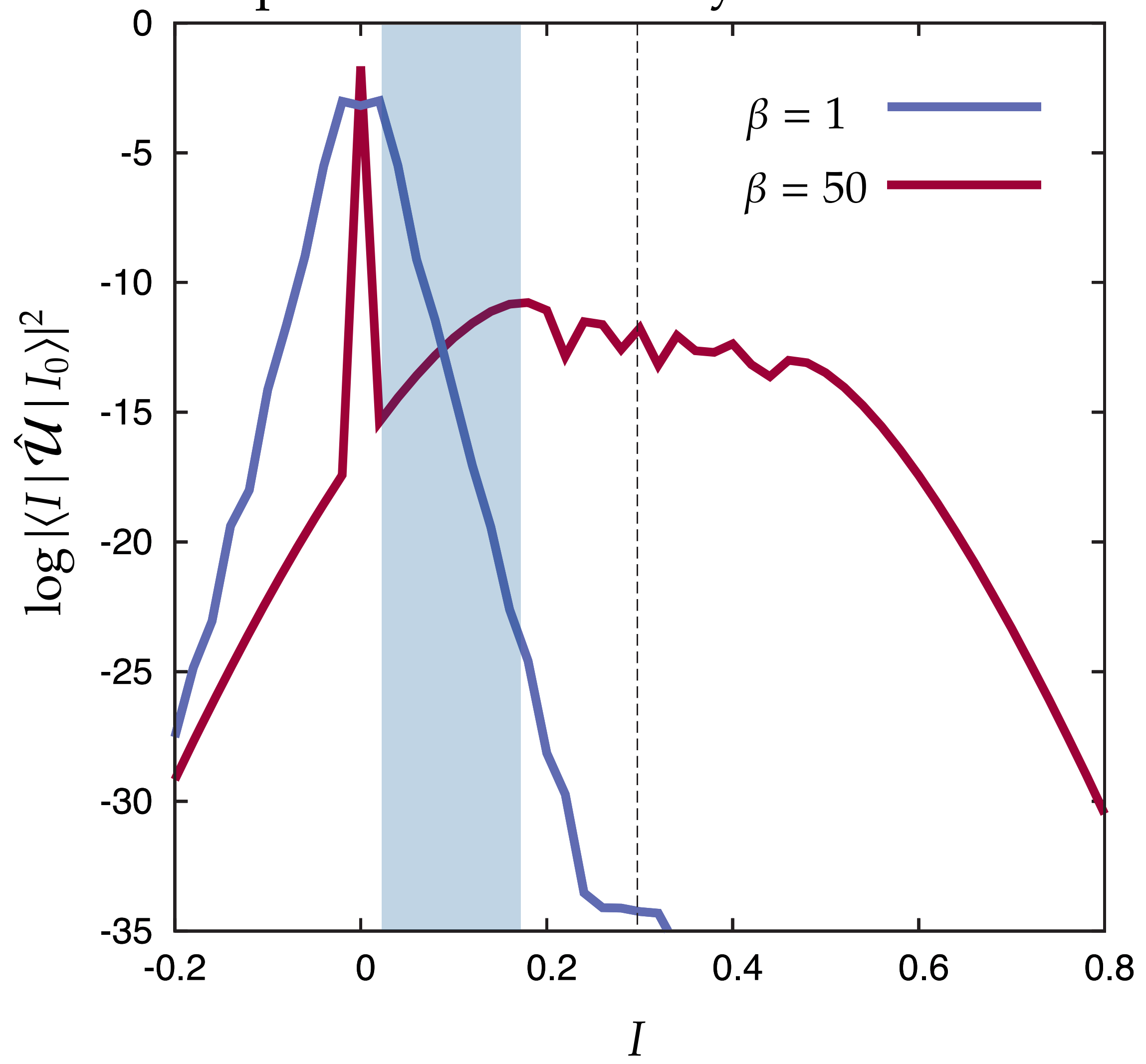
Here $|I^{(M)}\rangle$ denotes the eigenfunction of the integrable Hamiltonian $\hat{H}_{\rm eff}^{(M)}$:

$$\hat{H}_{\rm eff}^{(M)}|\,I^{(M)}\rangle = E_{\rm eff}^{(M)}|\,I^{(M)}\rangle$$

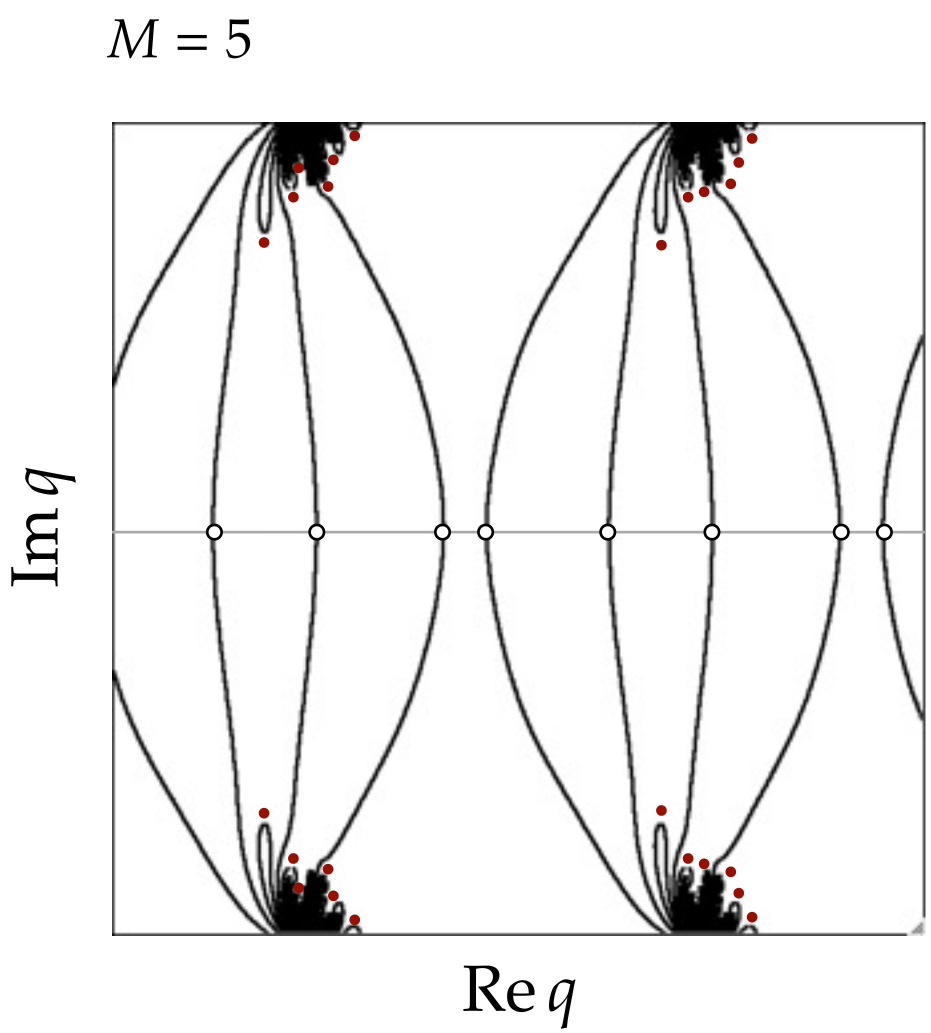
Standard map

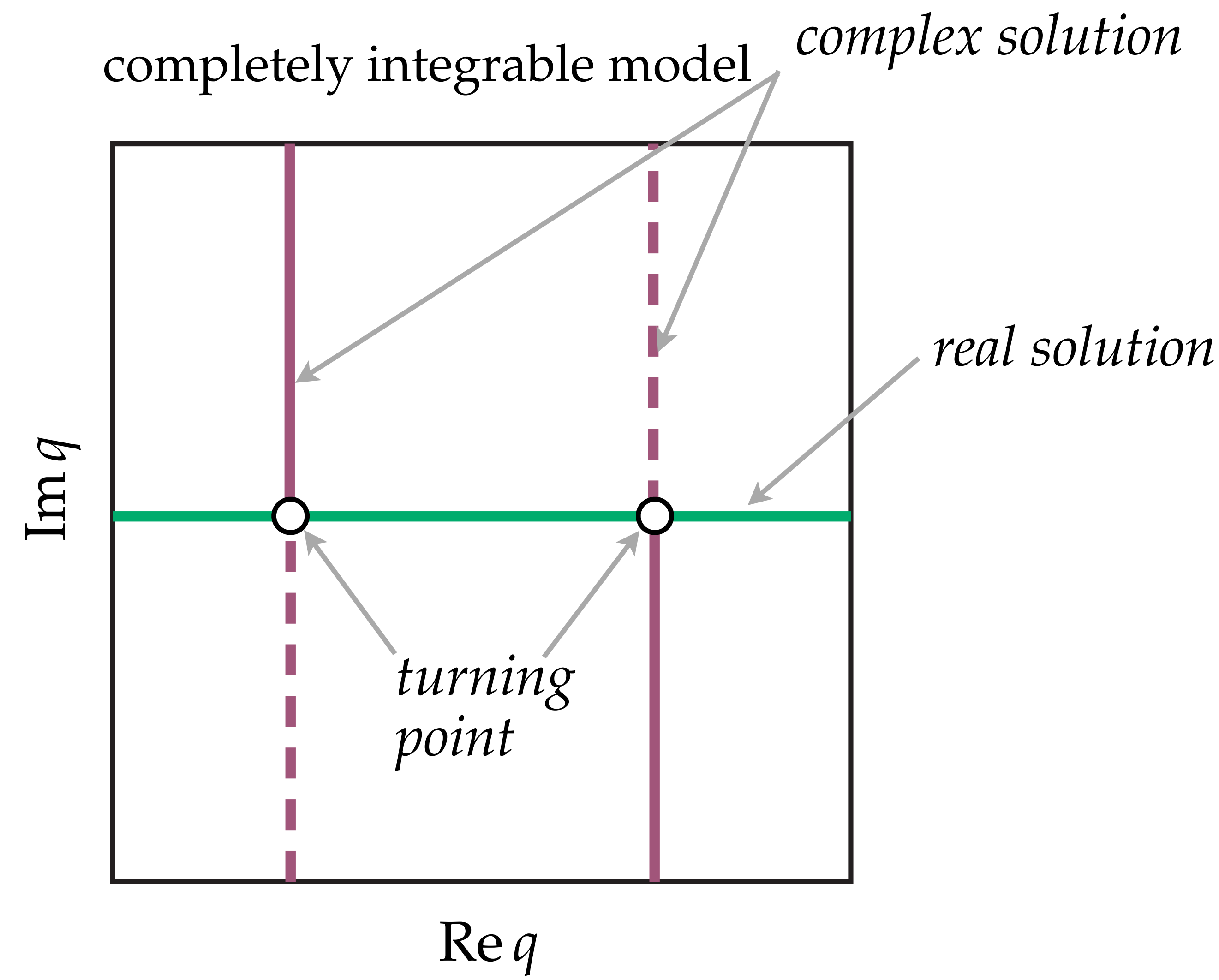



Map with discontinuity



# A set of saddle points



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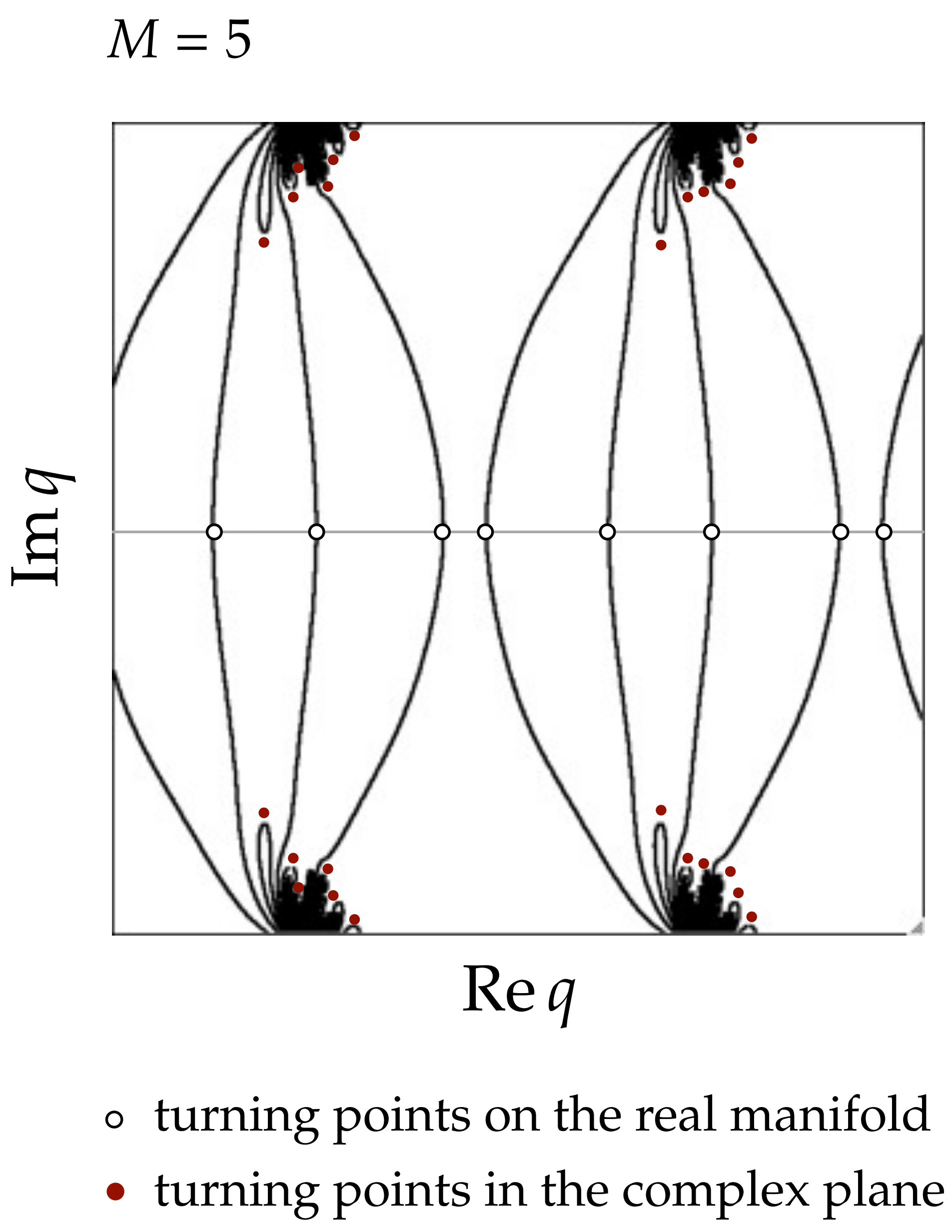


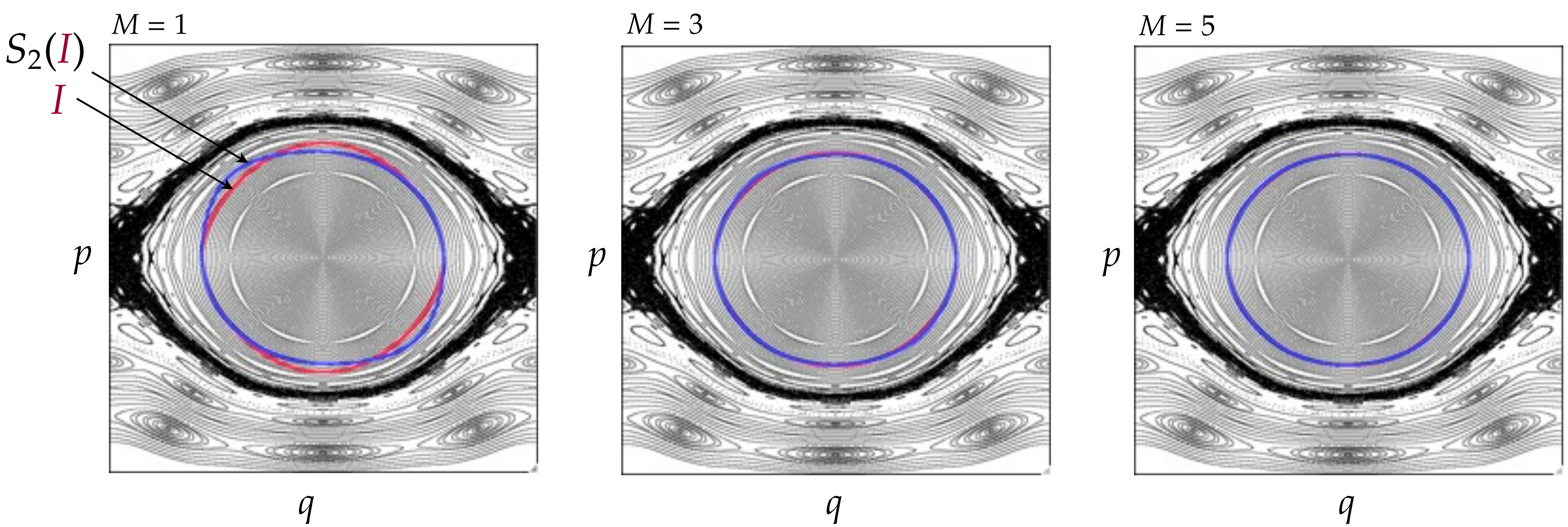

1. **Turning points on the real manifold**
   locally highly degenerated, reflecting tangency between $I$ and $F(I)$

2. **Turning points in the complex plane**
   increases as $M$ gets large, reflecting the increase of singularities, and possibly the existence of *natural boundaries*

# Integrable approximation

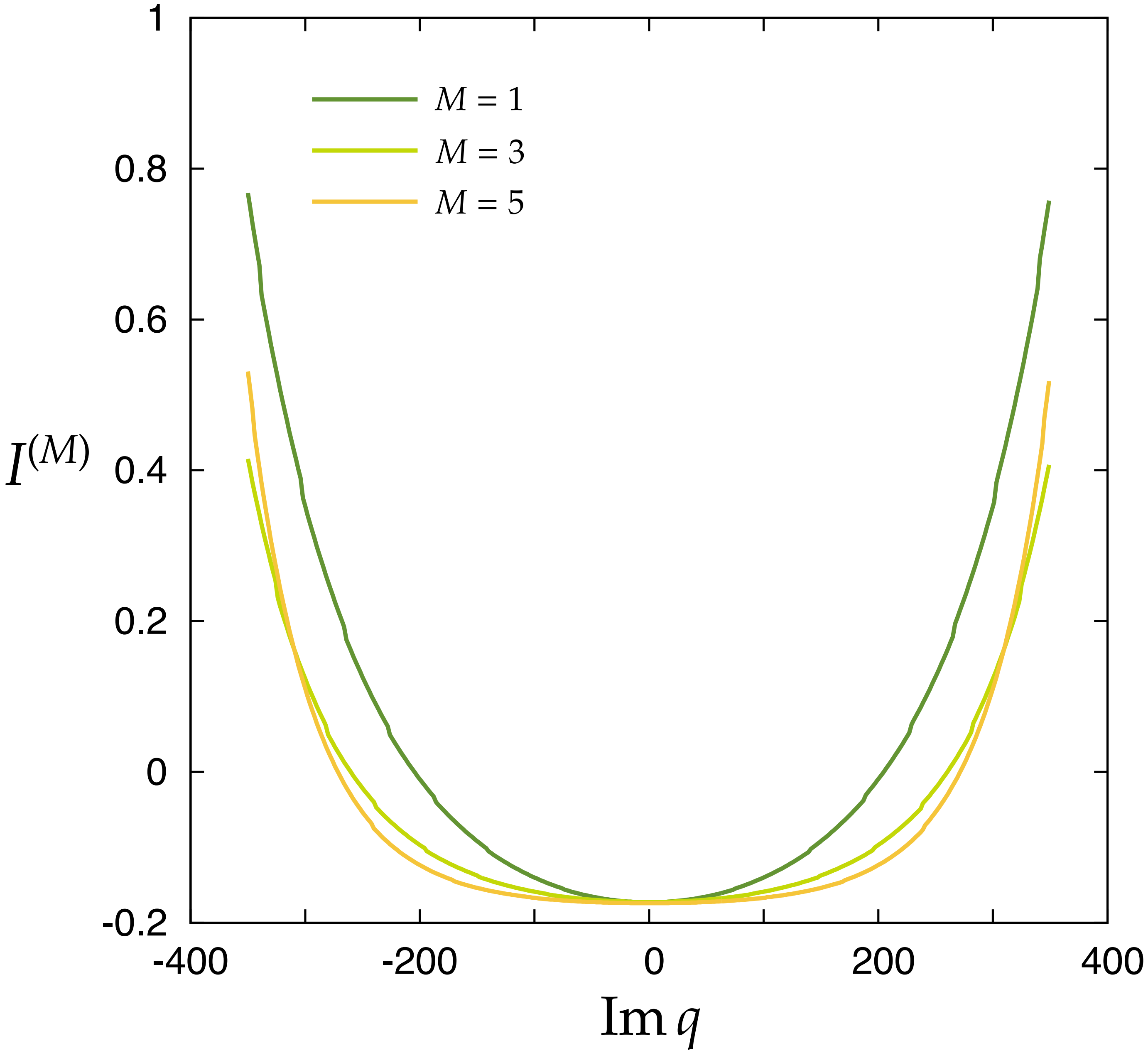
**$I$ : An invariant curve of the BCH integrable Hamiltonian**

**$S_2(I)$ : 1-step time evolution of $I$**



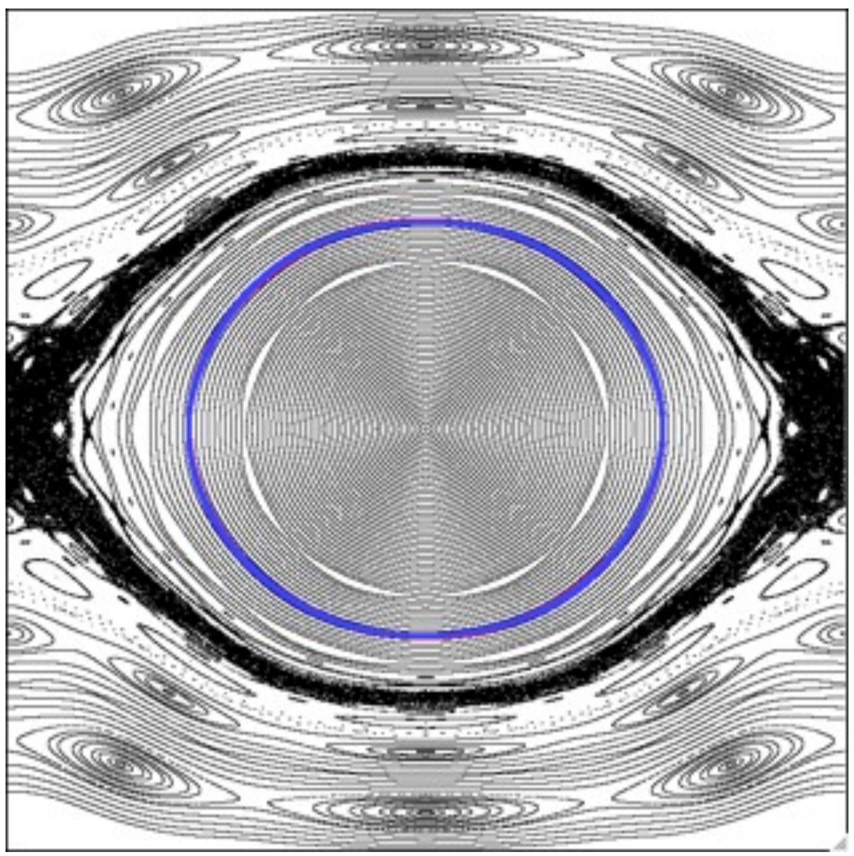

With increase in $M$, the initial manifold $I$ comes closer to KAM curves, and moves only slightly within a single step.

# Around the turning point on the real manifold



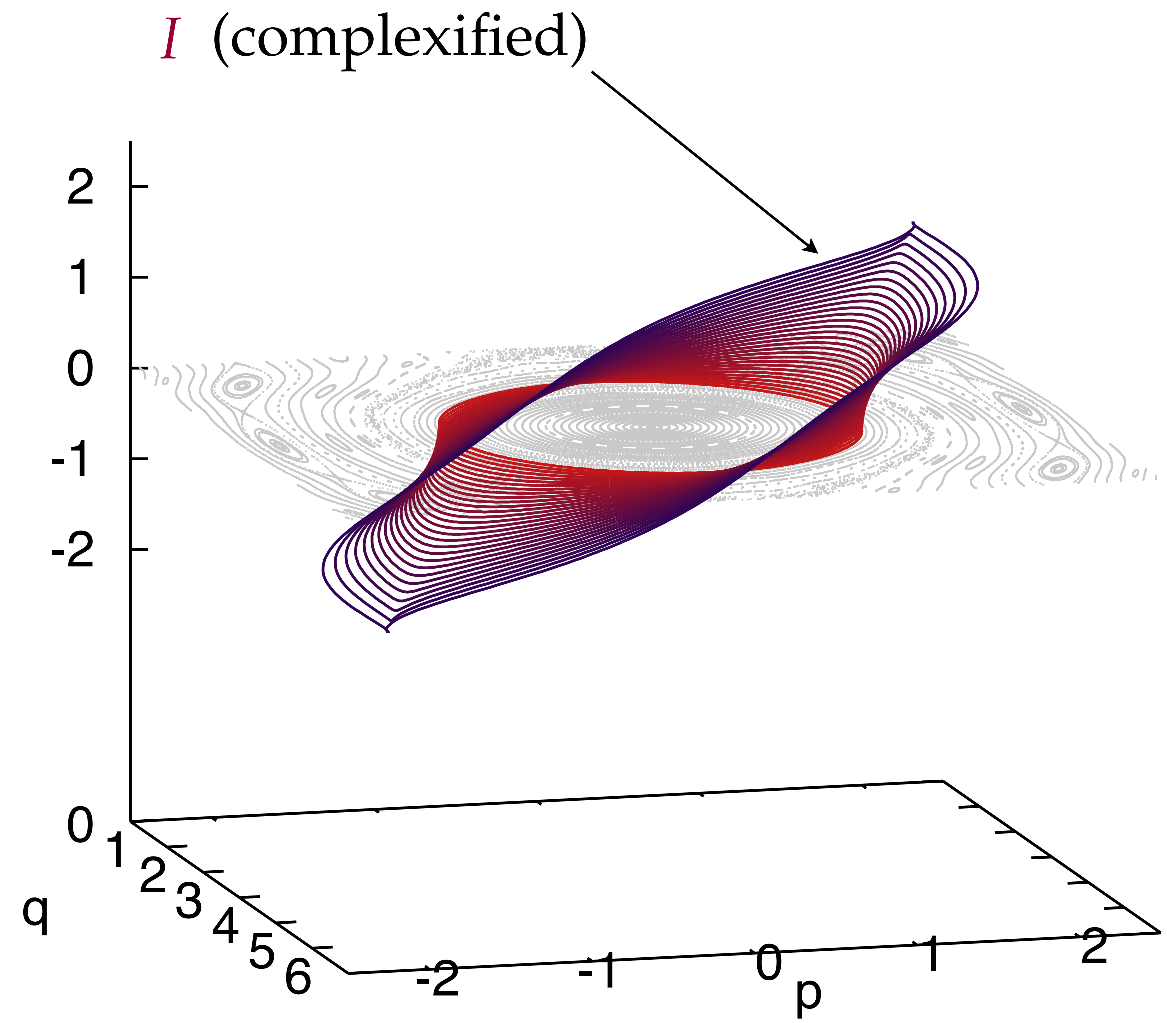
# Large deformation in the complex plane

$M = 5$



$p$

$q$



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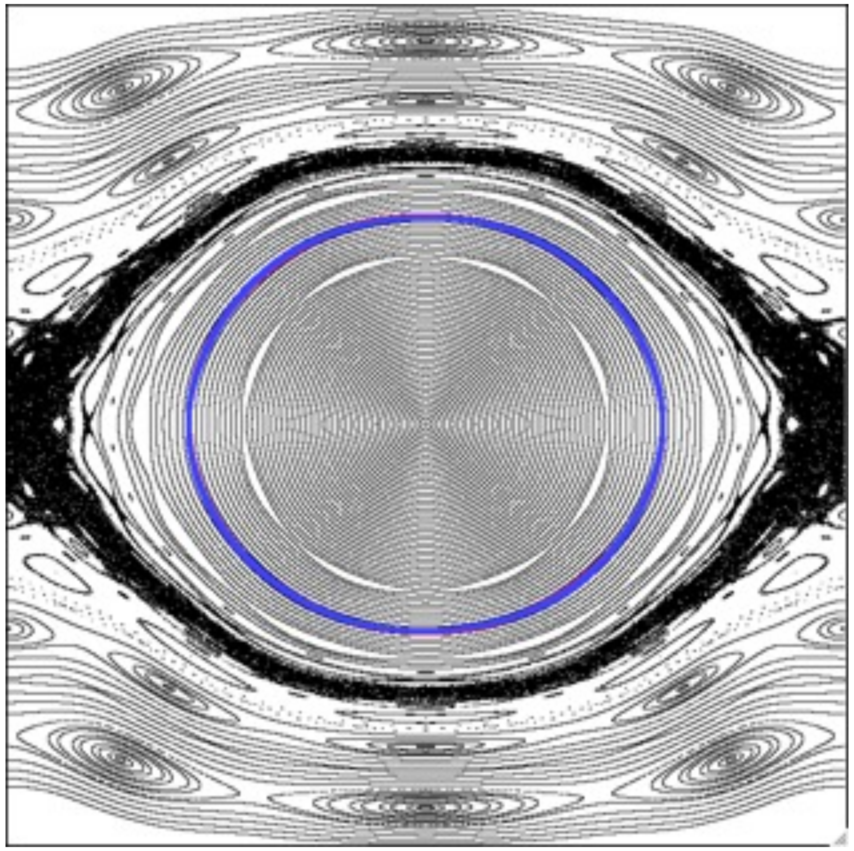


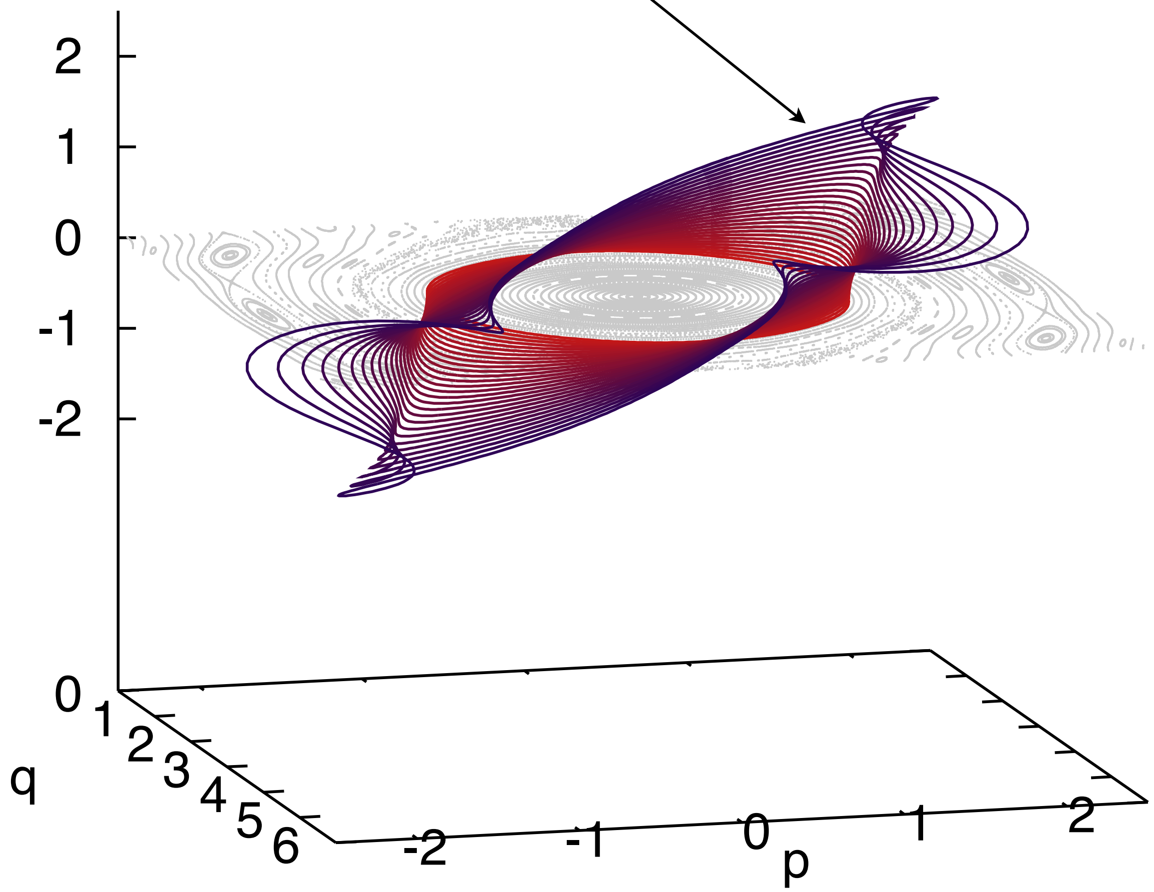

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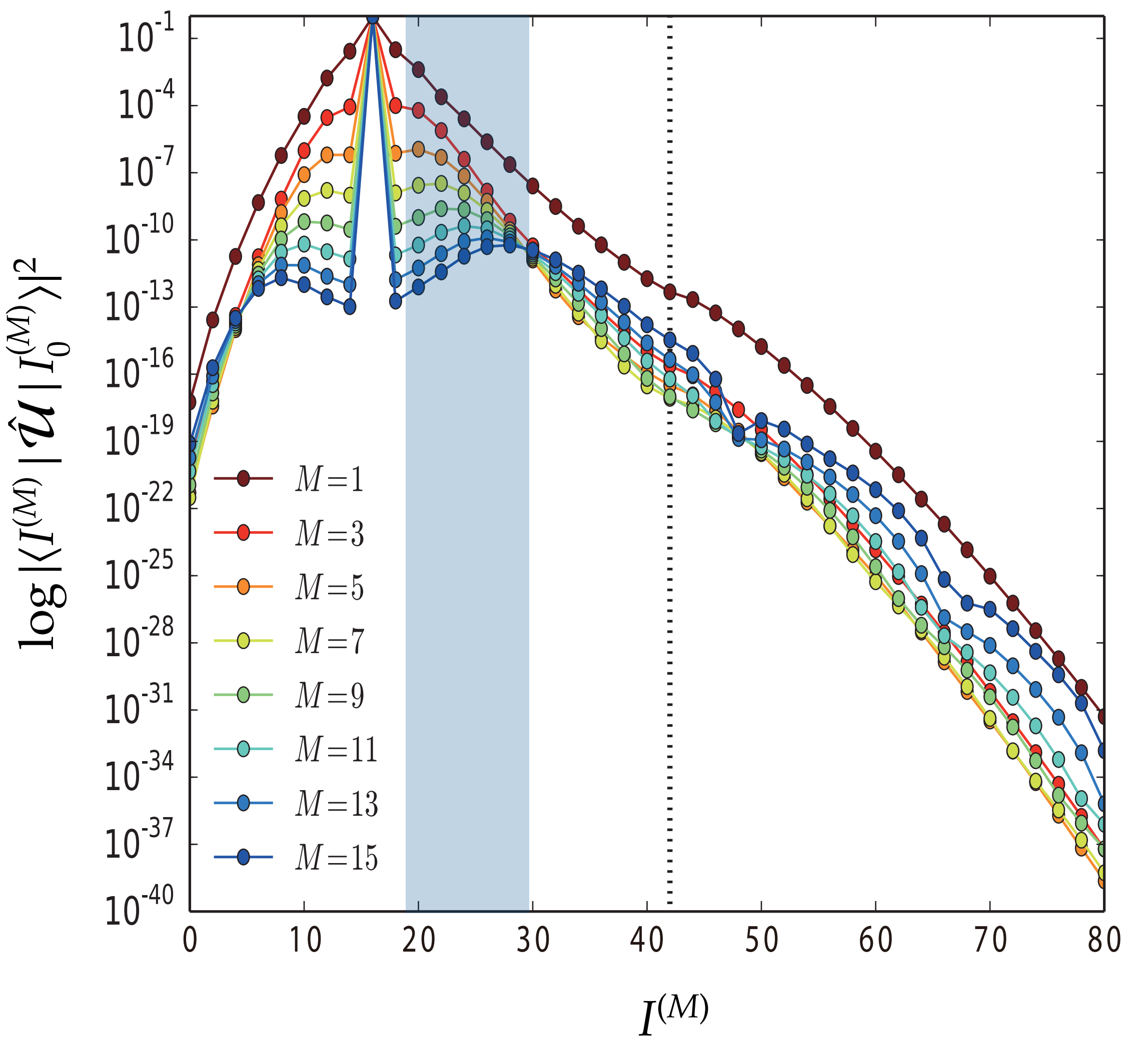
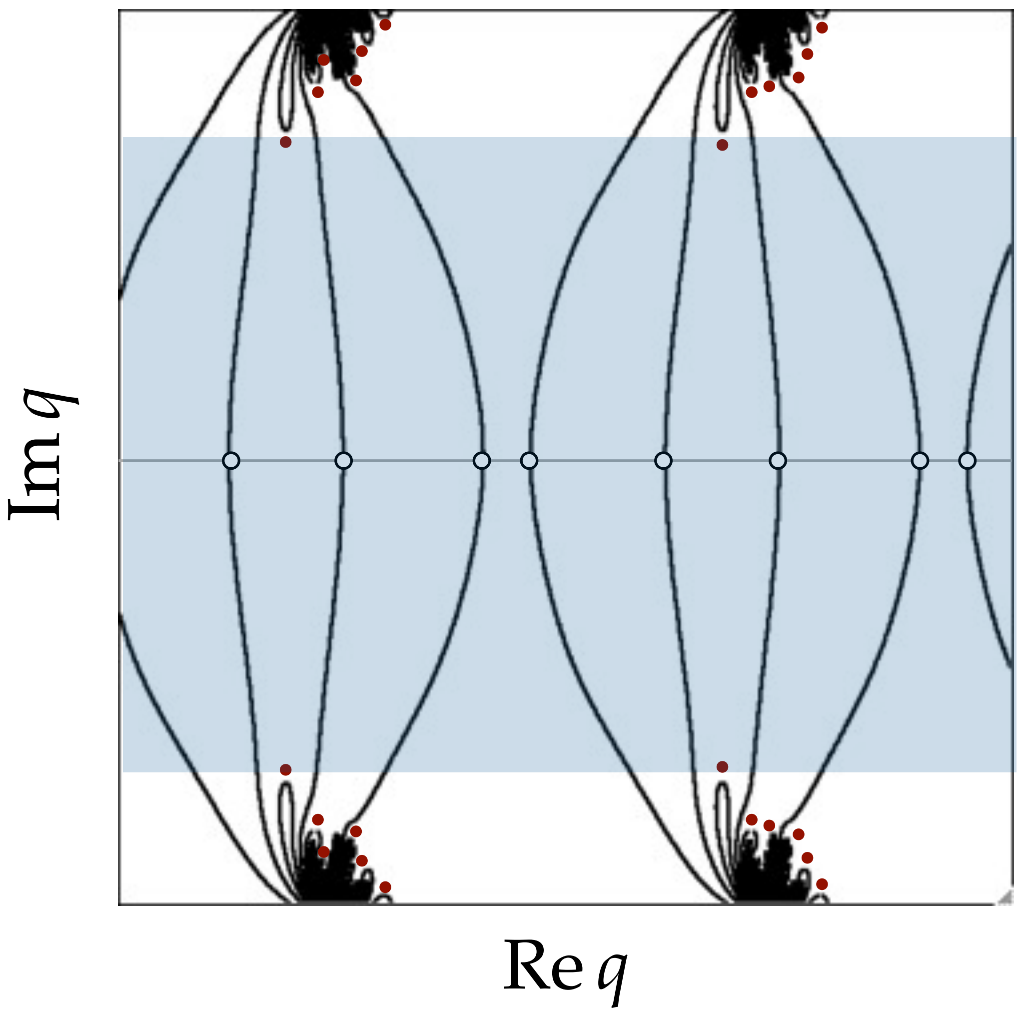



$M = 5$




- ○ turning points on the real manifold
- ● turning points in the complex plane

## Summary

- Semiclassical approximation (leading-order) in a single step propagator breaks down in the integrable representation

- Transition from one torus to another or to chaotic regions occurs under a purely quantum mechanism and cannot be described even by complex classical dynamics.

- Purely quantum regions are sandwiched between highly degenerated turning points and turning points associated with singularities of complexified tori, and possibly with *natural boundaries*.

- Observed diffractive phenomena are global and beyond the treatment based on local diffraction integrals such as a series of diffraction catastrophes.