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Efficient semiclassical approach for time delays

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Quantum chaos: fundamentals and applications
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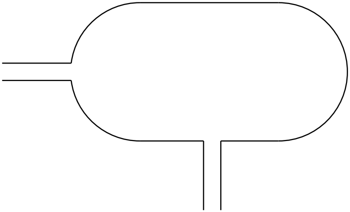
Quantum chaotic transport

Chaotic cavity to which two semi-infinite leads are attached

Asymptotic solutions in a lead of width w_i

$$C \sin(k_\perp x_\perp) \exp(\pm \mathrm{i} k_\parallel x_\parallel)$$



where $k_\perp = 1, \ldots, M_i$ and

$$k_\perp^2 + k_\parallel^2 = k^2 = \frac{2mE}{\hbar^2}\,, \quad M_i = \left\lfloor \frac{kw_i}{\pi} \right\rfloor$$

The $M \times M$ **scattering matrix** S connects the M (flux normalised) incoming modes to the M outgoing modes, where $M = M_1 + M_2$. Due to flux conservation S is unitary, $S^\dagger S = 1$, and it has the block structure

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$

r and t refer to reflection and transmission for incoming waves in lead 1
r' and t' refer to reflection and transmission for incoming waves in lead 2

Transmission eigenvalues

The eigenvalues of $tt^\dagger$ are the **transmission eigenvalues**

$$T_1, \ldots, T_n\,, \qquad T_j \in [0,1]\,, \qquad n = \min(M_1, M_2)$$

Quantities of interest:

- conductance $G = G_0 \left\langle \mathrm{tr}(tt^\dagger) \right\rangle = G_0 \left\langle \sum_j T_j \right\rangle$ (Landauer formula)
- conductance variance
- shot noise $P = \left\langle \mathrm{tr}(tt^\dagger) - \mathrm{tr}(tt^\dagger)^2 \right\rangle = \left\langle \sum_j T_j(1 - T_j) \right\rangle$
- moments of transmission $\mathcal{M}_k = \left\langle \mathrm{tr}[(tt^\dagger)^k] \right\rangle = \left\langle \sum_j T_j^k \right\rangle$

$G_0 = 2e^2/h$. We will set it in the following equal to one

Time delays

Other statistics are related to the **Wigner-Smith matrix** Q and its eigenvalues, the **proper time delays** $\boldsymbol{\tau_j}$

$$Q = -\mathrm{i}\hbar\, S^\dagger \frac{\partial S}{\partial E}\,, \qquad Q = Q^\dagger \quad \Longrightarrow \quad \tau_1, \ldots, \tau_M$$

Quantities of interest

- Wigner time delay $\qquad \tau_W = \frac{1}{M}\,\mathrm{tr}\,Q = \frac{1}{M}\sum_j \tau_j$
- Wigner time delay variance
- moments of proper time delays $\qquad m_k = \frac{1}{M}\left\langle \mathrm{tr}(Q^k)\right\rangle = \frac{1}{M}\left\langle \sum_j \tau_j^k \right\rangle$

The **Wigner time delay** is related to the total scattering phase shift

$$\tau_W(E) = -\frac{\mathrm{i}\hbar}{M}\frac{\mathrm{d}}{\mathrm{d}E}\ln\det S(E) = \frac{\hbar}{M}\frac{\mathrm{d}}{\mathrm{d}E}\Phi$$

Other statistics involve the **diagonal elements** $q_c = Q_{cc}$ and the **partial time delays** $t_c = \hbar \mathrm{d}\phi_c/\mathrm{d}E$.

Random matrix theory

RMT is effective for describing quantum transport in the regime $\tau_D > \tau_E$ where τ_D is the **dwell time** and τ_E the **Ehrenfest time**.

Two different approaches: In the "Heidelberg approach" the scattering matrix is related to a Hamiltonian of an inner system coupled to the outside [Verbaarschot, Weidenmüller, Zirnbauer '85]

$$S(E) = 1 - \mathrm{i}V^\dagger \frac{1}{E - \mathcal{H}_{\mathrm{eff}}} V \,, \qquad \mathcal{H}_{\mathrm{eff}} = H - \frac{\mathrm{i}}{2} V V^\dagger \,.$$

where H is an $N \times N$ Hermitian matrix, $\mathcal{H}_{\mathrm{eff}}$ is an effective non-Hermitian Hamiltonian of the open system, and V is an $N \times M$ coupling matrix

Remark: Formalism can be translated to cavities ($N \to \infty$). $(E - \mathcal{H}_{\mathrm{eff}})^{-1}$ corresponds to resolvent of the open cavity and V projects onto leads.

In the "Mexico approach" the scattering matrix is modelled directly by a random matrix . For **perfect coupling**, the relevant ensembles are the **Circular Ensembles**, the CUE ($\beta = 2$), COE ($\beta = 2$) and CSE ($\beta = 2$). [Mello, Pereyra, Seligman '85]

Random matrix theory

Joint probability density function of the transmission eigenvalues (Jacobi ensemble) [Baranger, Mello '94; Jalabert, Pichard, Beenakker '94]

$$P(T_1, T_2, \ldots, T_n) = \mathcal{N}_\beta \prod_{j=1}^{n} T_j^\alpha \prod_{1 \le j < k \le n} |T_j - T_k|^\beta$$

where $\alpha = \beta/2(|M_2 - M_1| + 1) - 1$ and $\mathcal{N}_\beta$ is a normalisation constant

Results for conductance

$$G = \begin{cases} \frac{M_1 M_2}{M} & \text{for CUE} \\ \frac{M_1 M_2}{M+1} = \frac{M_1 M_2}{M} - \frac{M_1 M_2}{M^2} + \frac{M_1 M_2}{M^3} - \ldots & \text{for COE} \end{cases}$$

$\frac{M_1 M_2}{M}$ is classical conductance

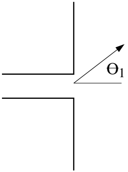
Variance of the conductance

$$\lim_{n \to \infty} \mathrm{Var(G)} = \frac{1}{8\beta}$$

Semiclassical approximation

Semiclassical approximation [Jalabert, Baranger, Stone '00]

$$t_{ba} \approx \sum_{\gamma:a\to b} A_\gamma \, e^{iS_\gamma/\hbar}$$



The channels (a, b) determine the angles with which the trajectories enter and leave the cavity

Semiclassical conductance

$$G = \left\langle \sum_{a,b} t_{ba}\, t_{ba}^* \right\rangle_E \approx \left\langle \sum_{a,b} \sum_{\gamma,\gamma':a\to b} A_\gamma \, A_{\gamma'}^* \, e^{i(S_\gamma - S_{\gamma'})/\hbar} \right\rangle_E$$

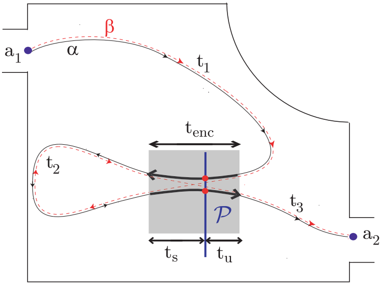
Relevant contribution only from correlated trajectories

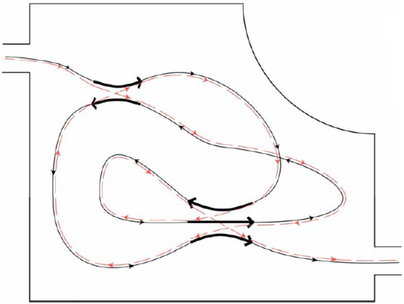
Diagonal approximation considers $\gamma = \gamma'$ [Baranger, Jalabert, Stone'93]

$$G_{\text{diag}} = \sum_{a,b} \sum_{\gamma:a\to b} |A_\gamma|^2 \sim \frac{M_1 M_2}{M}$$

using a classical sum rule. This is the **classical conductance**.

Higher orders in $1/M$





Correlated pairs [Richter, MS '02; Heusler, Müller, Braun, Haake '06]

Diagrammatic rules for each orbit configuration

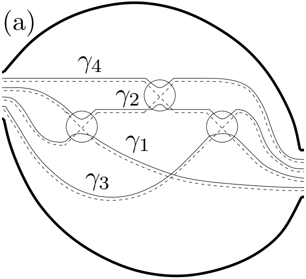
- $1/M$ for every link
- $-M$ for every encounter
- multiply by number of ingoing and outgoing channels $M_1 M_2$

Example: Left "diagram" contributes $(1/M)^3(-M)M_1 M_2 = -\dfrac{M_1 M_2}{M^2} = G^{(2)}$

Higher moments $\mathcal{M}_k$

The approximation of the k-th moment $\mathcal{M}_k$ requires $2k$ trajectories.

Example of a set of trajectories that contributes to the fourth moment $\mathcal{M}_4$.



Half of the trajectories connect the channels i_j and o_j, $j = 1, \ldots, k$,
the other half connect the channels i_{j+1} and o_j, $j = 1, \ldots, k$ ($i_{k+1} \equiv i_1$).

Same diagrammatic rules apply $\Longrightarrow$ Combinatorial problem

Semiclassical results

- [Richter, MS '02]: First correction to $\mathcal{M}_1$.
- [Schanz, Pulhmann, Geisel '03]: Leading order of $\mathcal{M}_2$.
- [Heusler, Müller, Braun, Haake '06]: All orders for $\mathcal{M}_1$ and $\mathcal{M}_2$.
- [Berkolaiko, Harrison and Novaes '08]: Leading order of all $\mathcal{M}_k$.
- [Berkolaiko, Kuipers '11]: Second order of all $\mathcal{M}_k$.
- [Berkolaiko, Kuipers '13; Novaes '13, '15]: All orders of all $\mathcal{M}_k$.

Time delays

Joint probability density function of the inverses of the proper time delays $\gamma_j = 1/\tau_j$ (Laguerre ensemble) [Brouwer, Frahm, Beenakker '97]

$$P(\gamma_1, \gamma_2, \ldots, \gamma_n) = \mathcal{N}_\beta \prod_{j=1}^{n} \gamma_j^{\beta M/2} \, \mathrm{e}^{-\beta\gamma_j/2} \prod_{1 \leq j < k \leq n} |\gamma_j - \gamma_k|^\beta$$

where $\mathcal{N}_\beta$ is a normalisation constant

The first moment, the average Wigner time delay, is equal to the classical dwell time: $m_1 = \langle \tau_W \rangle = \bar{\tau}_W = \tau_D$

Results for the second moment [Mezzadri, Simm '11,'12]

$$m_2 = \frac{1}{M} \langle \mathrm{tr}(Q^2) \rangle = \frac{2\,\beta\,M^2\,\bar{\tau}_W^2}{(M+1)\,(\beta M - 2)} \,.$$

Moments of the proper time delays

One can base a semiclassical approximations of the moments of the proper time delays

$$m_k = \langle \frac{1}{M} \mathrm{Tr} Q^k \rangle = \langle \frac{1}{M} \sum_{j=1}^{M} \tau_j^k \rangle$$

on the definition

$$Q = -\mathrm{i}\hbar \, S^\dagger \frac{\partial S}{\partial E}$$

This expresses the moments in terms of the same kind of lead connecting trajectories as for the transmission moments. However, the semiclassical computations are considerably more complicated due to energy derivative. There are no simple diagrammatic rules as in the transmission problem.

Semiclassical results

- [Kuipers, MS '08]: All orders for m_1.
- [Berkolaiko, Kuipers '10]: Leading order of all m_k.
- [Berkolaiko, Kuipers '11]: Next two orders of all m_k.

New semiclassical approach

Alternatively, one can base a semiclassical approximation on a different representation of the Wigner-Smith matrix [Sokolov, Zelevinsky '89]

$$Q = \hbar\, V^\dagger \frac{1}{(E - \mathcal{H}_{\rm eff})^\dagger}\, \frac{1}{(E - \mathcal{H}_{\rm eff})} V$$

The matrix element Q_{ab} can be interpreted as overlap of the internal parts $(E - \mathcal{H}_{\rm eft})^{-1} V$ of the scattering wave functions in incident channels a and b.

Semiclassical approximation [Kuipers, Savin, MS '14]

$$\langle \boldsymbol{r} | \frac{1}{E - \mathcal{H}_{\rm eff}} V_c \approx \frac{1}{\sqrt{\hbar}} \sum_{\gamma(c \to \boldsymbol{r})} A_\gamma \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \mathcal{S}_\gamma} \,,$$

Involves trajectories that connect channel c with an interior point $\boldsymbol{r}$.

Similar types of trajectories occur in problems involving the survival probability, the current density or the fidelity. [Waltner et al.08, Kuipers et al.09, Gutierrez et al.'09, Gutkin et al.'10]

Average Wigner time delay

Semiclassical approximation for Wigner time delay

$$\tau_W \approx \frac{1}{M} \sum_{c=1}^{M} \int \mathrm{d}^2 \boldsymbol{r} \sum_{\gamma,\gamma'(c\to \boldsymbol{r})} A_\gamma A^*_{\gamma'} \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\mathcal{S}_\gamma - \mathcal{S}_{\gamma'})} \,,$$

Note that this is the *third* semiclassical approximation after the one over lead connecting (transmission) trajectories

$$\tau_W \approx \frac{1}{M} \sum_{i,o=1}^{M} \sum_{\gamma,\gamma'(i\to o)} T_\gamma \tilde{A}_\gamma \tilde{A}^*_{\gamma'} \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\mathcal{S}_\gamma - \mathcal{S}_{\gamma'})} \,,$$

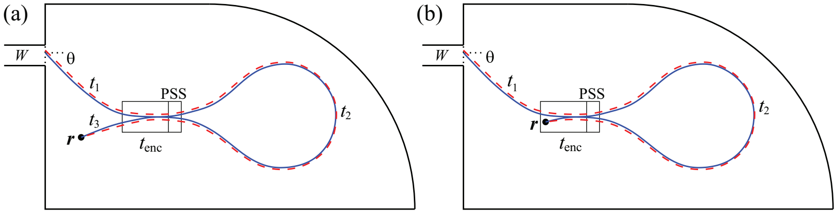
and the Gutzwiller type formula due to relation of τ_W to density of states

$$\tau_W \approx \bar{\tau}_W + \frac{1}{M} \mathrm{Re} \sum_p A_p \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}_p} \,,$$

where the sum is over all periodic orbits of the open cavity.

The diagonal approximation in the first case already yields: $\langle \tau_W \rangle \approx \tau_D$

Off-diagonal terms for average Wigner time delay



Trajectories can end in an encounter (one-leg loops) or not.

Simple diagrammatic rules [Kuipers, Savin, MS ’14].

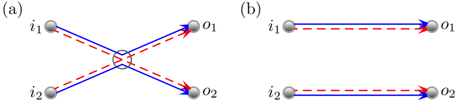
- $1/M$ for every link
- M for each incoming channel
- $-M$ for every encounter, unless it contains an end point
- 1 for every encounter that contains one end point
- 0 for every encounter that contains more than one end point

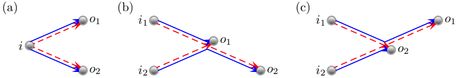
Example for second moment and results

Example: leading order for m_2

$$\left(\frac{-M^3}{M^4} + \frac{M}{M^2} + 2\frac{M^2}{M^3}\right) M\,\bar{\tau}_W^2$$

$$= 2\bar{\tau}_W^2\,.$$





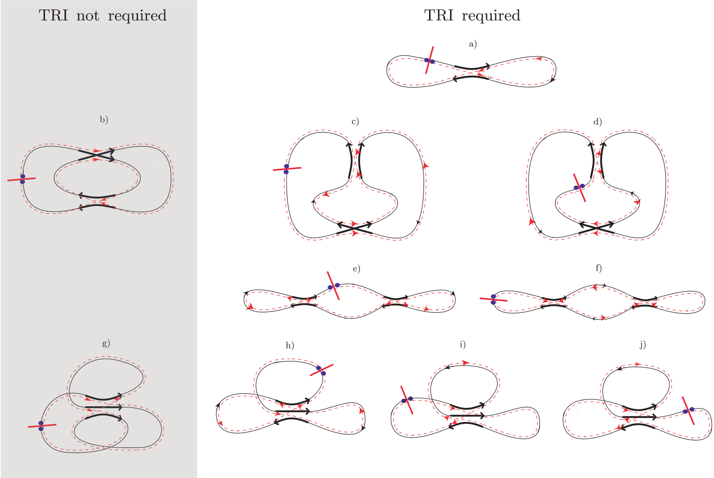
Results

- All orders for second moments of proper time delays m_2 (and other time delays), and for the variance of the Wigner time delay.
- Leading five orders for all moments m_k

For systems with and without time-reversal symmetry.

The highest two orders of m_k have not yet been calculated in RMT.

Higher orders in $1/M$



Müller, Heusler, Braun, Haake (2007)

Semiclassical evaluations become a combinatorial problem.