Statistical Theory of Random (and Chaotic) lasers

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S. Rotter – TU Wien
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RMT in Optics/E&M

- Analysis of multiple scattering problems (including wave chaos) => Extremal eigenvalue problems for non-hermitian or non-unitary matrices

- Random/QC lasing: Non-unitary (non-linear) S-matrix
  - Realistic technological application for theory

- Control of transmission/absorption/focusing in diffusive scattering media (another talk)
  SLM-based Focusing thru opaque white media: Mosk et al. PRL 2007
  “Hidden Black”, Y.D. Chong and ADS, PRL 107, 163901, 2011
  “Filtered Random Matrices”, A. Goetschy and ADS, PRL 111, 063901 (2013);
  effect of incomplete “channel” control => Free probability theory
  “Control of Total Transmission”, Popoff, Cao et al. PRL 112, 133903 (2014):
  \[<T> = 5\% \Rightarrow T_{\text{max}} = 18\%\]
Pioneering Random Lasers

Lawandy, Balachandran, Gomes & Sauvain, Nature 368, 436 (1994) (following early ideas from Letokhov)
ZnO Nanorods and Powders

Average particle diameter ~ 100 nm

Also confirmed by photon statistics

Why Interesting? Not due to Anderson
Localized High Q modes – Diffusive regime

Thouless # \[ N_T = \frac{\gamma}{\delta \nu} \gg 1 \]

\[ N_T = g = 1/f \]
\[ DRL \ has \ f << 1 \]

Passive cavity scattering spectrum shows no isolated resonances – not within standard laser theory

Resonances are strongly overlapping spatially and spectrally.
Modes are pseudo-random in space – not based on periodic orbits

Tureci, Ge, Rotter, ADS, Science 320, 643 (2008)
SALT-based calculations

Similar to Wave-chaotic Lasers

“Ray and Wave Chaos in Asymmetric Resonant Optical Cavities”,
J. U. Nöckel, A. D. Stone,
Open wave-chaotic systems

Hard Chaos

KAM Transition to ray chaos
Theory for lasers with complex geometry

Chaotic-ARC

microdisk

microtoroid

Photonic Crystal Lasers
Universal: Lasers as scattering systems

Non-hermitian Eq.  
Flux not conserved  

\[ \n(r) = \sqrt{\varepsilon_c(r)} + 4\pi \chi_g(r) \]

\[ \mathbf{S}(n(r)k) \cdot \alpha = \beta \]

\[ \mathbf{E}(r) = 0 \]

\[ k \equiv \omega/c \]

\[ \chi_g \text{ is complex } \Rightarrow n(r) \text{ complex, } n_2 < 0 \text{ (amplifying)} \]

Non-unitary non-linear scattering problem, \( \chi_g = \chi_g(E) \)
Threshold lasing modes

\[ S(n(\vec{r})k) \cdot \alpha = \beta \]

Laser: lasing mode \( \beta \) goes out, nothing in

⇒ Poles of the S-matrix

Passive cavity: \( n = (\varepsilon_c)^{1/2} \), \( S \) unitary, poles complex.

Simple example: 1D uniform dielectric cavity:

Now add gain medium + pump, \( n = n_c + \Delta n_g \)

TLM stabilized by non-linearity!

complex sine inside, purely outgoing outside

Pump harder \( \Rightarrow \) multimode lasing
Semiclassical lasing theory + SALT

Cavity arbitrary

\[ \nabla^2 E(x, t) - \frac{\varepsilon_c(x)}{c^2} \frac{\partial^2}{\partial t^2} E(x, t) = \frac{4\pi}{c^2} \frac{\partial^2}{\partial t^2} P_g(x, t) \]

\[ P_g = \chi_g E \]

Any cavity, gain medium, N-levels, M ind. transitions, non-uniform pumping

Simplest case: 2-level atoms

\[ P = n_a \text{Tr} p \rho \]
\[ D = n_a (\rho_{22} - \rho_{11}) \]

Not studying dynamical chaos
Look for non-linear steady-state, with purely outgoing BC

Maxwell-Bloch equations
Haken(1963), Lamb (1963) – the standard model

\[ \varepsilon_c \dot{E}^+ = \nabla^2 E^+ - 4\pi \dot{P}^+ \]
\[ \dot{P}^+ = (-i\omega_a - \gamma_\perp) P^+ - ig^2 E^+ D \]
\[ \dot{D} = \gamma_\parallel (D_0 - D) + 2i(E^+P^{+*} - E^{+*}P^+) \]

\[ \gamma_\perp = 1/T_2, \quad \gamma_\parallel = 1/T_1 \]

\[ D_0 = \text{pump strength} \]
\[ g^2 = \text{dipole coupling} \]
\[ \frac{dD/dt}{\approx 0, \text{ in steady-state}} \Rightarrow \text{SALT Eqs} \]

\[ \gamma_{\text{perp}} \ll \Delta, \gamma_{\text{par}} \Rightarrow \text{good approx for microlasers} \]

\[
\left[ \nabla^2 + \left( \epsilon_c + \tilde{D}_0 \gamma_\mu \right) k_\mu^2 \right] \Psi_\mu = 0
\]

\[ \tilde{D}_0(\mathbf{r}) \equiv \frac{D_0(\mathbf{r})}{1 + \sum_\nu^N |\gamma_\nu \Psi_\nu(\mathbf{x})|^2} \]

Non-linear coupled time-independent wave equations with outgoing BC

\[ \mu = 1, 2, \ldots, N \]

\[ \Psi_\mu \sim e^{ikr}/r, \quad r \to \infty \]
Saline Solution

- Specialized TLM/TCF (non-hermitian biorthogonal) basis set method (Tureci, ADS, Ge, Rotter, Chong)

\[ \Psi_\mu(\vec{r}) = \sum_{n} a^\mu_n u_n(\vec{r}, k_\mu) \]

Lasing Map

\[ D_0 \sum_{n'} T_{n'n}^\mu a^\mu_{n'} = a^\mu_n \]

- Solve by iterative method (rapidly convergent).

- SALT for DRL: approx sum by a single term, soln in terms of evales of Green fcn for this non-herm eq.

\[ \eta_n(k_\mu) = \eta_1 - i\eta_2 \]

Freq shift

\[ \Lambda_n \equiv \frac{1}{\eta_n} \approx \frac{+i}{\eta_2} \quad (RL) \]
Why SALT it is good for you

- General theory of CW steady-state lasing, partially analytic and analytic approx => physical insight
- Computationally tractable, no time integration
- Cavities/modes of arbitrary complexity and openness.
- Non-linear hole-burning interactions to infinite order
- How well does it work? (it has an approximation)
Test: SALT and FDTD agree for 1D random laser

Modal Field Intensities inside cavity

5 mode lasing
Other FDTD tests of SALT: 2D PCSEEL and 3D PC defect mode laser, coupled cavities; also multiple transitions, and injected signals

No FDTD on 2D RLs (yet), SALT studies:

Many modes with similar thresholds as kR gets large
SALT for 2D random laser


Also, L. Ge, PhD thesis (diffusive regime), and A. Cerjan and A. Goetschy (in preparation) – focus on diffusive results
Diffusive Random Laser

Note: decreasing power slope

4 mode lasing
Non-linear interactions

All modes lasing without interactions
Analytic Theory for DRL
Goetschy, Cerjan, ADS, in preparation

Linearized dynamics: \[ \{ \nabla^2 + k^2 [1 + \delta \epsilon_c(r, \omega) + \delta \epsilon_g(\omega)] \} E(r, \omega) = 0 \]

\( k = \omega/c \)

Disorder

Gain curve

Outgoing solution:

\[ G(\omega) = \frac{-k^2}{\nabla^2 + k^2 \epsilon_c(r) + i0^+} \]

Pump profile

Green's function of the disorder

Lasing threshold:

\[ \Lambda_n = \frac{1}{\delta \epsilon_g(\omega)} \]

\[ \Rightarrow \text{Im}\{\Lambda_n\} = \frac{1}{\tilde{D}_0} \]
\[ G(\omega) = \frac{-k^2}{\nabla^2 + k^2 \varepsilon_c(r)} + i0^+ \]
\[ \Lambda_n = \frac{\omega^2}{(\omega_n - i\Gamma_n/2)^2 - \omega^2} \]

Effective \( H \) similar to open chaotic cavities
\[ G(\omega) = \frac{-k^2}{k^2 - H^e(\omega)} \]
\[ H^e = -\nabla^2 - k^2 \delta \varepsilon_c(r) + H^{BC} \]

Eigenvalue \( \omega_n - i\Gamma_n/2 \)

Lasing threshold

Spectrum of the Green’s operator

Diffusion theory
\[ \sim 1/\Gamma^{Th} \]

Numerical solution obtained from the wave equation in open 2D disk (1 disorder realization)
Statistical averages

Single pole approx to SALT:

\[ \Psi_\mu (\vec{r}) = \sum_n a^\mu_n u_n (\vec{r}, k_\mu) \approx a_\mu u_\mu (\vec{r}) \]

\( \mu \rightarrow m, \ u_\mu \rightarrow R_m \)

\[ \Phi_m = |a_m|^2 \int dr |R_m(r)|^2, \]

\[ \omega_m = \omega_a + \frac{\text{Re} \Lambda_m(\omega_m)}{\text{Im} \Lambda_m(\omega_m)} \gamma_\perp, \]

\[ \sum_{p=1}^{N_L} \alpha_{mp} \Gamma_p \Phi_p = \tilde{D}_0 \text{Im} \Lambda_m(\omega_m) - 1, \]

\[ \alpha_{mp} = \frac{\int dr F(r) R_m(r)^2 |R_p(r)|^2}{\int dr |R_m(r)|^2}. \]

Constrained linear Eq. for modal intensities

Approx real

Self-averaging gaussian approx

\[ \alpha_{mp} \simeq \langle \alpha_{mp} \rangle \simeq \frac{1 + 2 \delta_{mn}}{V}, \]
\[ \Phi_m = \frac{V}{2\Gamma_m} \left[ \tilde{D}_0 \text{Im}\Lambda_m(\omega_m) - Y_{\text{thr}} \right]. \]

Express all properties of interest in terms of \( P(\text{Im}\{\Lambda\}) \)

Eventually saturates with pump

Modal interactions

Predicts monotonically decreasing modal slopes

\[ \langle Y_{\text{thr}} \rangle \]

\[ \Sigma_m |a_m|^2 \]

Pump Strength \( D \)
Need prob dist of $\text{Im}\{\Lambda\} \sim \Gamma$

Quasi-modes statistics
(simulations in open 2D disk)

$\Gamma^\text{Th} = D/R^2$

Signature of diffusion
$1/\Gamma^{3/2}$

$$P(\text{Im}\Lambda) \approx \sqrt{2} \frac{k R \sqrt{k \ell}}{(\text{Im}\Lambda)^{3/2}} \ln \left( \frac{\sqrt{\omega/\Gamma^\text{Th}}}{\text{Im}\Lambda} - 1 + \sqrt{\omega/\Gamma^\text{Th}/\text{Im}\Lambda} \right)$$
Two limits

Gain width: $\gamma_{\perp}$
Results $\gamma_\perp < \bar{\Gamma}$

In 2D

$\langle N_{\text{modes}} \rangle \sim g^{2/5}$
Total Intensity $\gamma_\perp < \bar{\Gamma}$

$slope \sim (kR)^2$

Useful for unresolved random laser emission
Results: $\gamma_\perp > \bar{\Gamma}$

Scaling parameter is:

$$g \iff \gamma_\perp < \bar{\Gamma}$$

$$g' = g \frac{\gamma_\perp}{\bar{\Gamma}} \iff \gamma_\perp > \bar{\Gamma}$$
Comparison of total intensities

Mean Total Intensity vs. Pump $D_0$

- kl = 20
- kR = 100

Mean mode volume = $\pi(kR)^2$

Left graph:
- kl = 2.7
  - kR = 60
  - kR = 50
  - kR = 40

Right graph:
- kl = 2.7
  - kR = 60
  - kR = 50
  - kR = 40

Mean Flux in a.u. vs. Pump $D_0$
Savior er App for Random Lasers: Exploiting spatial incoherence
Spatial Coherence

If wavefronts at different points have a stable phase relationship there will be interference fringes

⇒ Always true of single mode lasing
⇒ Not true of multimode
Controlling Spatial Coherence in RL by Varying Pump Volume

- MFP = 500 µm d = 215 µm
- MFP = 500 µm d = 290 µm
- MFP = 500 µm d = 390 µm
Imaging applications of RL?

Optical coherence tomography (OCT)

Sample

Light source

Detector

Mirror

Brandon Redding,
Res. Scientist (Cao Group)

Prof. Michael Choma, MD.
PhD, Yale Medicine
Spatial cross talk

**Coherent illumination**

\[ I = |E|^2 = |E_1 + E_2|^2 = |E_1|^2 + |E_2|^2 + 2|E_1|E_2 \cos(\theta) \]

**Incoherent illumination**

\[ I = I_1 + I_2 \]

*Much reduced artifacts*
Full-Field OCT

## Ideal Illumination Source for Imaging

### Photon Degeneracy/ Spectral Radiance

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<th>Low</th>
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<td>Light emitting diodes</td>
<td>Pinhole filtered white light</td>
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<td>Lasers</td>
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### Spatial Coherence

- Low
- High
Speckle-free Laser Imaging

On-chip Electrically-Pumped Semiconductor Random laser

Development of a New Light Source for Massive Parallel Confocal Microscopy and Optical Coherence Tomography
Do we really want to use a random laser?

No – a simpler chaotic shape is easier to fabricate

Fabricated on chip AlGaAs - GaAs QW structure.

B. Redding, A. Cerjan, X. Huang, ADS, M. L. Lee, M. A. Choma, H. Cao, PNAS, in press
What Specific D-Shape is best?

Want max number of lasing modes at lowest power level – perfect for SALT

Effect comes from:
1) Flat dist of passive cavity Q
2) Weaker mode comp for more uniform chaotic states

Which is best?

- $r_0 = 0.5R$
- $r_0 = 0.3R$
- $r_0 = 0.7R$
D-laser characterization
Results?

(a) Fabry-Perot laser: $C=0.6, M\sim 3$

(b) Microdisk Laser: $C=0.25, M\sim 15$

(c) Chaotic Laser: $C=0.05, M\sim 400$

Success!
## Ideal illumination Source for Imaging

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- **Random Lasers**: Chaotic Lasers
- **Lasers**: Superluminescent Diodes
- **Thermal white light**: Light emitting diodes
- **Pinhole filtered white light**:
Thanks!

Yidong Li

Hui

Hakan

Stefan

Arthur

Alex

Brandon