“Phase Diagram” of a mean field game

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Outline

- Brief introduction to mean field games
- Study of a toy model
  - The “seminar problem”
  - Phase diagram
- Work in progress

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Mean Field Games

[Hawk and dove]

A simple game:

2 players

2 strategies

	Hawk	Dove
Hawk	(V-C)/2 , (V-C)/2	V,0
Dove	0,V	V/2, V/2

- As the number of players and strategies becomes large, the study of such games becomes quickly intractable.
- However:
  - « continuum » of strategy
  - very large number of « small » players

→ **Mean Field (differentiable) Games**

General structure (e.g: model of population distribution)

[Guéant, Lasry, Lions (2011)]

- N agents $i = 1, 2, \cdots, N \qquad (N \gg 1)$
- state of agent $i \longrightarrow$ real vector $\mathbf{X}^i$ (here just physical space)

$$m(\mathbf{x}, t) \equiv \frac{1}{N} \sum_{1}^{N} \delta(\mathbf{x} - \mathbf{X}_t^i) \qquad \text{density of agents}$$

- agent's dynamic

$$dX_t^i = a_t^i dt + \sigma dW_t^i$$

$dW_t^i \equiv$ white noise
drift $a_t^i \equiv$ control parameter

- agent tries to optimize (by the proper choice of a_t^i) the cost function

$$\int_t^\infty d\tau e^{-\lambda(\tau - t)} \left[\frac{1}{2} (a_\tau^i)^2 + g[m](X_\tau^i, \tau) \right]$$

Mean Field Game = coupling between a (collective) stochastic motion and an (individual) optimization problem through the mean field $g[m](\mathbf{x},t)$

e.g. $$g[m](\mathbf{x}) \equiv f(\mathbf{x}) + \mu \int d\mathbf{y}\, m(\mathbf{y},t) \exp\left[-(\mathbf{y}-\mathbf{x}))^2/2\Sigma^2\right]$$

Examples of mean field games

- Pedestrian crowds [Dogbé (2010), Lachapelle & Wolfram (2011)]
- Production of an exhaustible resource [Guéant, Lasry, Lions (2011)]

  (agents = firms, ***X*** = yearly production)
- Order book dynamics [Lasry et al. (2015)]

  (agents = buyers or sellers , ***X*** = value of the sell or buy order)

Two main avenues of research

- Proof of existence and uniqueness of solutions

  [cf Cardaliaguet's notes from Lions collège de France lectures]

- Numerical schemes to compute exact solutions of the problem

  [eg: Achdou & Cappuzzo-Dolcetta (2010), Lachapelle & Wolfram (2011), etc …]

⇨ Our (physicist) approach : develop a more "qualitative" understanding of the MFG (extract characteristic scales, find explicit solutions in limiting regimes, etc..)

For starters : study of a simple toy model “At what time does the meeting start ?”:

[O. Guéant, J.M. Lasry, P.L. Lions]

$$\bar{t} \equiv \text{official time of the seminar}$$

$$\tilde{\tau}_i \equiv \text{time at which the agent arrives in the seminar room}$$

$$T \equiv \text{actual times at which the seminar begins}$$

(T determined through a quorum condition)

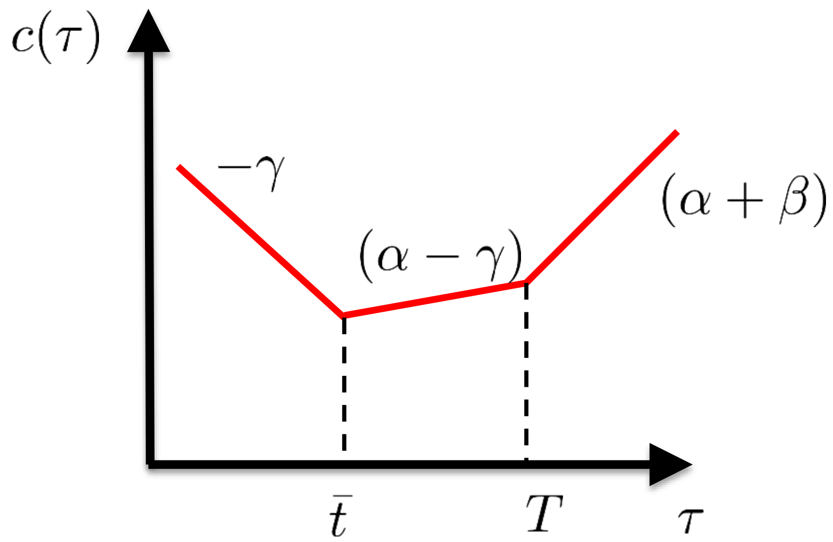
cost

$$c(\tilde{\tau}_i) = \alpha[\tilde{\tau}_i - \bar{t}]_+ + \beta[\tilde{\tau}_i - T]_+ + \gamma[T - \tilde{\tau}_i]_+$$

concerns for the agent’s reputation

desire not to miss the begining

reluctance to useless waiting



Shape of the cost function

$$\left.\begin{matrix}\alpha \\ \beta \\ \gamma \\ \bar{t}\end{matrix}\right\} \equiv \text{parameters of the problem}$$

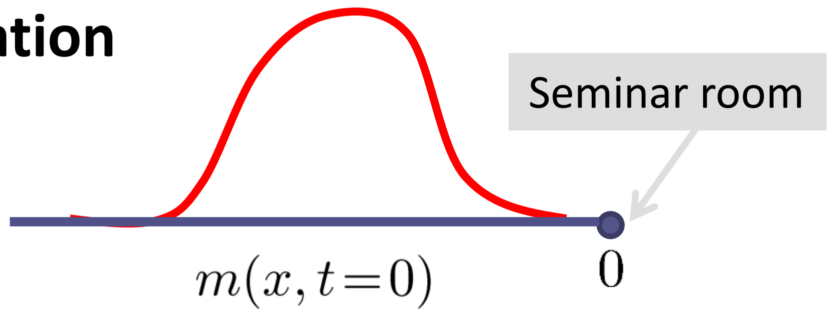
$$T \equiv \text{unknown}$$

Two other parameters to come

$$\sigma \equiv \text{strenght of the noise}$$

$$m_0(x) \equiv \text{initial density of agents}$$

Agents' dynamics & optimization



$$dX_t^i = a_t^i dt + \sigma dW_t^i$$
$$(dW_t^i \equiv \text{white noise})$$

drift a_t has a quadratique cost : $\frac{1}{2}a_t^2$

$$\hookrightarrow \qquad u[X_t^i, t] \equiv \min_{a(.)} E\left[c(\bar{t}, T, \tilde{\tau}_i) + \frac{1}{2}\int_t^{\tilde{\tau}_i} a(\tau)^2 d\tau\right] \qquad \text{(value function)}$$

Bellman: $$u[X_t, t] = \min_{a_t} E\left[\frac{1}{2}a_t^2\delta t + u[X_{t+\delta t}, t+\delta t]\right]$$

$$\hookrightarrow \qquad \partial_t u + \min_a \left[\frac{1}{2}a^2 + a\partial_x u\right] + \frac{\sigma}{2}\partial_{xx}^2 u = 0$$

$$\Rightarrow (a = -\partial_x u)$$

H.J.B. $$\partial_t u - \frac{1}{2}(\partial_x u)^2 + \frac{\sigma}{2}\partial_{xx}^2 u = 0$$ backward propagation

boundary condition : $u(x=0, \cdot) = c(\cdot)$

In practice, one must thus solve the system of coupled PDE :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{2}\left(\dfrac{\partial u}{\partial x}\right)^2 + \dfrac{\sigma^2}{2}\dfrac{\partial^2 u}{\partial x^2} = 0 \\ u(x=0,t) = c(t;T,\bar{t}) \end{cases} \qquad \text{(Hamilton-Jacobi-Bellman)}$$

$$\begin{cases} \dfrac{\partial m}{\partial t} + \dfrac{\partial am}{\partial x} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 m}{\partial x^2} = 0 \\ m(x=0,t) = 0 \\ m(x,t=0) = m_0(x) \end{cases} \qquad \text{(Kolmogorov) .}$$

Kolmogorov coupled to HJB through the drift $a(x,t) = -\partial_x u(x,t)$

HJB coupled to Kolmogorov through the quorum condition

$$\begin{cases} N(T) = \displaystyle\int_{-\infty}^{0} m(x,T) = \bar{\theta} & \text{(if } T > \bar{t}\text{)} \\ \qquad\qquad\qquad\qquad\quad \leq \bar{\theta} & \text{(if } T = \bar{t}\text{)} \end{cases}$$

“mean field” $\equiv T$

NB : system of coupled PDE in the generic case

$$\left\{ \frac{\partial u}{\partial t} - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = -\nabla g[m](x,t) \right. \qquad \text{(Hamilton-Jacobi-Bellman)}$$

$$\left\{ \begin{aligned} \frac{\partial m}{\partial t} + \frac{\partial a m}{\partial x} - \frac{\sigma^2}{2}\frac{\partial^2 m}{\partial x^2} &= 0 \\ m(x{=}0,t) &= 0 \\ m(x,t{=}0) &= m_0(x) \end{aligned} \right. \qquad \text{(Kolmogorov)}\ .$$

Kolmogorov coupled to HJB through the drift $a(x,t) = -\partial_x u(x,t)$

HJB coupled to Kolmogorov through the mean field $g[m][x,t)$

General strategy

Let

$$G(x,t|x_0) \equiv \text{solution for a point source } m_0(x) = \delta(x - x_0)$$
$$\rho(x_0,t) \equiv \int_{-\infty}^{0} dx\, G(x,t|x_0)$$

Kolmogorov equation linear $\Rightarrow$ $m(x,t) = \int_{-\infty}^{0} dx_0\, G(x,t|x_0) m_0(x_0)$.

Quorum condition reads

$$\boxed{\int_{-\infty}^{0} dx_0\, \rho(x_0,T) m_0(x_0) = \bar{\theta}} \qquad (*)$$

$\Downarrow$

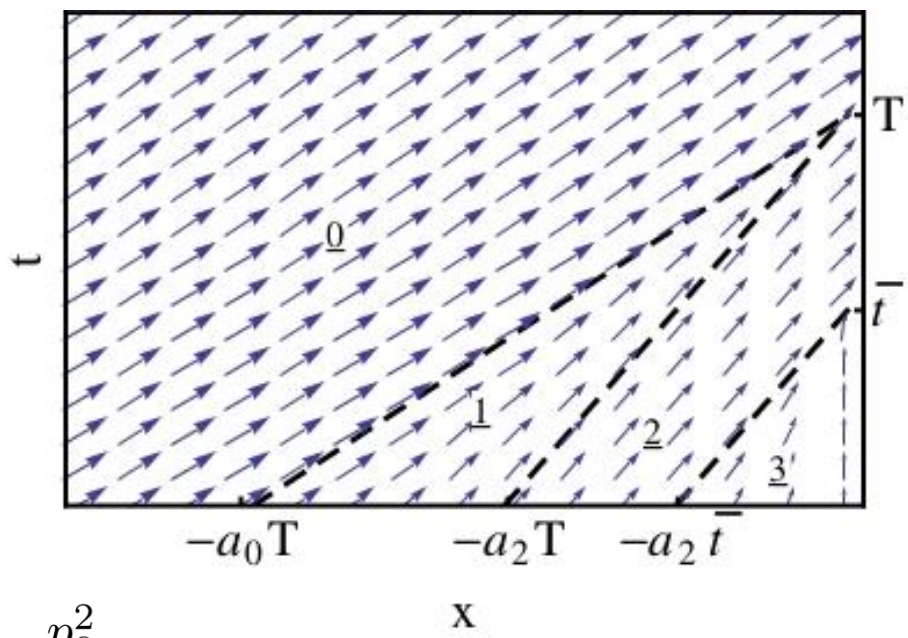
Two steps process

- first step : compute $\rho(x_0,T)$ for arbitrary T.
- second step : solve the self-consistent equation (*)

Hamilton Jacobi Bellman (HJB) equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{2}\left(\dfrac{\partial u}{\partial x}\right)^2 + \dfrac{\sigma^2}{2}\dfrac{\partial^2 u}{\partial x^2} = 0 \\ u(x=0,t) = c(t;T,\bar{t}) \end{cases}$$

σ → 0 limit



- $\dfrac{\partial u}{\partial t} \equiv E \qquad \dfrac{\partial u}{\partial x} \equiv p$
- $H = \dfrac{p^2}{2}$ (free motion)
- boundary condition $E_0 \equiv \dfrac{dc}{dt} = \dfrac{p_0^2}{2}$

$(a_0 = \sqrt{2(\alpha+\beta)} \quad a_2 = \sqrt{2(\alpha-\gamma)})$

σ → ∞ limit

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 u}{\partial x^2} = 0 \\ u(x{=}0,t) = c(t;T,\bar{t}) \end{cases}$$

(backward diffusion equation
with strange boundary conditions)

One way to solve this : go back to original optimization pb

$$u(x,t) = \min_{a_i(t)} \left\{ E\left[c(\tilde{\tau}) + \frac{1}{2}\int_t^{\tilde{\tau}} a_i^2(\tau)\ \mathrm{d}\tau \right] \right\}$$

$$\lim_{\sigma\to\infty} u(x,t) = E\left[c(\tilde{\tau})\right] = \int_{t_0}^{\infty} d\tau\, \tilde{c}(\tau) P(\tau)$$

$$= -x \int_0^{\infty} d\tau\, \frac{\tilde{c}(\tau+t)}{\tau} G_0(x,\tau)\ ,$$

distribution of
first passage
At x=0

Arbitrary σ

Cole-Hopf transformation : $u(x,t) = -\sigma^2 \ln \phi(x,t)$

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 \phi}{\partial x^2} = 0 \\ \phi(x=0,t) = e^{-\frac{c(t)}{\sigma^2}} \end{cases}$$

$$c(t) = \alpha[t-\bar{t}]_+ + \beta[t-T]_+ + \gamma[T-t]_+$$

$$\phi(x,t) = -x\int_0^\infty \frac{e^{-\frac{c(t+\tau)}{\sigma^2}}}{\tau} G_0(x,\tau)\ \mathrm{d}\tau$$

$$\frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{x^2}{2\sigma^2 t}\right)$$

Kolmogorov equation

$$\begin{cases} \dfrac{\partial m}{\partial t} + \dfrac{\partial am}{\partial x} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 m}{\partial x^2} = 0 \\ m(x{=}0,t) = 0 \\ m(x,t{=}0) = m_0(x) \end{cases}$$

$$a(x,t) = -\partial_x u(x,t)$$

Igor's magical trick

$$m(x,t) = \exp\left(-\frac{u(x,t)}{\sigma^2}\right)\Gamma(x,t)$$

$$\sigma^2\partial_t\Gamma - \frac{\sigma^4}{2}\partial^2_{xx}\Gamma = \Gamma\underbrace{\left(\frac{\partial u}{\partial t} - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}\right)}_{=0}$$

$$G(x,t|x_0) = \frac{\phi(x,t)}{\phi(x_0,t{=}0)} \times G_0^{\text{abs}}(x,t|x_0)$$

$$G_0^{\text{abs}}(x,t|x_0) = (G_0(x,t|x_0) - G_0(x,t|-x_0))$$

Self consistency

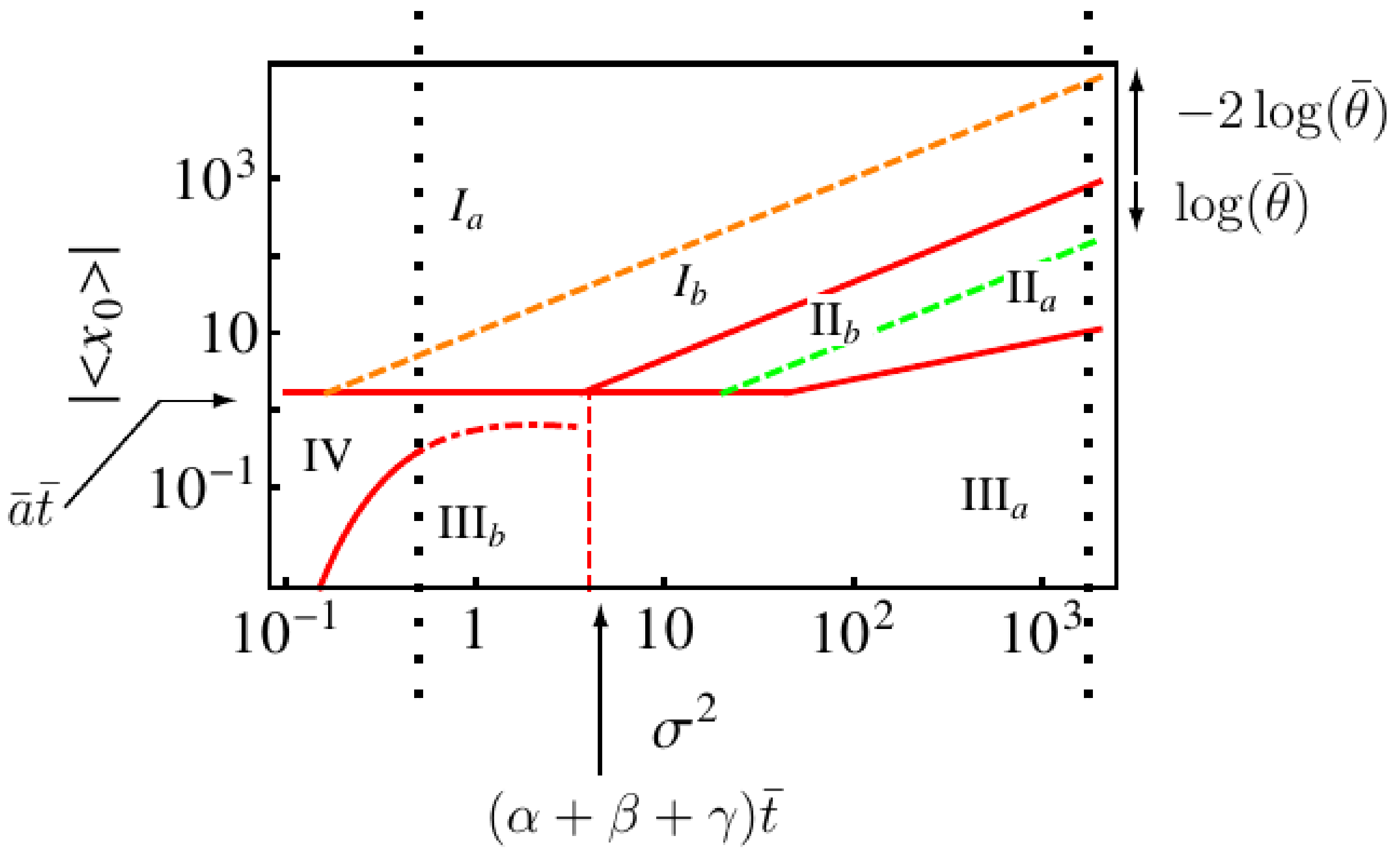
$$\int_{-\infty}^{0} dx_0\, \rho(x_0, T) m_0(x_0) = \bar{\theta}$$

($\bar{\theta}$ a priori small)

$$\rho(x_0, t) \quad \equiv \quad \int_{-\infty}^{0} dx\, G(x, t|x_0)$$

$$m_0(x_0) \text{ characterized by } \begin{cases} \text{mean position } \langle x_0 \rangle \\ \quad \text{variance} \qquad \Sigma \end{cases}$$

"phase diagram" of the small Σ regime



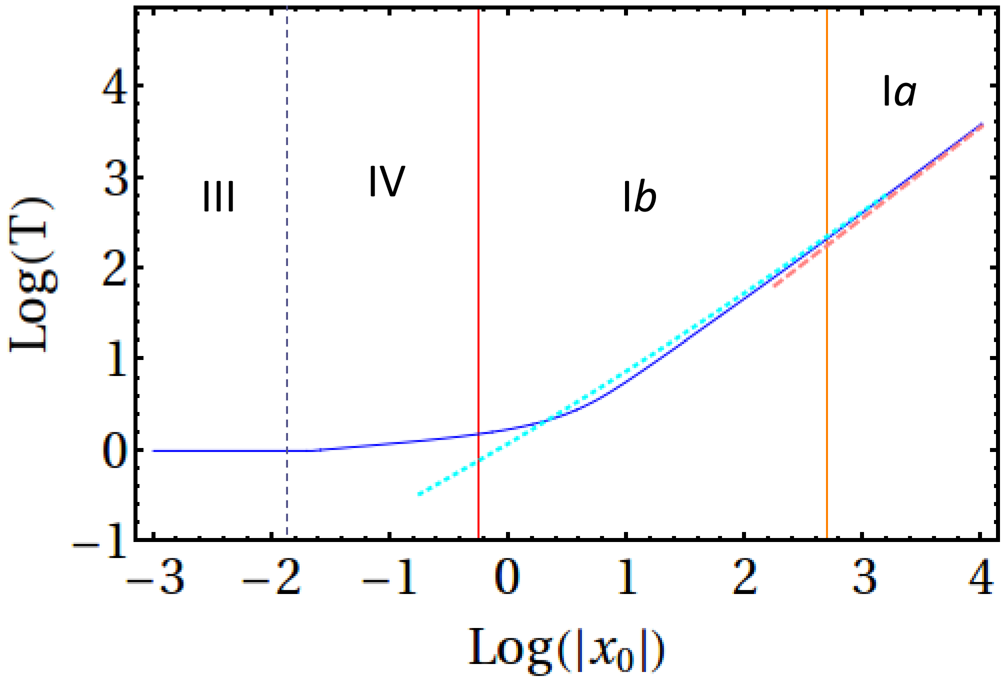
I. Convection regime $T \simeq \dfrac{|\langle x_0 \rangle|}{\bar{a}(\theta)}$

II. Diffusion regime $T \simeq 2\dfrac{\langle x_0 \rangle^2}{\pi \sigma^2 \theta^2}$

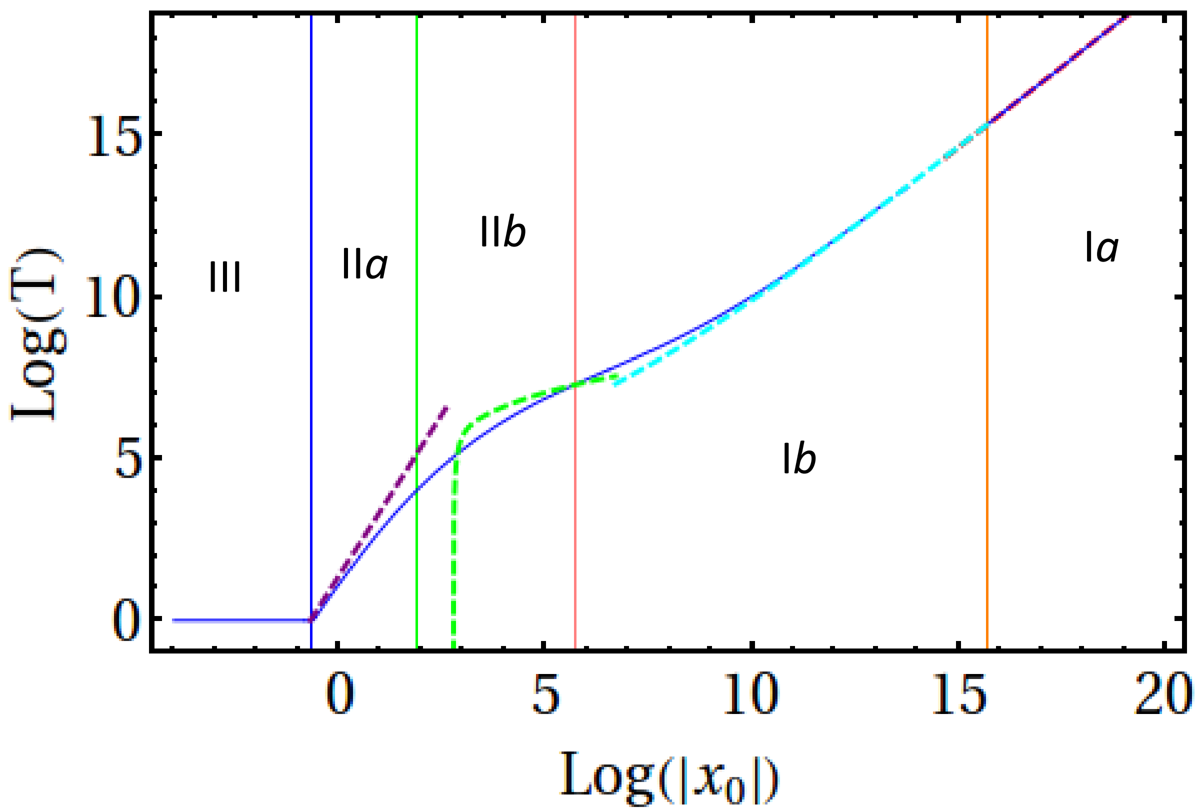
III. $T = \bar{t}$

IV. $T \approx \bar{t}$

Cut at small σ



Cut at large σ



Summary for the toy model

- Relevant velocity scales related to the slope of the cost function *c(t).*

$$(a_0 = \sqrt{2(\alpha+\beta)} \quad a_2 = \sqrt{2(\alpha-\gamma)})$$

- Limiting regimes :
  - Convective vs Diffusive : $t_{\mathrm{drift}} \equiv \frac{|x_0|}{a_{0,2}} \quad \lessgtr \quad t_{\mathrm{diff}} \equiv \frac{x_0^2}{\sigma^2}$
  - Close vs far: $t_{\mathrm{drift}}, t_{\mathrm{diff}} \quad \lessgtr \quad \bar{t}$
  - Etc ..
- “Phase diagram”

[arXiv:1503.01591]

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Of course not …

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- Geometry a bit simplistic.
- Dynamics = some version of the spherical cow.

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Well …. this is just a toy model

Going toward more relevant problems

Under what condition can a MFG model teach us something ?

- Dynamics, control parameter and cost function should bare some resemblance with reality (cf Lucas & Prescott model, or book order model).
- The optimization part should be "simple enough" (you may assume that agents are 'rational', you cannot expect all of them to own a degree in applied math).

Preference for present time

Function to optimize: $$\int_t^{\infty} d\tau e^{-\lambda(\tau - t)} \left[\frac{1}{2}(a^i_\tau)^2 + g[m](X^i_\tau, \tau) \right]$$

Two "simple" limiting cases :

- $\lambda \to \infty$: optimization on $m(x,t)$ ($t =$ now).
- $\lambda \to 0$: optimization on ergodic $m^*(x)$

Work in progress :

- Characterize these regimes and see how much one can "integrate out" the the optimization part of the game.
- Investigate how chaos may increase the speed at which the system relaxes to its ergodic state
  $\to$ MGF on a compact surface of const negative curvature.