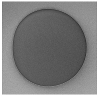
Asymmetric backscattering in deformed microcavities: fundamentals and applications

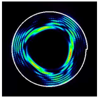
Jan Wiersig

Otto-von-Guericke-Universität Magdeburg: J. Kullig, A. Eberspächer, J.-B. Shim (now Liège)

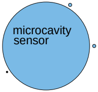
Collaborations: S. W. Kim (Busan), M. Hentschel (Ilmenau), J.-W. Ryu (Daegu), S. Shinohara (Kyoto),
H. Schomerus (Lancaster), H. Cao (Yale), R. Sarma (Yale), L. Ge (New York)



Introduction to deformed microcavities



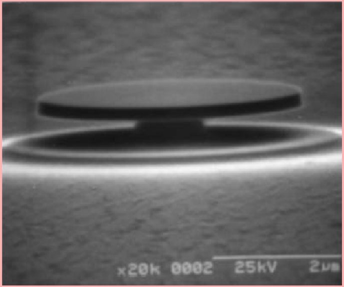
Asymmetric backscattering: fundamentals



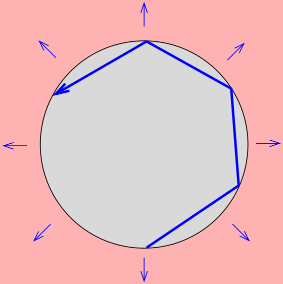
Asymmetric backscattering: applications

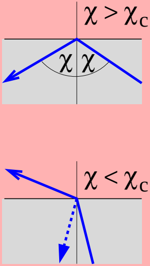
Introduction to deformed microcavities

Light confinement by total internal reflection



P. Michler et al.

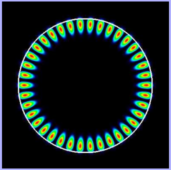


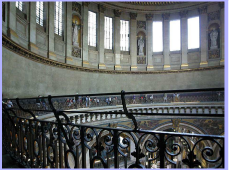


Optical modes: solutions of Maxwell's equations with harmonic time dependence

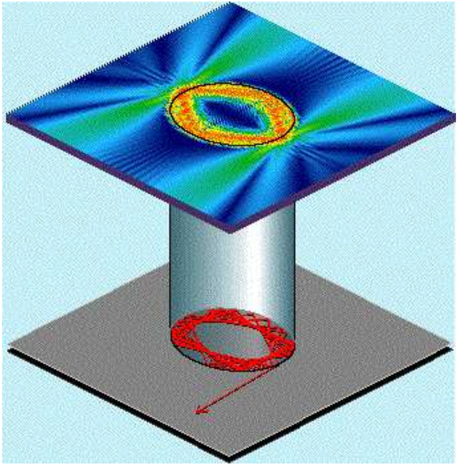
High $Q = \omega\tau$ with frequency ω and lifetime τ

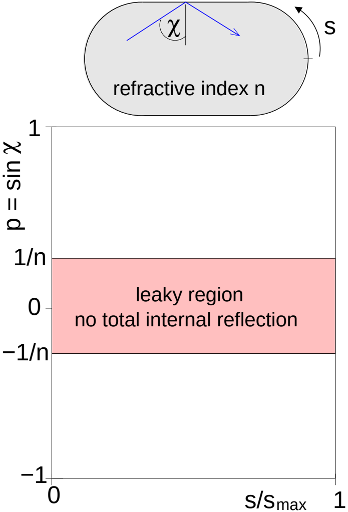
Applications: microlasers, single-photon sources, sensors, filters, ...





J.U. Nöckel und A.D. Stone, Nature **385**, 45 (1997)

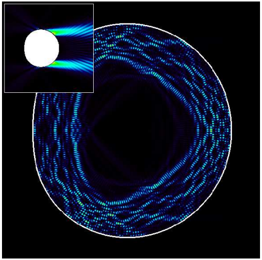


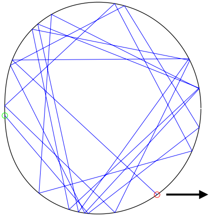


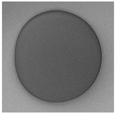
Limaçon of Pascal

J. Wiersig and M. Hentschel, PRL **100**, 033901 (2008)

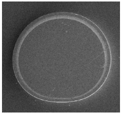
$$\rho(\phi) = R(1 + \varepsilon \cos \phi)$$



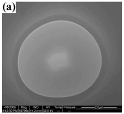




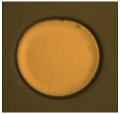
T. Harayama et al., Kyoto



F. Capasso et al., Harvard



H. Cao et al., Yale



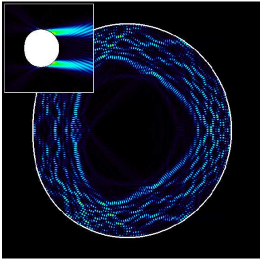
C.M. Kim et al., Seoul

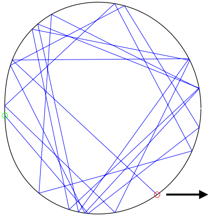
- unidirectional emission along the unstable manifold of the chaotic saddle

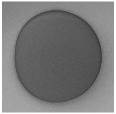
Limaçon of Pascal

J. Wiersig and M. Hentschel, PRL **100**, 033901 (2008)

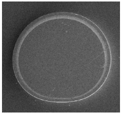
$$\rho(\phi) = R(1 + \varepsilon \cos \phi)$$



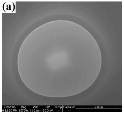




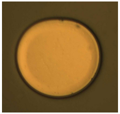
T. Harayama et al., Kyoto



F. Capasso et al., Harvard



H. Cao et al., Yale

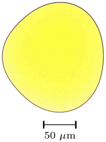


C.M. Kim et al., Seoul

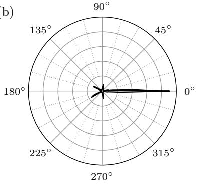
- unidirectional emission along the unstable manifold of the chaotic saddle

Shortegg

(a)



(b)



M. Schermer, S. Bittner, G. Singh, C. Ulysee, M. Lebental, and J. Wiersig, APL **106**, 101107 (2015)

Optical microcavities are open wave systems

- mode frequencies ($\widehat{=}$ energy eigenvalues) $\in \mathbb{C}$
- modes ($\widehat{=}$ energy eigenstates) are nonorthogonal
- modes may not form a complete basis

Optical microcavities are open wave systems

- mode frequencies ($\widehat{=}$ energy eigenvalues) $\in \mathbb{C}$
- modes ($\widehat{=}$ energy eigenstates) are nonorthogonal
- modes may not form a complete basis

Exceptional point (EP)

Point in parameter space at which two (or more) eigenvalues and eigenstates of a non-Hermitian linear operator coalesce. EP $\neq$ diabolic point

T. Kato, Perturbation Theory for Linear Operators (1966)

Optical microcavities are open wave systems

- mode frequencies ($\widehat{=}$ energy eigenvalues) $\in \mathbb{C}$
- modes ($\widehat{=}$ energy eigenstates) are nonorthogonal
- modes may not form a complete basis

Exceptional point (EP)

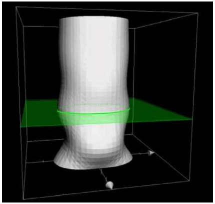
Point in parameter space at which two (or more) eigenvalues and eigenstates of a non-Hermitian linear operator coalesce. EP $\neq$ diabolic point

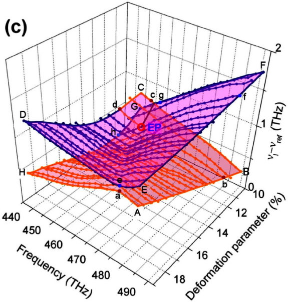
T. Kato, Perturbation Theory for Linear Operators (1966)

microwave cavity C. Dembowski *et al.*, PRL **86**, 787 (2001)

deformed microcavity (liquid jet containing laser dyes)

S.B. Lee *et al.*, PRL **103**, 134101 (2009)

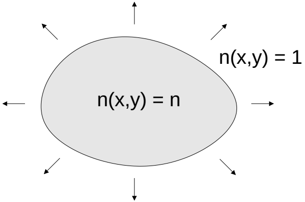




Effective index approximation

$$\left[\nabla^2 + n(x,y)^2 k^2\right] \psi(x,y) = 0$$

$$\text{Re}[\psi(x,y)e^{-i\omega t}] = \begin{cases} E_z & \text{TM} \\ H_z & \text{TE} \end{cases}$$



Continuity conditions at the cavity's boundary

TM : ψ and $\partial\psi$
TE : ψ and $\frac{1}{n^2}\partial\psi$

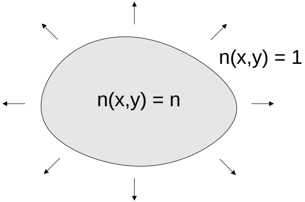
Outgoing wave condition at infinity

$\implies \omega \in \mathbb{C}$, quasibound state with lifetime $\tau = -\frac{1}{2\text{Im}(\omega)}$

Effective index approximation

$$\left[\nabla^2 + n(x,y)^2 k^2\right] \psi(x,y) = 0$$

$$\mathrm{Re}[\psi(x,y)e^{-i\omega t}] = \begin{cases} E_z & \text{TM} \\ H_z & \text{TE} \end{cases}$$



Continuity conditions at the cavity's boundary

TM : ψ and $\partial\psi$
TE : ψ and $\frac{1}{n^2}\partial\psi$

Outgoing wave condition at infinity

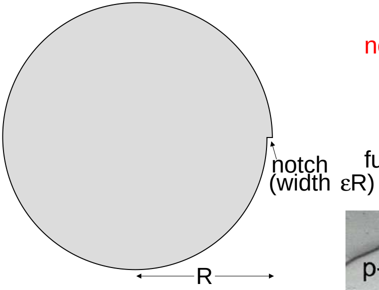
$\implies \omega \in \mathbb{C}$, quasibound state with lifetime $\tau = -\frac{1}{2\mathrm{Im}(\omega)}$

Boundary element method J. Wiersig, J. Opt. A: Pure Appl. Opt. **5**, 53 (2003)
S-matrix approach/wave matching e.g. M. Hentschel and K. Richter, PRE **66**, 056207 (2002)

Review on deformed microcavities H. Cao and J. Wiersig, RMP **87**, 61 (2015)

Asymmetric backscattering:

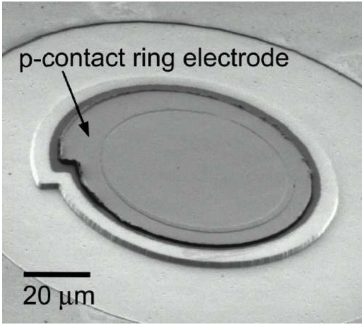
Fundamentals



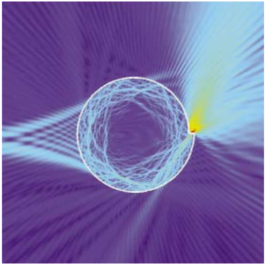
no mirror symmetry

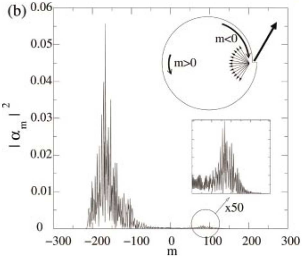
$$\rho(\phi) = R\left(1 - \frac{\varepsilon}{2\pi}\phi\right) \quad ; \varepsilon > 0$$

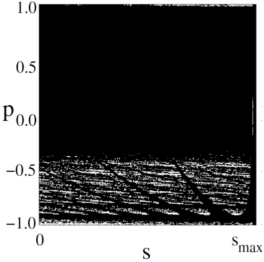
fully chaotic ray dynamics



M. Kneissl *et al.*, APL **84**, 2485 (2004)







G. D. Chern *et al.*, APL **83**, 1710 (2003)

S.-Y. Lee *et al.*, PRL **93**, 164102 (2004)

Angular momentum representation (inside the cavity)

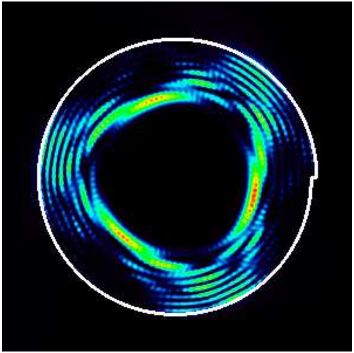
$$\psi(r,\phi) = \sum_{m=-\infty}^{\infty} \alpha_m J_m(nkr) \exp(im\phi)$$

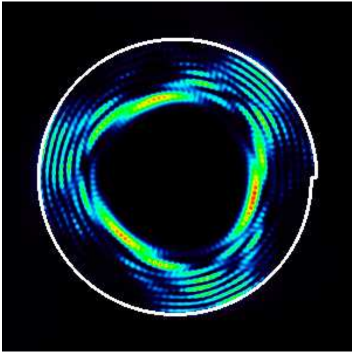
Chirality: mainly traveling wave instead of standing wave

Experimental confirmation M. Kim *et al.*, Opt. Lett. **39**, 2423 (2014)

J. Wiersig, S.W. Kim, and M. Hentschel, PRA **78**, 053809 (2008)

TE polarization, $n = 2$, and small deformation $\varepsilon = 0.04$ (spiral has been flipped)

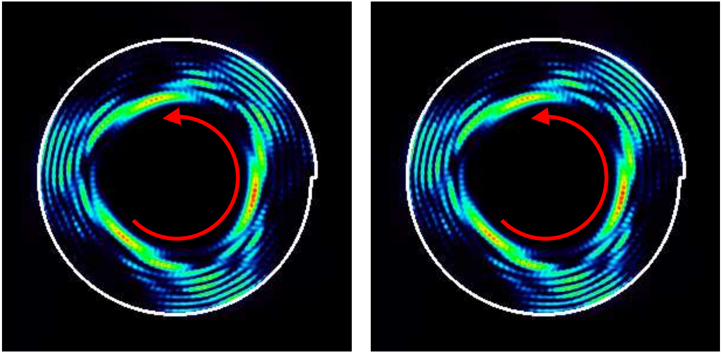




$\Omega = \frac{\omega}{c}R = kR = 41.4674 - i0.03419$ $\Omega = 41.4625 - i0.03469$; $Q = \frac{\mathrm{Re}(kR)}{2\mathrm{Im}(kR)}$

J. Wiersig, S.W. Kim, and M. Hentschel, PRA **78**, 053809 (2008)

TE polarization, $n = 2$, and small deformation $\varepsilon = 0.04$ (spiral has been flipped)

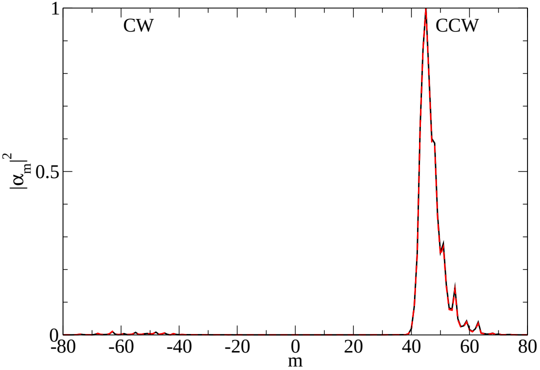


$\Omega = \frac{\omega}{c} R = kR = 41.4674 - i0.03419$ $\Omega = 41.4625 - i0.03469$; $Q = \frac{\mathrm{Re}(kR)}{2\mathrm{Im}(kR)}$

copropagation: both modes have the same dominant propagation direction

Asymmetric backscattering: Fundamentals

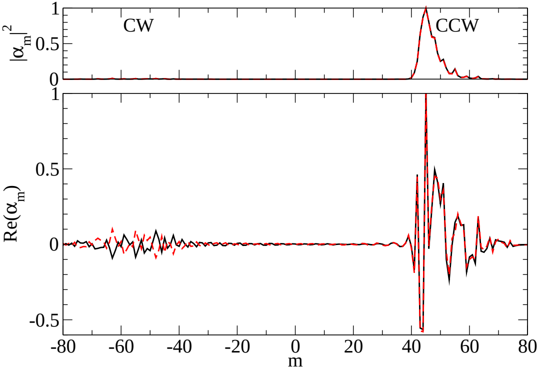
Angular momentum representation



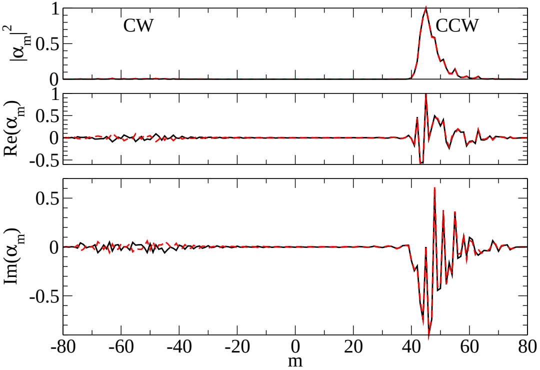
- chirality
- copropagation

Asymmetric backscattering: Fundamentals

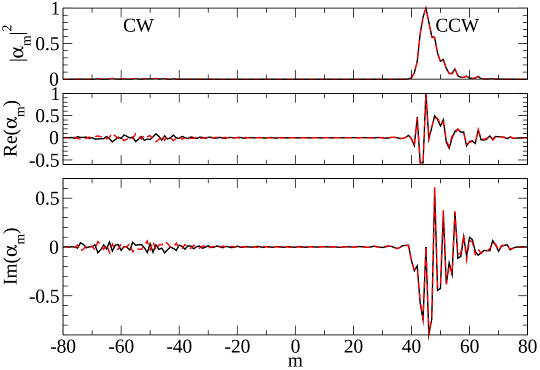
Angular momentum representation



- chirality
- copropagation



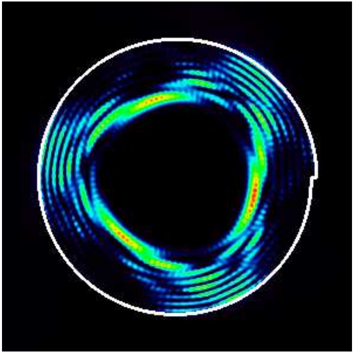
- chirality
- copropagation



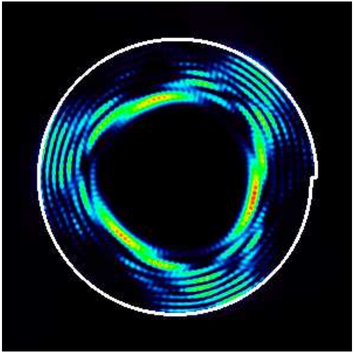
- chirality
- copropagation

Chirality

$$\alpha = 1 - \frac{\min\left(\sum_{m=-\infty}^{-1} |\alpha_m|^2, \sum_{m=1}^{\infty} |\alpha_m|^2\right)}{\max\left(\sum_{m=-\infty}^{-1} |\alpha_m|^2, \sum_{m=1}^{\infty} |\alpha_m|^2\right)} \approx \begin{cases} 0.978 \\ 0.967 \end{cases}$$



$\Omega = 41.4674 - i0.03419$



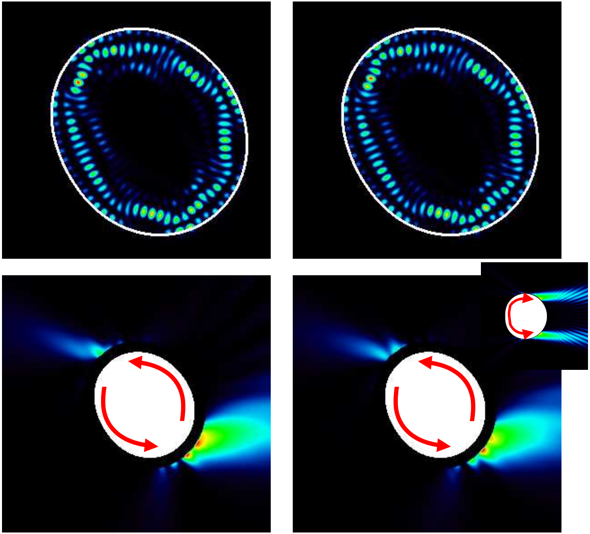
$\Omega = 41.4625 - i0.03469$

Normalized overlap integral

$$S = \frac{|\int_{\mathcal{C}} dxdy\ \psi_1^* \psi_2|}{\sqrt{\int_{\mathcal{C}} dxdy\ \psi_1^* \psi_1}\sqrt{\int_{\mathcal{C}} dxdy\ \psi_2^* \psi_2}} \approx 0.972$$

almost collinear!

$\rho = R\,[1 + \varepsilon_1 \cos\phi + \varepsilon_2 \cos(2\phi + \delta)]$ J. Wiersig *et al.*, PRA **84**, 023845 (2011)



Overlap $S \approx 0.72$

Field inside

Field outside

Chirality $\alpha \approx 0.84$

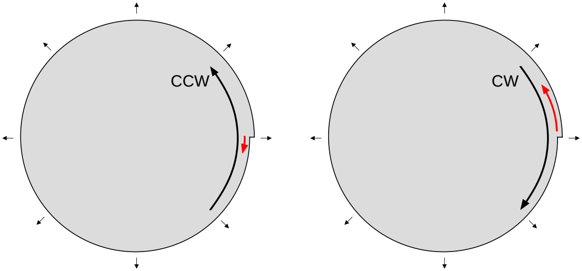
$\Omega_+ = 12.31981 - i0.00089$ $\Omega_- = 12.31985 - i0.0009$

How to explain the chirality, copropagation, and nonorthogonality?

Asymmetric backscattering: Fundamentals
A toy model

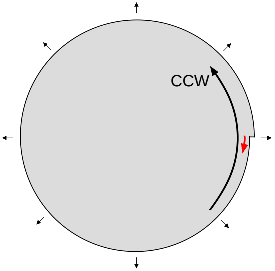
How to explain the chirality, copropagation, and nonorthogonality?

asymmetric backscattering of CW and CCW traveling waves



How to explain the chirality, copropagation, and nonorthogonality?

asymmetric backscattering of CW and CCW traveling waves

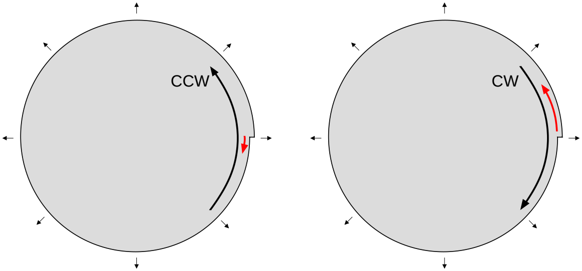


Effective non-Hermitian Hamiltonian in (CCW,CW) basis

$$H_{\text{eff}} = \begin{pmatrix} \Omega & A \\ B & \Omega \end{pmatrix} \quad \text{with } \Omega, A, B \in \mathbb{C} \text{ and } |A| \neq |B|$$

How to explain the chirality, copropagation, and nonorthogonality?

asymmetric backscattering of CW and CCW traveling waves



Effective non-Hermitian Hamiltonian in (CCW,CW) basis

$$H_{\text{eff}} = \begin{pmatrix} \Omega & A \\ B & \Omega \end{pmatrix} \quad \text{with } \Omega, A, B \in \mathbb{C} \text{ and } |A| \neq |B|$$

open quantum/wave systems with weak CW-CCW coupling and no mirror symmetries

J. Wiersig, PRA **89**, 012119 (2014)

$$H_{\text{eff}} = \begin{pmatrix} \Omega & A \\ B & \Omega \end{pmatrix} \quad ; \; |A| \neq |B|$$

Complex eigenvalues and (right hand) eigenvectors

$$\Omega_{\pm} = \Omega \pm \sqrt{AB}$$

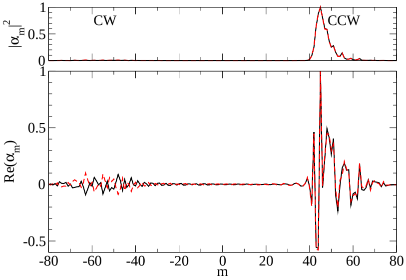
$$\vec{\psi}_{\pm} = \begin{pmatrix} \psi_{\text{ccw},\pm} \\ \psi_{\text{cw},\pm} \end{pmatrix} = \begin{pmatrix} \sqrt{A} \\ \pm\sqrt{B} \end{pmatrix}$$

$$H_{\text{eff}} = \begin{pmatrix} \Omega & A \\ B & \Omega \end{pmatrix} \quad ; \; |A| \neq |B|$$

Complex eigenvalues and (right hand) eigenvectors

$$\Omega_{\pm} = \Omega \pm \sqrt{AB}$$

$$\vec{\psi}_{\pm} = \begin{pmatrix} \psi_{\text{CCW},\pm} \\ \psi_{\text{CW},\pm} \end{pmatrix} = \begin{pmatrix} \sqrt{A} \\ \pm\sqrt{B} \end{pmatrix}$$



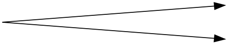
$$H_{\text{eff}} = \begin{pmatrix} \Omega & A \\ B & \Omega \end{pmatrix} \quad ; \; |A| \neq |B|$$

Complex eigenvalues and (right hand) eigenvectors

$$\Omega_{\pm} = \Omega \pm \sqrt{AB}$$

$$\vec{\psi}_{\pm} = \begin{pmatrix} \psi_{\text{CCW},\pm} \\ \psi_{\text{CW},\pm} \end{pmatrix} = \begin{pmatrix} \sqrt{A} \\ \pm\sqrt{B} \end{pmatrix}$$

$|A| > |B|$:



- CCW component $>$ CW component

$\Longrightarrow$ chirality
$\Longrightarrow$ copropagation
$\Longrightarrow$ nonorthogonality

Asymmetric backscattering: Fundamentals

Properties of the effective Hamiltonian

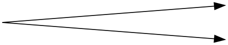
$$H_{\text{eff}} = \begin{pmatrix} \Omega & A \\ B & \Omega \end{pmatrix} \quad ; \; |A| \neq |B|$$

Complex eigenvalues and (right hand) eigenvectors

$$\Omega_{\pm} = \Omega \pm \sqrt{AB}$$

$$\vec{\psi}_{\pm} = \begin{pmatrix} \psi_{\text{CCW},\pm} \\ \psi_{\text{CW},\pm} \end{pmatrix} = \begin{pmatrix} \sqrt{A} \\ \pm\sqrt{B} \end{pmatrix}$$

$|A| > |B|$:



- CCW component $>$ CW component

  $\Longrightarrow$ chirality

  $\Longrightarrow$ copropagation

  $\Longrightarrow$ nonorthogonality

$|A| < |B|$: CW $\leftrightarrow$ CCW

Asymmetric backscattering: Fundamentals

Relation between overlap and chirality

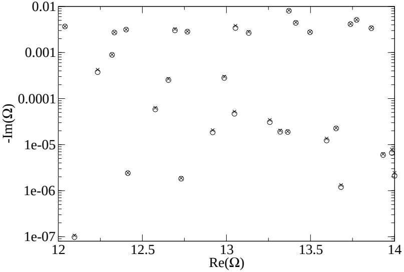
Effective Hamiltonian $\Longrightarrow$ relation between overlap and chirality

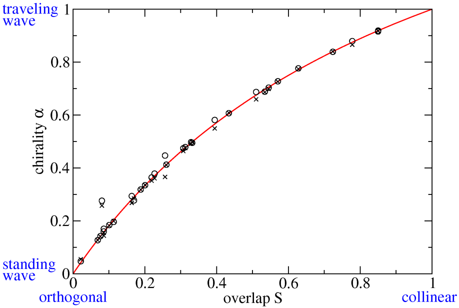
$$\alpha = \frac{2S}{1+S}$$

Effective Hamiltonian $\Longrightarrow$ relation between overlap and chirality

$$\alpha = \frac{2S}{1+S}$$

Asymmetric Limaçon cavity





Effective Hamiltonian explains the relation between chirality and nonorthogonality

$$H_{\text{eff}} = \begin{pmatrix} \Omega & A \\ B & \Omega \end{pmatrix} \quad ; \quad \Omega_{\pm} = \Omega \pm \sqrt{AB} \quad ; \quad \vec{\psi}_{\pm} = \begin{pmatrix} \sqrt{A} \\ \pm\sqrt{B} \end{pmatrix}$$

Fully asymmetric backscattering: $B \to 0$ with $A \neq 0$

$$H_{\text{eff}} = \begin{pmatrix} \Omega & A \\ 0 & \Omega \end{pmatrix} \quad ; \quad \Omega_{\pm} = \Omega \quad ; \quad \vec{\psi} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Jordan block

- splitting $\to 0$
- only one linearly independent eigenvector $\hat{=}$ CCW traveling-wave mode
- exceptional point

$$H_{\mathrm{eff}} = \begin{pmatrix} \Omega & A \\ B & \Omega \end{pmatrix} \quad ; \quad \Omega_{\pm} = \Omega \pm \sqrt{AB} \quad ; \quad \vec{\psi}_{\pm} = \begin{pmatrix} \sqrt{A} \\ \pm\sqrt{B} \end{pmatrix}$$

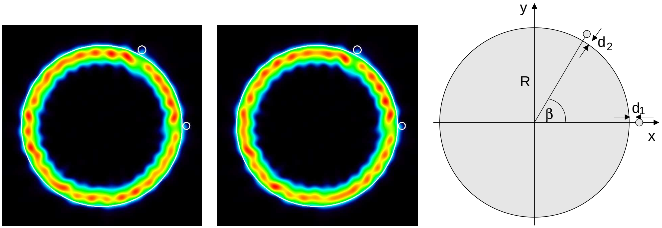
Fully asymmetric backscattering: $B \to 0$ with $A \neq 0$

$$H_{\mathrm{eff}} = \begin{pmatrix} \Omega & A \\ 0 & \Omega \end{pmatrix} \quad ; \quad \Omega_{\pm} = \Omega \quad ; \quad \vec{\psi} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

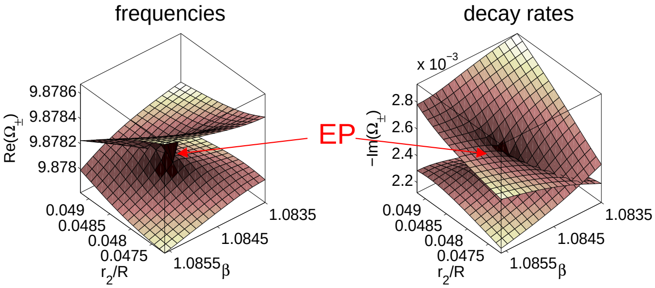
Jordan block

- splitting $\to 0$
- only one linearly independent eigenvector $\hat{=}$ CCW traveling-wave mode
- exceptional point

$A \to 0$ with $B \neq 0$: CW $\leftrightarrow$ CCW



J. Wiersig, PRA **84**, 063828 (2011)



complex-square-root topology at EP due to fully asymmetric backscattering

Ray dynamics: chirality ✓ S.-Y. Lee *et al.*, PRL **93**, 164102 (2004)

What about copropagation and nonorthogonality? ongoing work by J. Kullig

Ray dynamics: chirality ✓ S.-Y. Lee *et al.*, PRL **93**, 164102 (2004)

What about copropagation and nonorthogonality? ongoing work by J. Kullig

discrete time evolution of phase-space density ρ with Frobenius-Perron operator $\mathcal{F}$

$$\rho_{n+1}(s,p) = \mathcal{F}\rho_n(s,p)$$

for maps see e.g. J. Weber *et al.*, PRL **85**, 3620 (2000), K. Frahm and D. Shepelyansky, EPL **75**, 299 (2010)

Ray dynamics: chirality ✓ S.-Y. Lee *et al.*, PRL **93**, 164102 (2004)

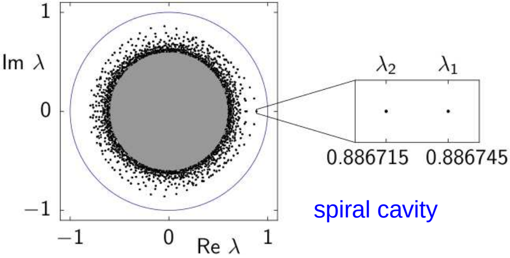
What about copropagation and nonorthogonality? ongoing work by J. Kullig

discrete time evolution of phase-space density ρ with Frobenius-Perron operator $\mathcal{F}$

$$\rho_{n+1}(s,p) = \mathcal{F}\rho_n(s,p)$$

for maps see e.g. J. Weber *et al.*, PRL **85**, 3620 (2000), K. Frahm and D. Shepelyansky, EPL **75**, 299 (2010)

- weight to incorporate reflectivity
  $\Longrightarrow$ $\mathcal{F}$ is sub-unitary

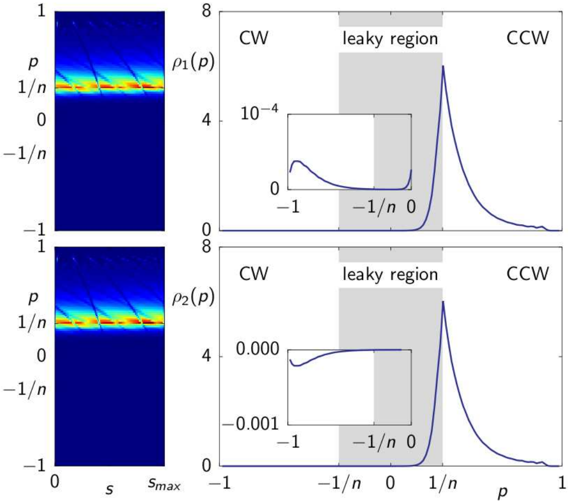


spiral cavity

the two largest eigenvalues are nearly degenerate (eigenstate pair)

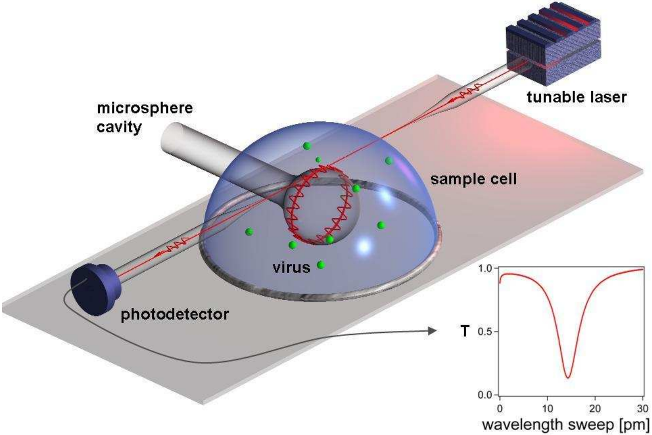
Asymmetric backscattering: Fundamentals

Frobenius-Perron eigenstate pair for the spiral cavity



Frobenius-Perron eigenstate pair show chirality, copropagation, and nonorthogonality

Asymmetric backscattering: Applications

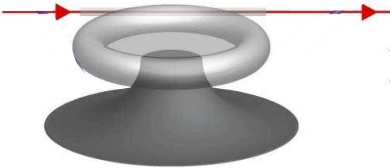


F. Vollmer *et al.*, PNAS **105**, 20701 (2008)

Measure frequency shift $\Longrightarrow$ particle detection

Measure frequency splitting of initially degenerate modes (diabolic point)

J. Zhu *et al.*, Nature Photonics **4**, 46 (2010)

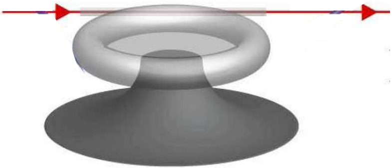


Asymmetric backscattering: Applications

Microcavity sensor based on frequency-splitting detection

Measure frequency splitting of initially degenerate modes (diabolic point)

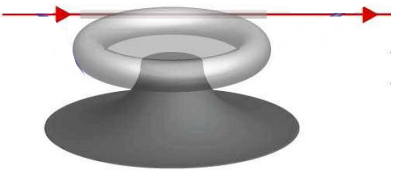
J. Zhu *et al.*, Nature Photonics **4**, 46 (2010)



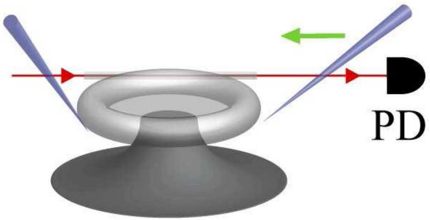
Problem: initial splitting due to fabrication imperfections

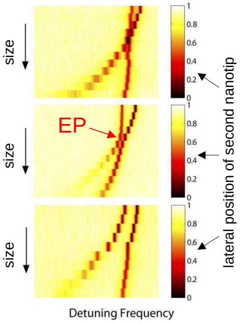
Measure frequency splitting of initially degenerate modes (diabolic point)

J. Zhu *et al.*, Nature Photonics **4**, 46 (2010)



Problem: initial splitting due to fabrication imperfections



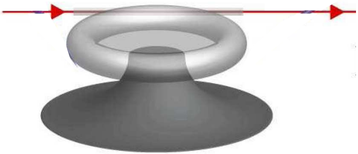


J. Zhu *et al.*, Opt. Express **18**, 23535 (2010)

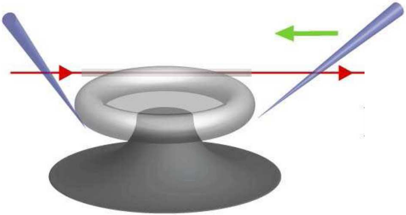
Asymmetric backscattering: Applications

Conventional degeneracy vs exceptional point

J. Wiersig, PRL **112**, 203901 (2014)



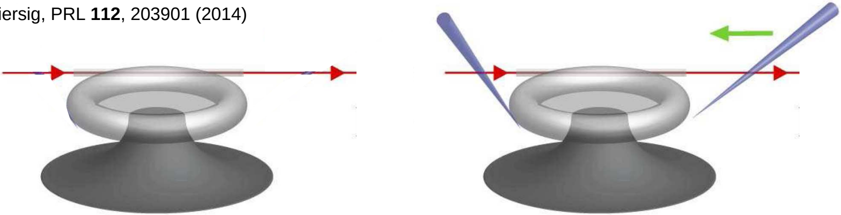
conventional (DP)



EP

Which one is better for sensing?

J. Wiersig, PRL **112**, 203901 (2014)



conventional (DP) EP

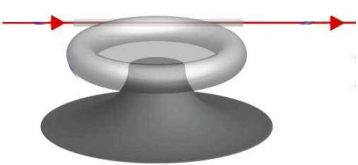
Which one is better for sensing?

Apply a perturbation of strength ϵ to a (two-fold) degeneracy

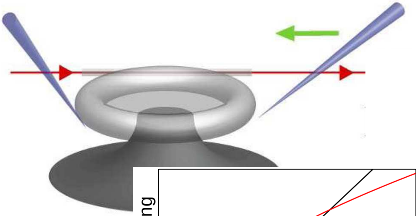
$$\Delta\Omega_{\text{DP}} = \mathcal{O}(\varepsilon) \qquad \Delta\Omega_{\text{EP}} = \mathcal{O}(\sqrt{\varepsilon})$$

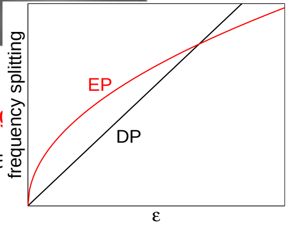
T. Kato (1966)

J. Wiersig, PRL **112**, 203901 (2014)



conventional (DP)





Which one is better for sensin

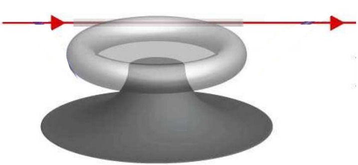
Apply a perturbation of strength ϵ to a (two-fold) degenera

$$\Delta\Omega_{\text{DP}} = \mathcal{O}(\varepsilon)$$

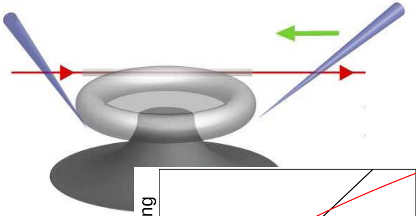
$\Delta\Omega_{\text{EP}}$

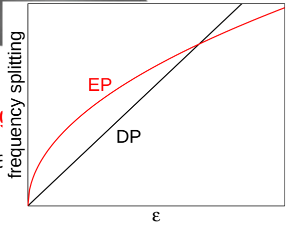
T. Kato (1966)

J. Wiersig, PRL **112**, 203901 (2014)



conventional (DP)





T. Kato (1966)

Which one is better for sensin

Apply a perturbation of strength ϵ to a (two-fold) degenera

$$\Delta\Omega_{\text{DP}} = \mathcal{O}(\varepsilon)$$

$\Delta\Omega_{\text{EP}}$

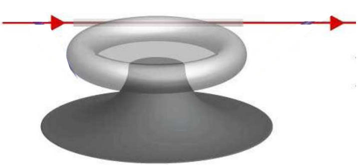
Enhancement factor of sensitivity for splitting detection

$$\frac{\Delta\Omega_{\text{EP}}}{\Delta\Omega_{\text{DP}}} = \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}}\right) \quad \text{for sufficiently small } \varepsilon$$

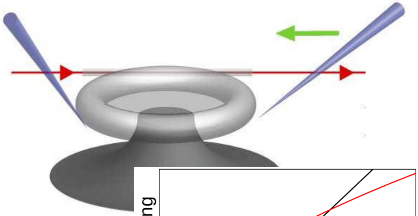
Asymmetric backscattering: Applications

Conventional degeneracy vs exceptional point

J. Wiersig, PRL **112**, 203901 (2014)



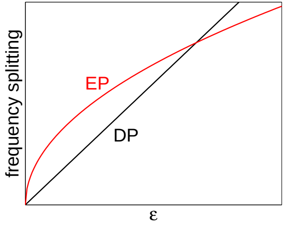
conventional (DP)



Which one is better for sensin

Apply a perturbation of strength ϵ to a (two-fold) degenera

$$\Delta\Omega_{\mathrm{DP}} = \mathcal{O}(\varepsilon) \qquad \Delta\Omega_{\mathrm{EP}}$$

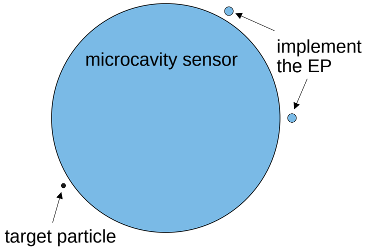


T. Kato (1966)

Enhancement factor of sensitivity for splitting detection

$$\frac{\Delta\Omega_{\mathrm{EP}}}{\Delta\Omega_{\mathrm{DP}}} = \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}}\right) \quad \text{for sufficiently small } \varepsilon$$

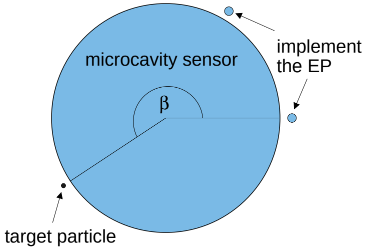
Price to pay: $\Delta\Omega_{\mathrm{EP}} \in \mathbb{C} \implies$ frequency and linewidth splitting



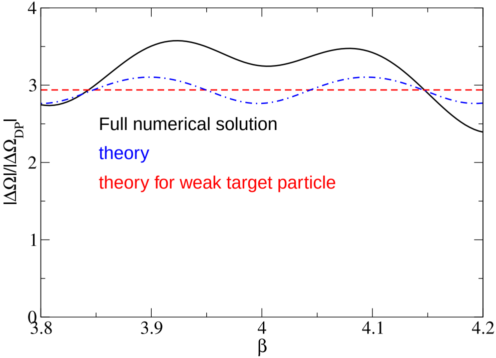
EP is due to fully asymmetric backscattering

Asymmetric backscattering: Applications

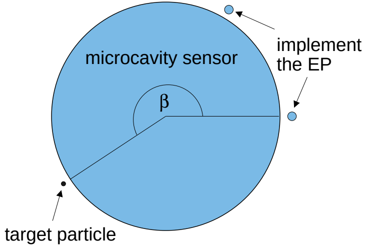
Results for a microcavity sensor at an EP



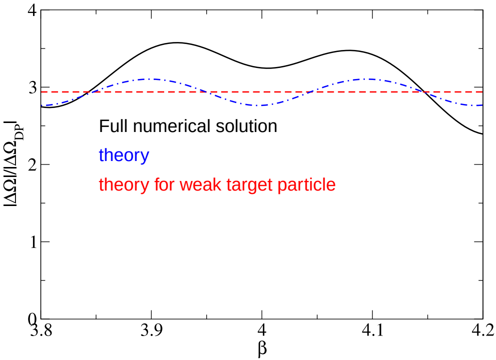
EP is due to fully asymmetric backscattering



- 3 to 3.5 fold enhancement of sensitivity
- Splitting $|\Delta\Omega|$ is nearly independent on β



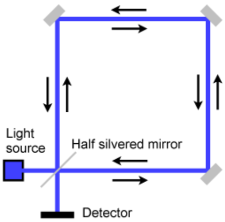
EP is due to fully asymmetric backscattering



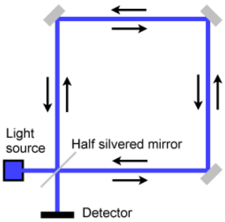
- 3 to 3.5 fold enhancement of sensitivity
- Splitting $|\Delta\Omega|$ is nearly independent on β

Sensitivity of sensors based on frequency splitting detection can be enhanced at an EP

Sagnac effect: rotations leads to a frequency splitting of counterpropagating waves

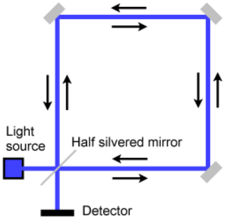


Sagnac effect: rotations leads to a frequency splitting of counterpropagating waves



EP does not help here

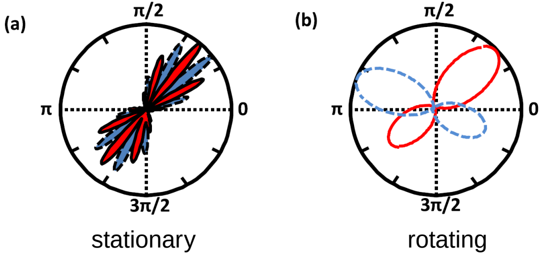
Sagnac effect: rotations leads to a frequency splitting of counterpropagating waves



EP does not help here

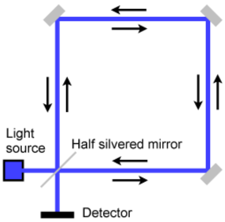
R. Sarma, L. Ge, J. Wiersig, and H. Cao, PRL **114**, 053903 (2015)

Asymmetric limaçon: chirality and copropagation



$\Longrightarrow$ far-field pattern is a sensitive measure of rotation

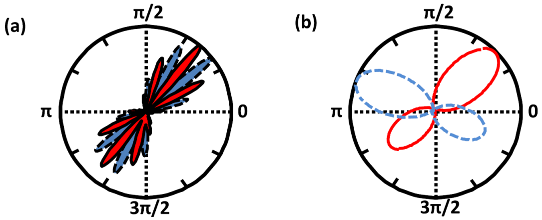
Sagnac effect: rotations leads to a frequency splitting of counterpropagating waves



EP does not help here

R. Sarma, L. Ge, J. Wiersig, and H. Cao, PRL **114**, 053903 (2015)

Asymmetric limaçon: chirality and copropagation



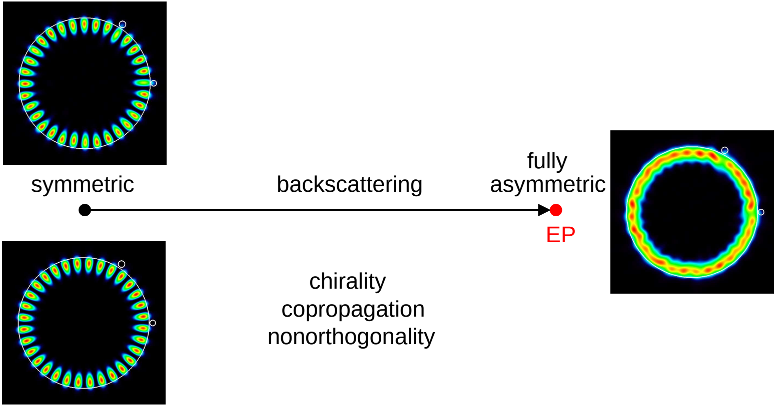
stationary rotating

$\Longrightarrow$ far-field pattern is a sensitive measure of rotation

3 orders of magnitude more sensitive than the Sagnac effect!

Summary

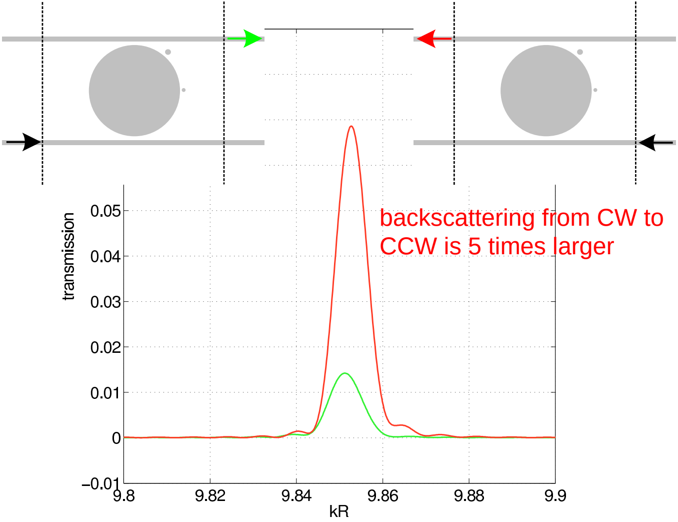
Fundamentals



Applications

- enhancing the sensitivity of microcavity sensors for particle detection
- enhancing the sensitivity of microcavity gyroscopes

FDTD simulations of a waveguide-coupled microcavity Johannes Kramer, diploma thesis 2014



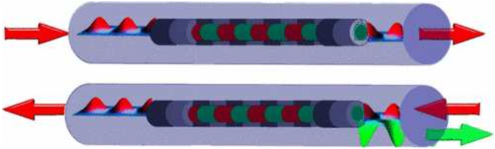
Bonus

Bragg reflectors with aborption/gain

- “Irreversible coupling by use of dissipative optics” (theory)

  M. Greenberg and M. Orenstein, Opt. Lett. **29**, 5 (2004), Opt. Express **12**, 4013 (2004)
- “Unidirectional invisibility induced by PT-symmetric periodic structures” (theory)

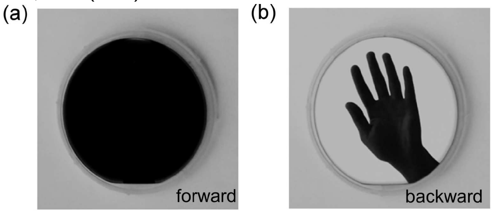
  Z. Lin *et al.*, PRL **106**, 213901 (2011)



- “~~Nonreciprocal~~ light propagation” (experiment)

  L. Feng *et al.*, Science **333**, 729 (2011)
- “Unidirectional reflectionless light transport” (experiment)

  L. Feng *et al.*, Opt. Express **22**, 1760 (2014)



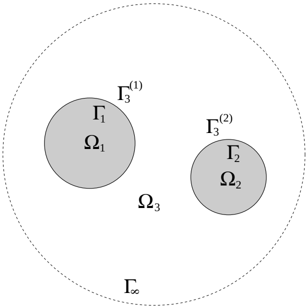
J. Wiersig, J. Opt. A: Pure Appl. Opt. **5**, 53 (2003)

2D PDE $\rightarrow$ 1D boundary integral equations

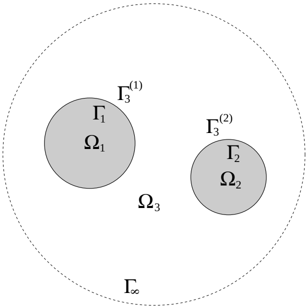
$$\psi(\mathbf{r}') = \oint_{\Gamma_j} ds[\psi(s)\partial G(s,\mathbf{r}';k) - G(s,\mathbf{r}';k)\partial\psi(s)]$$

with (outgoing) Green's function

$$G(\mathbf{r},\mathbf{r}';k) = -\frac{i}{4}H_0^{(1)}(n_j k|\mathbf{r}-\mathbf{r}'|)$$



J. Wiersig, J. Opt. A: Pure Appl. Opt. **5**, 53 (2003)



2D PDE $\rightarrow$ 1D boundary integral equations

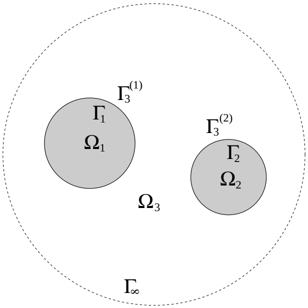
$$\psi(\mathbf{r}') = \oint_{\Gamma_j} ds[\psi(s)\partial G(s, \mathbf{r}'; k) - G(s, \mathbf{r}'; k)\partial\psi(s)]$$

with (outgoing) Green's function

$$G(\mathbf{r}, \mathbf{r}'; k) = -\frac{i}{4} H_0^{(1)}(n_j k|\mathbf{r} - \mathbf{r}'|)$$

- outgoing wave condition $\rightarrow$ Γ_∞ does not contribute

J. Wiersig, J. Opt. A: Pure Appl. Opt. **5**, 53 (2003)



2D PDE $\rightarrow$ 1D boundary integral equations

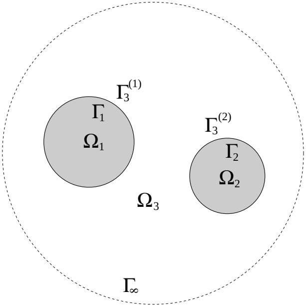
$$\psi(\mathbf{r}') = \oint_{\Gamma_j} ds[\psi(s)\partial G(s, \mathbf{r}'; k) - G(s, \mathbf{r}'; k)\partial\psi(s)]$$

with (outgoing) Green's function

$$G(\mathbf{r}, \mathbf{r}'; k) = -\frac{i}{4} H_0^{(1)}(n_j k|\mathbf{r} - \mathbf{r}'|)$$

- outgoing wave condition $\rightarrow$ Γ_∞ does not contribute
- spurious solutions: interior Dirichlet problem with $n_j = 1$

J. Wiersig, J. Opt. A: Pure Appl. Opt. **5**, 53 (2003)



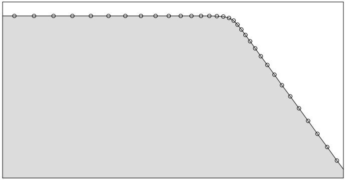
2D PDE $\rightarrow$ 1D boundary integral equations

$$\psi(\mathbf{r}') = \oint_{\Gamma_j} ds[\psi(s)\partial G(s,\mathbf{r}';k) - G(s,\mathbf{r}';k)\partial\psi(s)]$$

with (outgoing) Green's function

$$G(\mathbf{r},\mathbf{r}';k) = -\frac{i}{4}H_0^{(1)}(n_j k|\mathbf{r}-\mathbf{r}'|)$$

- outgoing wave condition $\rightarrow$ Γ_∞ does not contribute
- spurious solutions: interior Dirichlet problem with $n_j = 1$
- discretization $0 = M(k_{\text{res}})\vec{x}$ with $\vec{x} = (\partial\psi\big|_{s_1}, \ldots, \psi\big|_{s_1}, \ldots)$



1. initial guess k_0

$$0 = M(k_0 + \delta k)\vec{x} \approx \left[M(k_0) + \delta k\, M'(k_0)\right] \vec{x}$$

$\Longrightarrow$ generalized eigenvalue equation

$$M(k_0)\vec{x} = -\delta k\, M'(k_0)\vec{x}$$

1. initial guess k_0

$$0 = M(k_0 + \delta k)\vec{x} \approx \left[M(k_0) + \delta k\, M'(k_0)\right] \vec{x}$$

$\Longrightarrow$ generalized eigenvalue equation

$$M(k_0)\vec{x} = -\delta k\, M'(k_0)\vec{x}$$

2. find eigenvector $\vec{x}$ with smallest eigenvalue $|\delta k|$
3. $k_1 = k_0 + \delta k$
4. iterate until δk is small enough

Bonus

Boundary element method for dielectric microcavities

1. initial guess k_0

$$0 = M(k_0 + \delta k)\vec{x} \approx \left[M(k_0) + \delta k\, M'(k_0)\right]\vec{x}$$

$\Longrightarrow$ generalized eigenvalue equation

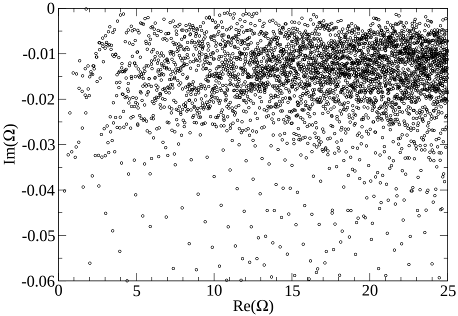
$$M(k_0)\vec{x} = -\delta k\, M'(k_0)\vec{x}$$

2. find eigenvector $\vec{x}$ with smallest eigenvalue $|\delta k|$
3. $k_1 = k_0 + \delta k$
4. iterate until δk is small enough

stadium (3772 resonances)

J. Wiersig and J. Main, PRE **77**, 036205 (2008)

Normalized frequency $\Omega = \frac{\omega}{c}R = kR$



Bonus

What does the coalescence of two eigenstates mean dynamically?

Eigenvalue equation

$$E_j|\phi_j\rangle = H_{\text{eff}}|\phi_j\rangle$$

Bonus

What does the coalescence of two eigenstates mean dynamically?

Eigenvalue equation

$$E_j|\phi_j\rangle = H_{\text{eff}}|\phi_j\rangle$$

Schrödinger equation

$$i\frac{d}{dt}|\psi\rangle = H_{\text{eff}}|\psi\rangle$$

Bonus

What does the coalescence of two eigenstates mean dynamically?

Eigenvalue equation

$$E_j|\phi_j\rangle = H_{\text{eff}}|\phi_j\rangle$$

Schrödinger equation

$$i\frac{d}{dt}|\psi\rangle = H_{\text{eff}}|\psi\rangle$$

2-by-2 Hamiltonian at EP

- eigenvalue equation: one solution

$$\vec{\phi}_{\text{EP}} \ , E_{\text{EP}}$$

Bonus

What does the coalescence of two eigenstates mean dynamically?

Eigenvalue equation

$$E_j|\phi_j\rangle = H_{\text{eff}}|\phi_j\rangle$$

Schrödinger equation

$$i\frac{d}{dt}|\psi\rangle = H_{\text{eff}}|\psi\rangle$$

2-by-2 Hamiltonian at EP

- eigenvalue equation: one solution

$$\vec{\phi}_{\text{EP}}\ , E_{\text{EP}}$$

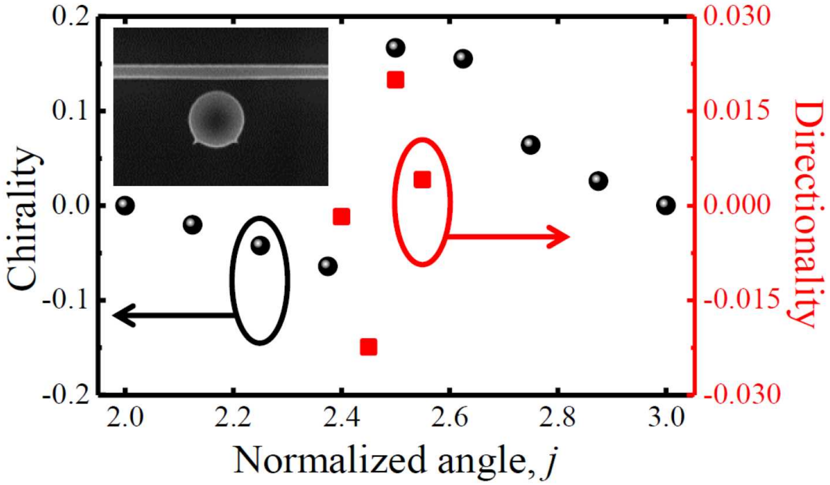
- Schrödinger equation: two solutions

$$\begin{aligned}\vec{\psi}_1(t) &= \vec{\phi}_{\text{EP}}e^{-iE_{\text{EP}}t}\\ \vec{\psi}_2(t) &= \left(\vec{\phi}_0 + t\vec{\phi}_{\text{EP}}\right)e^{-iE_{\text{EP}}t}\end{aligned}$$

B. Dietz *et al.*, PRE **75**, 027201 (2007), W. D. Heiss, Eur. Phys. J. D **60**, 257 (2010)

Bonus

Experimental confirmation of chirality



M. Kim *et al.*, Opt. Lett. **39**, 2423 (2014)