Systems with off–diagonal disorder on a lattice

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in collaboration with

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Quantum Chaos, **Luchon**, March 17, 2015

Some spectral properties of quantum systems

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Random matrices: applications in quantum & classical physics

A) Quantum Chaos and Unitary Dynamics:

'Quantum chaology'

Quantum analogues of classically chaotic dynamical systems can be described by random matrices

a) autonomous systems – Hamiltonians:
Gaussian ensembles of random Hermitian matrices, (GOE, GUE, GSE)

b) periodic systems – evolution operators:
Dyson circular ensembles of random unitary matrices, (COE, CUE, CSE)

Universality classes

Depending on the symmetry properties of the system one uses ensembles form
orthogonal ($\beta = 1$);
unitary ($\beta = 2$) and
symplectic ($\beta = 4$) ensembles.

The exponent β determines the level repulsion,

$$P(s) \sim s^{\beta}$$

for $s \to 0$ where s stands for the (normalised) level spacing, $s_i = \phi_{i+1} - \phi_i$.

see e.g. F. Haake, *Quantum Signatures of Chaos*

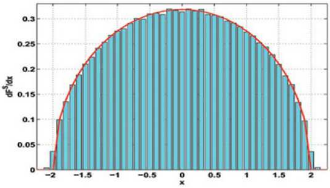
Wigner Semicircle Law

Spectral density $P(x)$ for random hermitian matrices

can be obtained by integrating out all eigenvalues but one from jpd.
For all three **Gaussian ensembles** of Hermitian random matrices one obtains (asymptotically, for $N \to \infty$) the **Wigner Semicircle Law** (1955)

$$P(x) = \frac{1}{2\pi}\sqrt{2 - x^2}$$

where x denotes a **normalized eigenvalue**, $x_i = \lambda_i/\sqrt{N}$



Normalised eigenvalue distribution of a random 100 × 100 GUE matrix. (Image by Alan Edelman.)

Extremal eigenvalues & Tracy–Widom Law

Statistics of **extremal** cases - the **largest eigenvalue** x_{max}

The normalized largest eigenvalue ("s" of Tracy–Widom)

$$s := (x_{max} - 2\sqrt{N})N^{-1/6}$$

of a **GUE random matrix** is (asymptotically) distributed according to the **Tracy-Widom** law (1994)

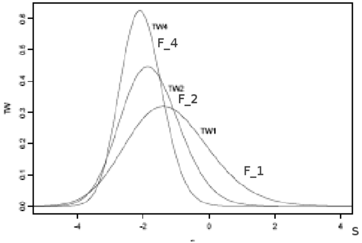
$$F_2(s) = \det(1 - K) \ ,$$

where K is the integral operator with the **Airy kernel**

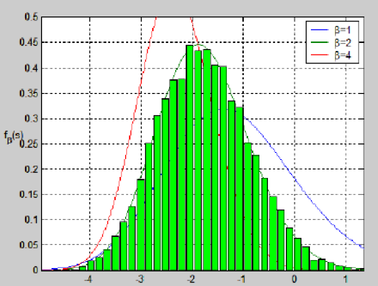
$$K(x, y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x - y}.$$

The **scaling behaviour** of the finite size effect (as $N^{-1/6}$) is due to **Bowick & Brezin** (1991) and **Forrester** (1991).

Tracy–Widom distributions $F_\beta(s)$



Distributions $F_\beta(s)$



and the largest eigenvalue of random **GUE matrices** (image by A. Edelman)

Level spacing distribution $P(s)$

Nearest neighbour spacing s

$s_i = \frac{x_{i+1} - x_i}{\Delta}$ ("s" of Wigner), where Δ is the mean spacing
a) Gaussian ensembles for $N = 2$ $\Rightarrow$ **Wigner surmise**

- $\beta = 1$ **GOE** (orthogonal) $P_1(s) = \frac{\pi}{2} s \exp(-\frac{\pi}{4} s^2)$
- $\beta = 2$ **GUE** (unitary) $P_2(s) = \frac{32}{\pi^2} s^2 \exp(-\frac{4}{\pi} s^2)$
- $\beta = 4$ **GSE** (symplectic) $P_4(s) = \frac{2^{18}}{3^6 \pi^3} s^4 \exp(-\frac{64}{9\pi} s^2)$

These distributions derived for $N = 2$ work well also for Gaussian ensembles in the asymptotic case, $N \to \infty$.

Random unitary matrices & Circular ensembles of Dyson

Uniform density of phases along the unit circle, $P(\phi) = 1/2\pi$.
Phase spacing, $s_i = \frac{N}{2\pi}[\phi_{i+1} - \phi_i]$ since $\Delta = 2\pi/N$.
For large matrices the **level spacing** distributions for **Gaussian ensembles** (Hermitian matrices) and **circular ensembles** (unitary matrices) coincide.

Extremal spacings
for **random unitary matrices**. Consider

a) **Minimal spacing** $s_{\min} = \min_j \{s_j\}_{j=1}^N$ (*how close to degeneracy?*)
and b) **Maximal spacing** $s_{\max} = \max_j \{s_j\}_{j=1}^N$

Minimal spacing distribution for $N = 4$ random unitary matrices

Two qubits & random local gates

Analytical results $P_{2\otimes 2}(t)$ for CUE(2) $\otimes$ CUE(2) case, where $t = s_{\min}$

$$P_{2\otimes 2}(t) = \frac{1}{4\pi}\Big(2\pi(1-t)\big(4-\cos(\frac{\pi t}{2})\big) - 3\sin(\frac{\pi t}{2}) + 8\sin(\pi t) - 3\sin(\frac{3\pi t}{2})\Big)$$

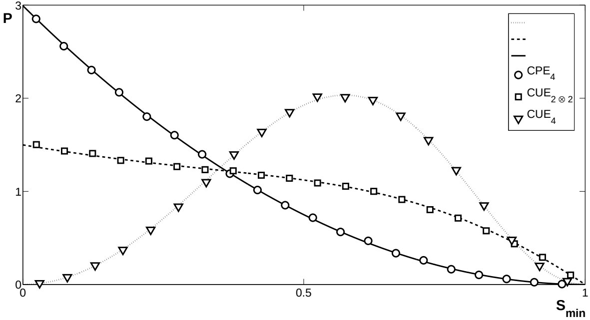
CUE, $\beta = 2$, CUE(4), $P_4^{(2)}(t) = \ldots$
explicit result to long to reproduce it here...

Poisson ensemble, $\beta = 0$, CPE(4), $P_4^{(0)}(t) = 3(1-t)^2$.

Minimal spacing $P(s_{\min})$ for $N = 4$ unitary matrices

Comparison of spacing distribution $P(s_{\min})$ for

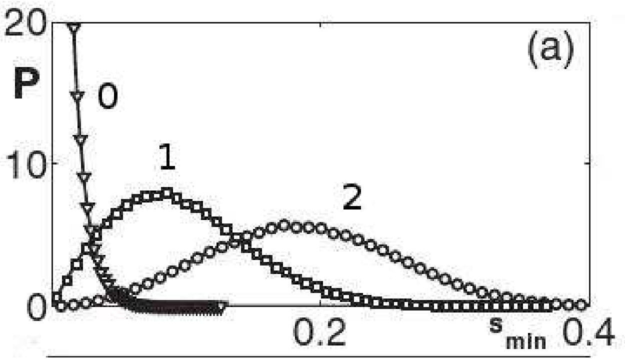
a) Poisson CPE(4), b) **CUE**(2) $\otimes$ CUE(2), c) CUE(4).



mean values: $\langle s_{\min} \rangle_{CPE4} = 1/4$, $\langle s_{\min} \rangle_{CUE2 \otimes CUE2} \approx 0.4$, $\langle s_{\min} \rangle_{CUE4} \approx 0.54$

Smaczyński, Tkocz, Kuś, Życzkowski Phys. Rev. **E** (2013)

Minimal spacing $P(s_{\min})$ for large unitary matrices



(here $N = 100$)

Minimal spacing distribution $P(s_{\min})$ for

0) Poisson CPE(N), $P_0(s_{\min}) = A_0 N e^{-N s_{\min}}$

1) **COE(N)**, $P_1(s_{\min}) = 2A_1^2 N s_{\min} e^{-A_1^2 N s_{\min}^2}$

2) CUE(N), $P_2(s_{\min}) = 3A_2^3 N s_{\min}^2 e^{-A_2^3 N s_{\min}^3}$.

Average minimal spacing $\langle s_{\min} \rangle$ for large unitary matrices

Approximation of independent spacings

Assume spacings s_j described by the distribution $P_\beta(s)$ are independent.

Minimal spacing

Since for **small** spacings $P_\beta(s) \sim s^\beta$ so the **integrated distribution**

$$I(s) = \int_0^s P(s')ds' \text{ behaves as } I_\beta(s) \sim s^{1+\beta}$$

Matrix of order N yields N spacings s_j. The **minimal** spacing $s_{\min}$ occurs for such an argument that $I_\beta(s_{\min}) \approx 1/N$.

Thus $(s_{\min})^{1+\beta} \approx 1/N \Longrightarrow s_{\min} \approx N^{-\frac{1}{\beta+1}}$

Average maximal spacing $\langle s_{\max} \rangle$

Approximation of independent spacings

Assume spacings s_j described by the distribution $P_\beta(s)$ are independent.

Mean maximal spacing for COE

Since for **large** spacings $P_\beta(s) \sim s^1 \exp(-s^2)$ so the **integrated distribution**

$$I_1(s) = \int_0^s P(s')ds' \text{ behaves as } I_1(s) \sim -\exp(-s^2)$$

Matrix of order N yields N spacings s_j. The **maximal** spacing $s_{\max}$ occurs for such an argument that $1 - I_1(s_{\max}) \approx 1/N$.

Thus $\exp[-(s_{\max})^2] \approx 1/N \Longrightarrow s_{\max} \approx \sqrt{\ln N}$

Smaczyński, Tkocz, Kuś, Życzkowski Phys. Rev. **E** (2013)

Some of these results (*and some other*) appeared in a preprint **arXiv:1010.1294** "Extreme gaps between eigenvalues of random matrices" by **Ben Arous and Bourgade**.

Classical probabilistic dynamics & Markov chains

Stochastic matrices

Classical states: N-point probability distribution, $\mathbf{p} = \{p_1, \dots p_N\}$,
where $p_i \geq 0$ and $\sum_{i=1}^{N} p_i = 1$
Discrete dynamics: $p'_i = S_{ij} p_j$, where S is a **stochastic matrix** of size N and maps the simplex of classical states into itself, $S: \ \Delta_{N-1} \to \Delta_{N-1}$.

Frobenius–Perron theorem

Let S be a **stochastic matrix**:

a) $S_{ij} \geq 0$ for $i, j = 1, \dots, N$,

b) $\sum_{i=1}^{N} S_{ij} \ = \ 1$ for all $j = 1, \dots, N$.

Then

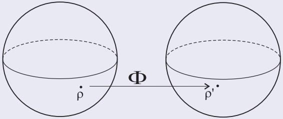
i) the spectrum $\{z_i\}_{i=1}^{N}$ of S belongs to the unit disk,

ii) the leading eigenvalue equals unity, $z_1 = 1$,

iii) the corresponding eigenstate $\mathbf{p}_{\rm inv}$ is invariant, $S\mathbf{p}_{\rm inv} = \mathbf{p}_{\rm inv}$.

B) Quantum Chaos & Nonunitary Dynamics

Quantum operation: linear, completely positive trace preserving map



$\Phi : \mathcal{M}_2 \rightarrow \mathcal{M}_2$

positivity: $\Phi(\rho) \geq 0, \quad \forall \rho \in \mathcal{M}_N$

complete positivity: $[\Phi \otimes \mathbb{1}_K](\sigma) \geq 0, \quad \forall \sigma \in \mathcal{M}_{KN}$ and $K = 2, 3, ...$

Enviromental form (interacting quantum system !)

$$\rho' = \Phi(\rho) = \mathrm{Tr}_E[U\,(\rho \otimes \omega_E)\;U^\dagger]\;.$$

where ω_E is an initial state of the environment while $UU^\dagger = \mathbb{1}$.

Kraus form

$\rho' = \Phi(\rho) = \sum_i A_i \rho A_i^\dagger, \quad$ where the Kraus operators satisfy $\sum_i A_i^\dagger A_i = \mathbb{1}$, which implies that the trace is preserved.

Quantum stochastic maps (trace preserving, CP)

Superoperator $\Phi : \quad \mathcal{M}_N \to \mathcal{M}_N$

A *quantum operation* can be described by a matrix Φ of size N^2,

$$\rho' = \Phi\rho \qquad \text{or} \qquad \rho'_{m\mu} = \Phi_{\substack{m\mu \\ n\nu}} \rho_{n\nu} \ .$$

The superoperator Φ can be expressed in terms of the Kraus operators A_i,
$\Phi = \sum_i A_i \otimes \bar{A}_i$.

Dynamical Matrix D: Sudarshan et al. (1961)

obtained by *reshuffling* of a 4–index matrix Φ is Hermitian,

$$D_{\substack{mn \\ \mu\nu}} := \Phi_{\substack{m\mu \\ n\nu}} , \quad \text{so that} \quad D_\Phi = D_\Phi^\dagger =: \Phi^R \ .$$

Theorem of Choi (1975). A map Φ is **completely positive** (CP) if and only if the dynamical matrix D_Φ is **positive**, $D \geq 0$.

Spectral properties of a superoperator Φ

Quantum analogue of the Frobenious-Perron theorem

Let Φ represent a stochastic quantum map, i.e.
a') $\Phi^R \geq 0$; **(Choi theorem)**
b') $\mathrm{Tr}_A \Phi^R = \mathbb{1} \Leftrightarrow \sum_k \Phi_{\substack{kk\\ij}} = \delta_{ij}$. **(trace preserving condition)**

Then
i') the spectrum $\{z_i\}_{i=1}^{N^2}$ of Φ belongs to the unit disk,
ii') the leading eigenvalue equals unity, $z_1 = 1$,
iii') the corresponding eigenstate (with N^2 components) forms a matrix ω of size N, which is positive, $\omega \geq 0$, normalized, $\mathrm{Tr}\omega = 1$, and is invariant under the action of the map, $\Phi(\omega) = \omega$.

Classical case

In the case of a **diagonal dynamical matrix**, $D_{ij} = d_i \delta_{ij}$ reshaping its diagonal $\{d_i\}$ of length N^2 one obtains a matrix of size N, where $S_{ij} = D_{\substack{ii\\jj}}$, of size N which is **stochastic** and recovers the standard F–P theorem.

Decoherence for quantum states and quantum maps

Quantum states → classical states = diagonal matrices

Decoherence of a state: $\rho \to \tilde{\rho} = \mathrm{diag}(\rho)$

Quantum maps → classical maps = stochastic matrices

Decoherence of a map: The **Choi matrix** becomes diagonal, $D \to \tilde{D} = \mathrm{diag}(D)$ so that the map $\Phi = D^R \to \tilde{D}^R \to S$ where for any Kraus decomposition defining $\Phi(\rho) = \sum_i A_i \rho A_i^\dagger$ the corresponding **classical map** S is given by the **Hadamard product**,

$$S = \sum_i A_i \odot \bar{A}_i$$

If a **quantum map** Φ is trace preserving, $\sum_i A_i^\dagger A_i = \mathbb{1}$
then the **classical map** S is **stochastic**, $\sum_j S_{ij} = 1$.

If additionally a **quantum map** Φ is unital, $\sum_i A_i A_i^\dagger = \mathbb{1}$
then the **classical map** S is **bistochastic**, $\sum_j S_{ij} = \sum_i S_{ij} = 1$.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Interacting quantum dynamical systems

Generalized **quantum baker map** with measurements

a) Quantisation of **Balazs and Voros** applied for the asymmetric map

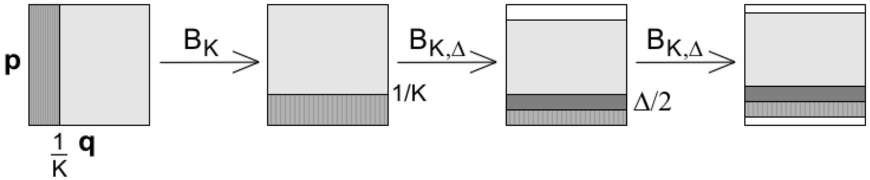
$$B = F_N^\dagger \begin{bmatrix} F_{N/K} & 0 \\ 0 & F_{N(K-1)/K} \end{bmatrix}, \quad \text{where} \quad N/K \in \mathbb{N}.$$

where F_N denotes the **Fourier matrix** of size N. Then $\rho' = B\rho_i B^\dagger$

b) M **measurement operators** projecting into orthogonal subspaces

Kraus form: $\rho_{i+1} = \sum_{i=1}^{M} P_i \rho' P_i$

c) vertical **shift** by $\Delta/2$ (**Łoziński, Pakoński, Życzkowski** 2004)

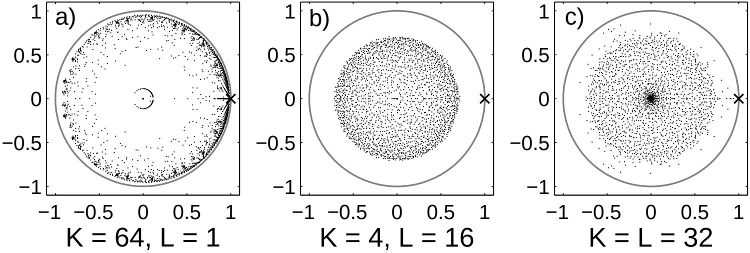


Standard classical model $K = 2$, **dynamical entropy** $H = \ln 2$;

Asymmetric model, $K > 2$, entropy decreases to zero as $K \to \infty$.

Classical limit: $N \to \infty$ with $K \le N$.

Exemplary spectra of superoperator for L–fold generalized baker map B^L & measurement with M **Kraus operators** for $N = 64$ and $M = 2$:



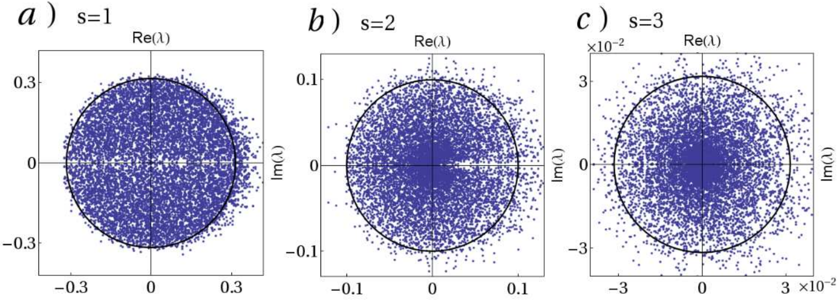
a) weak chaos ($K = 64$ and $L = 1$),

b) **strong chaos** ($K = 4$ and $L = 4$) – **'universal' behaviour**:
$\lambda_1 = 1$ and **uniform Girko disk of eigenvalues** of radius R,
(described by **real Ginibre** ensemble).

c) weak chaos ($K = 32$ and $L = 32$).

s–steps propagators ("s" of Fuss–Catalan)

Exemplary spectra of superoperator Φ^s for s–**steps** non-unitary evolution



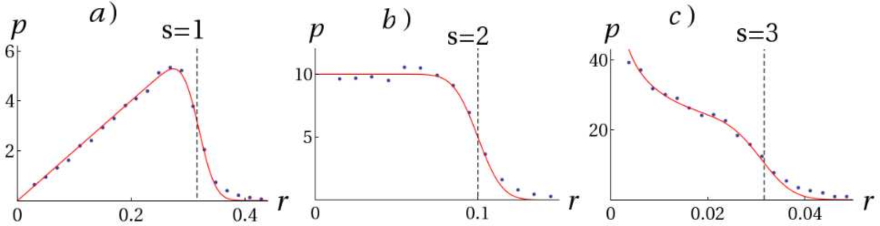
i) spectral properties of 1–**step** propagator Φ coincide with these of real random Ginibre matrices (uniform disk apart of the real axis)

ii) properties of s–**step** propagators Φ^s are similar as products of random matrices:

a) the radial density of complex eigenvalues $r = |z|$ of Φ^s

behaves asymptotically as the **algebraic law** for **products** of s random Ginibre matrices

of **Burda et al. 2010**: $P_s(r) \sim r^{-1+2/s}$



with an error-function Ansatz (red line) describing the **finite** N effects.

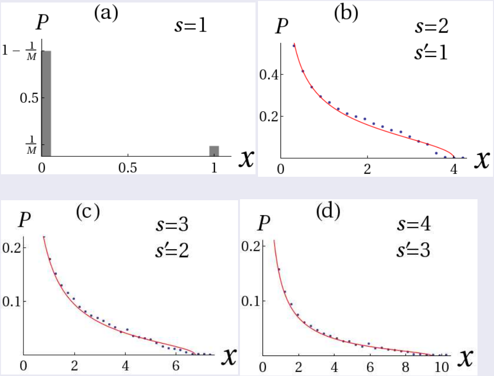
b) the squared singular values of Φ^s

can be described by **Fuss-Catalan distribution** of order $t = s - 1$.

Let $x = N^2\lambda$, where λ is an eigenvalue of $\Phi^s(\Phi^s)^\dagger$. Then

$s = 2$, $t = 1$ (Wishart)
$P_1(x) = \frac{\sqrt{1-x/4}}{\pi\sqrt{x}} \quad x \in [0, 4]$,
Marchenko–Pastur distrib.
(with moments given by the **Catalan** numbers);

$s \geq 3$, $t \geq 2$, the **Fuss–Catalan** distrib. $P_t(x)$ for $x \in [0, (t+1)^{t+1}/t^t]$
(with moments given by the **Fuss–Catalan** numbers)
expicitely derived in
Penson, K. Ż., 2011,
Młotkowski 2013



Concluding Remarks

- **Random Matrices**: a) offer a useful tool applicable in several branches of science including physics !
  b) display (asymptotically) **universal properties**, which depend on the symmetry with respect to orthogonal / unitary / symplectic transformations
- **Quantum Chaos**:
  a) in case of closed systems one studies unitary evolution operators and characterizes their spectral properties,
  b) for open, interacting systems one analyzes non–unitary time evolution described by quantum stochastic maps.
- We analyzed spectral properties of quantum stochastic maps and formulated a quantum analogue of the **Frobenius-Perron** theorem.
- **Non–unitary dynamics**: in case of **strong chaos** and **large interaction** with the environment the superoperators can be described by **real random Ginibre** matrices, while s–**step propagators** correspond to products (powers) of non-hermitian random matrices.