Wave chaos and regular modes in oscillation spectra of rapidly rotating stars

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Asteroseismology

⇒ Information obtained from the stars: spectra and luminosity

⇒ Luminosity is not constant: stars are pulsating

⇒ Detection precise enough to observe the presence of planets

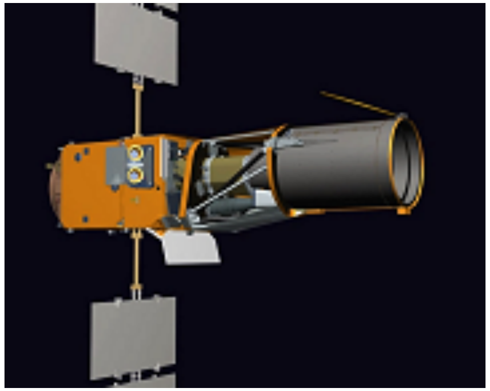
⇒ Fourier transform: oscillation spectrum

⇒ Star models: oscillation modes are inertial (Coriolis force), gravitational or acoustic

⇒ Focus of this talk: acoustic waves

⇒ New observations: space missions Corot, Kepler

⇒ unprecedented precision



Slowly rotating stars (e.g. the sun)

$\Rightarrow$Approximate spherical symmetry

$\rightarrow$ Asymptotic theory: modes are distributed according to the labelling (Tassoul 1980, Deubner and Gough 1984, Roxburgh and Vorontsov 2000) :

$$\int_{r_i}^{r_e} \left(\frac{\omega^2 - \omega_c^2}{c_s^2} - \frac{L^2}{r^2} \right)^{1/2} dr = (n - \frac{1}{2})\pi$$

$$L = \ell + \frac{1}{2}$$

$$L_z = m$$

$\Rightarrow$ Enables to associate spectra to specific properties of the star interior

$\Rightarrow$ No theory for rapidly rotating stars, not spherically symmetric

Rapidly rotating stars

- Stars with masses $M > 2M_\odot$ are expected to rotate rapidly (e.g. $v \simeq 150\mathrm{km.s}^{-1}$ at the surface).
- Equilibrium configuration is flattened by the centrifugal distortion.
- Drives circulation currents, turbulence, mass loss (not treated in this talk) ...

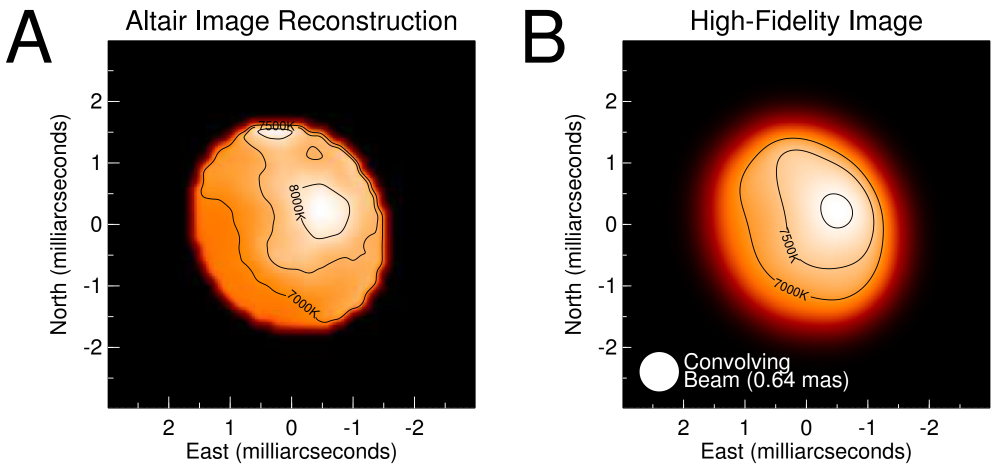


Figure: From Monnier *et al.*, 2007

Stellar model and wave equation

- Self-gravitating, uniformly rotating, monoatomic gas ($\gamma_{\rm ad} = 5/3$), assumed to verify a "polytropic" relation $P_0 = K\rho_0^\gamma$, with $\gamma = 4/3$.
- Small adiabatic perturbations. No perturbation of gravitational potential, no Coriolis force (frequency $\gg 2\Omega$).
- For high-frequencies, frequency $\gg$ buoyancy frequency, thus we discard terms corresponding to gravity waves.
- Cylindrical symmetry $\Rightarrow \Psi = \Psi_m \exp(im\phi)$

$\Rightarrow$ Helmholtz equation in the meridian plane in $\Phi_m = \sqrt{r \sin\theta}.\Psi_m$

$$-c_s^2 \Delta\Phi_m + \left(\omega_c^2 + \frac{c_s^2(m^2 - 1/4)}{d^2}\right)\Phi_m = \omega^2 \Phi_m \ .$$

c_s is the (inhomogeneous) sound velocity, d the distance to the rotation axis and ω_c the cut-off frequency of the stellar model. (Lignières & Georgeot, 2009)

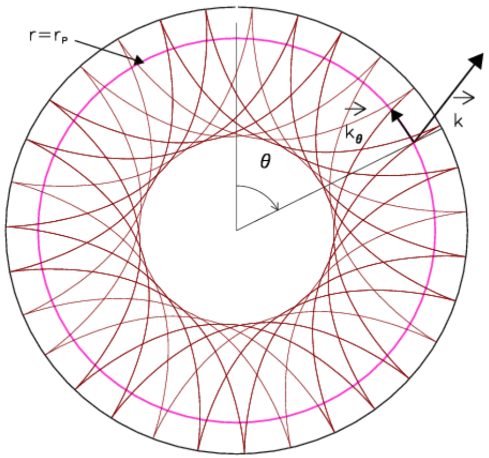
Model of acoustic waves

Numerical solution for the modes:

- self-gravitating uniformly rotating monatomic gas verifying a polytropic relation (give a reasonably good approximation of the relation between the pressure and the density in the star)
- Solved numerically, using an iterative scheme (Lignières, Rieutord and Reese 2006).
- For small rotations: system has spherical symmetry, modes close to spherical harmonics
- For larger rotations, system has only cylindrical symmetry; the star is deformed, equatorial radius larger than polar radius.
- We will focus on the properties of observable modes.

Ray limit of acoustic waves

$$\Omega = 0$$

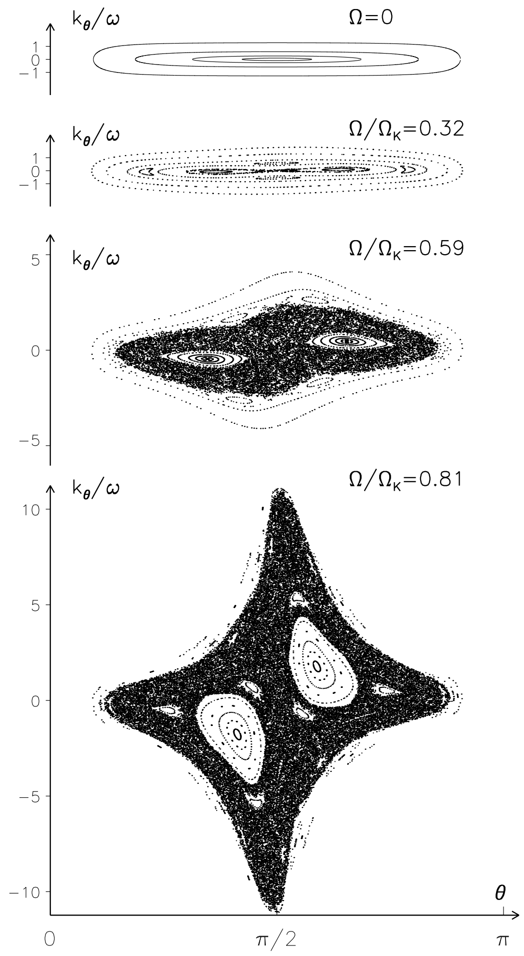


⇒ Acoustic ray: trajectory tangent to the wave vector **k** at the point **x** -> Hamiltonian classical equations of motion (Lighthill 78, Gough 93)

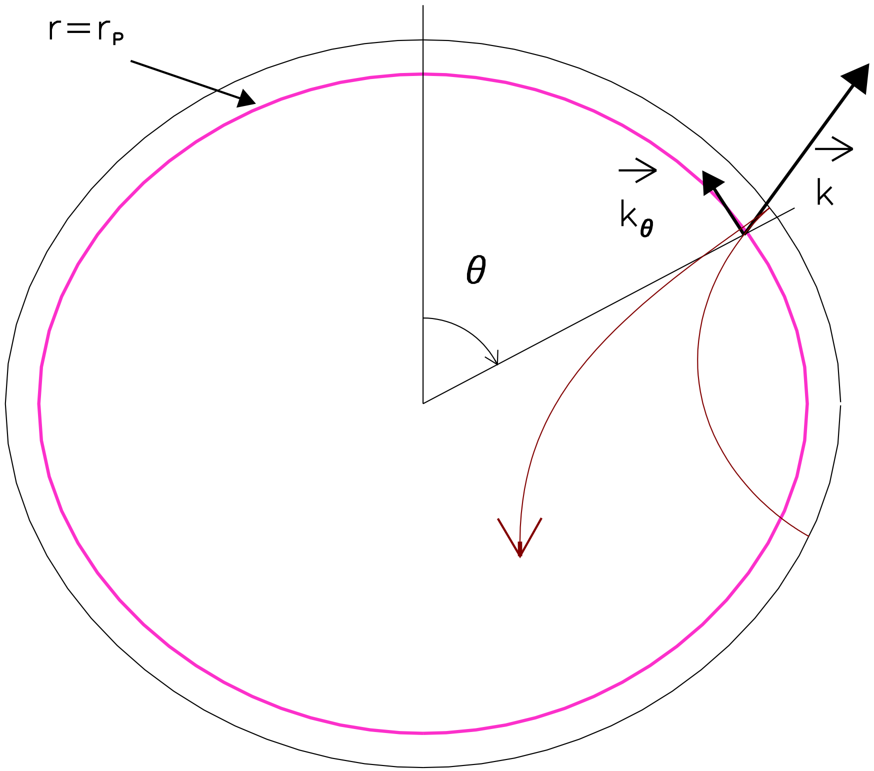
⇒ Limit of acoustic wave dynamics at high frequency, in the same way as classical mechanics is the limit of quantum mechanics for $\hbar \rightarrow 0$ or geometrical optics the limit of optics

Acoustic ray dynamics

- Three degrees of freedom $\longrightarrow$ six-dimensional phase space
- L_z conserved $\longrightarrow$ four-dimensional phase space
- Poincaré Surface of Section close to the boundary of the star $\Rightarrow$ enables to visualize phase space (two dimensions only)
- Result: transition from integrability ($\Omega = 0$) to increasing degree of chaos for rapid rotation (KAM-type transition)

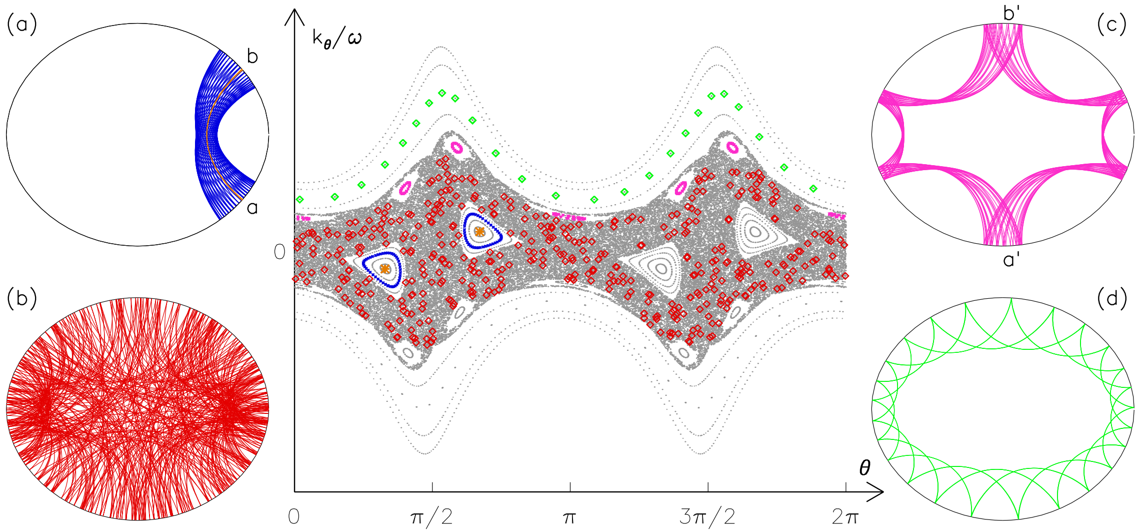


Poincaré surface of section



Phase space at high rotation

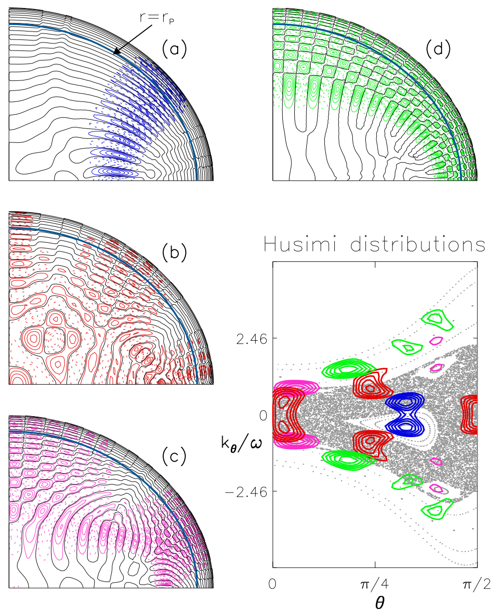
- Phase space displays integrable and chaotic zones (mixed systems).
- Integrable zones: two-period islands, (blue), six-period islands (pink), whispering gallery rays (green). Trajectories are stable and organized in low-dimensional tori
- chaotic zones (red): trajectories are unstable and ergodic.



Comparison with acoustic modes

⇒ High rotation: mixed systems: Percival, Berry-Robnik: modes should asymptotically be associated with different phase space regions
⇒ Phase space picture of the modes: Husimi distribution
$H(s_0, k_0) = |\int \Phi(s) \exp(-(s - s_0)^2/2\Delta) \exp(ik_0 s) ds|^2$
⇒ Enables to plot a representation of the modes in phase space, in order to associate them with specific regions in phase space
⇒ Result: Numerically computed modes fulfill the conjecture

Four modes and their phase space representation :
(a) a 2-period island mode (blue)
(b) a chaotic mode (red)
(c) a 6-period island mode (magenta)
(d) a whispering gallery mode (green).



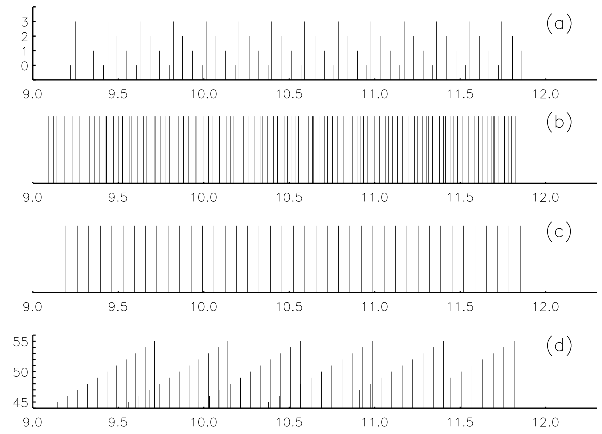
Consequences for spectra

⇒ Slow rotation: integrable system → EBK theory gives closed formulas for the frequencies (cf supra)

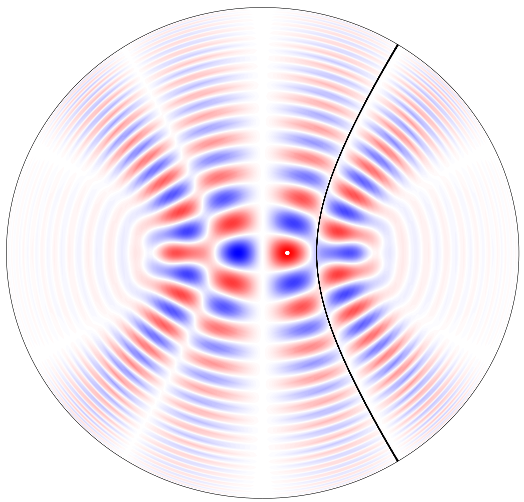
⇒ Moderate or high rotation: not integrable any more; should use the tools ofquantum chaos

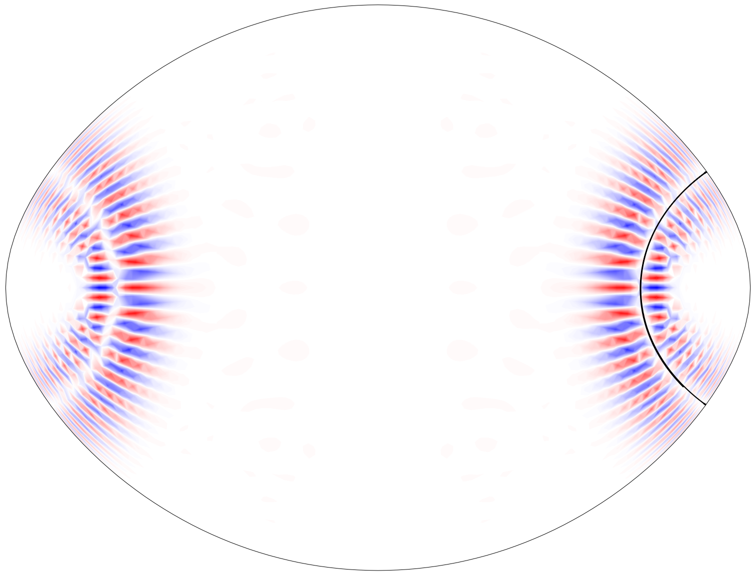
⇒ Prediction: Spectrum should be divided into well-defined subspectra

Frequency sub-spectra of four classes of modes :
(a) 2-period island modes
(b) chaotic modes
(c) 6-period island modes
(d) some whispering gallery modes
For sub-spectra (a) and (d), height of the vertical bar specifies one of the two quantum numbers.



Regular spectrum: 2-period island modes





Quantization of 2-period island modes: method of Babich (1968)

- Different methods for quantizing a stable periodic orbit were developed by Babich, 1968; Miller, 1975; Voros, 1975.
- From integration of the short-wavelength limit equations: the isolated stable periodic orbit in the main stable island is known.
- The method of Babich searches for a solution of an approximate wave equation in the close vicinity of this trajectory.
- It yields the mode frequencies and amplitude distributions.
- It was applied to light in dielectric cavities and electronic resonators in a magnetic field.

Outline of the method

- The wave equation is written in local coordinates (s, ξ) centered on the trajectory.
- WKB ansatz: $\Phi_m(s,\xi) = \exp\left(i\omega \int^s \frac{ds'}{\tilde{c}_s}\right) U_m(s,\xi)$.
- Assumption : $s = O(1)$, $\xi = O(1/\sqrt{\omega})$.
- ω-expansion.
- At the order ω, with $\nu = \sqrt{\omega}\xi$ and $V_m = U_m/\sqrt{\tilde{c}_s}$, we obtain:

$$\frac{\partial^2 V_m}{\partial \nu^2} - K(s)\nu^2 V_m + \frac{2i}{\tilde{c}_s}\frac{\partial V_m}{\partial s} = 0 \ ,$$

  with $K(s) = \frac{1}{\tilde{c}_s(s)^3} \frac{\partial^2 \tilde{c}_s}{\partial \xi^2}\Big|_{\xi=0}$.
- The terms in ν (transverse deviation) are similar to a quantum harmonic oscillator.

Gaussian beam solution

- The ground state solution is $V_m^0 = A(s)\exp\left(i\frac{\Gamma(s)}{2}\nu^2\right)$, $\Gamma \in \mathbb{C}$.
- If we define $z(s)$ and $p(s)$ as $\Gamma = p/z$ we obtain a Hamiltonian system for deviations

$$\frac{d}{d\tau}\begin{pmatrix} z \\ p \end{pmatrix} = \begin{pmatrix} 0 & \tilde{c}_s^2 \\ -\tilde{c}_s^2 K & 0 \end{pmatrix}\begin{pmatrix} z \\ p \end{pmatrix} .$$

- This system being time periodic, we can write

$$\begin{bmatrix} z(\tau + T_\gamma) \\ p(\tau + T_\gamma) \end{bmatrix} = M \begin{bmatrix} z(\tau) \\ p(\tau) \end{bmatrix} ,$$

  where M is called the monodromy matrix.
- If the trajectory is stable then $\Lambda^\pm = \exp(\pm i\alpha)$ where α is the stability angle.
- Higher-order solutions are Hermite-Gauss polynomials

$$V_m^\ell(s,\nu) = \left(\frac{\bar{z}}{z}\right)^{\ell/2} H_\ell(\sqrt{\mathrm{Im}(\Gamma)}\nu)\frac{\exp\left(i\frac{\Gamma}{2}\nu^2\right)}{\sqrt{z}} .$$

Result: frequency spacings

- Quantization condition on the phase yields

$$\omega_{n,\ell,m} \oint \frac{ds}{\tilde{c}_s} - \frac{\alpha + 2\pi N_r}{2} - (\alpha + 2\pi N_r)\ell = 2\pi n + \pi \ .$$

Formula for regular sub-spectrum in rapidly rotating stars

$$\omega_{n,l,m} = \delta_n(m) n + \delta_\ell(m)\ell + \beta(m) \ ,$$

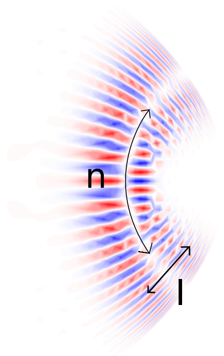
with

$$\delta_n(m) = \frac{2\pi}{\oint \frac{ds}{\tilde{c}_s}} \ , \quad \delta_\ell(m) = \frac{2\pi N_r + \alpha}{\oint \frac{ds}{\tilde{c}_s}} \ \text{ and } \ \beta(m) = \frac{\delta_n + \delta_\ell}{2} \ .$$

- $\oint \frac{ds}{\tilde{c}_s}$ is the propagation time for a period of the stable orbit.
- N_r and α are related to the stability of the periodic orbit.

Numerical results

- High-frequency modes computed with accurate code based on spectral methods (Reese et al. 2008).
- Polytropic stellar model with $\gamma = 4/3$.
- Individual modes are followed from $\Omega/\Omega_K = 0$ to $\Omega/\Omega_K = 0.896$ (where $\Omega_K = \sqrt{GM/R_{\rm eq}^3}$).
- $m = -1, 0, 1$ (angular momentum along the rotation axis).
- Mode labelling: n , ℓ are number of nodes in the meridian plane.



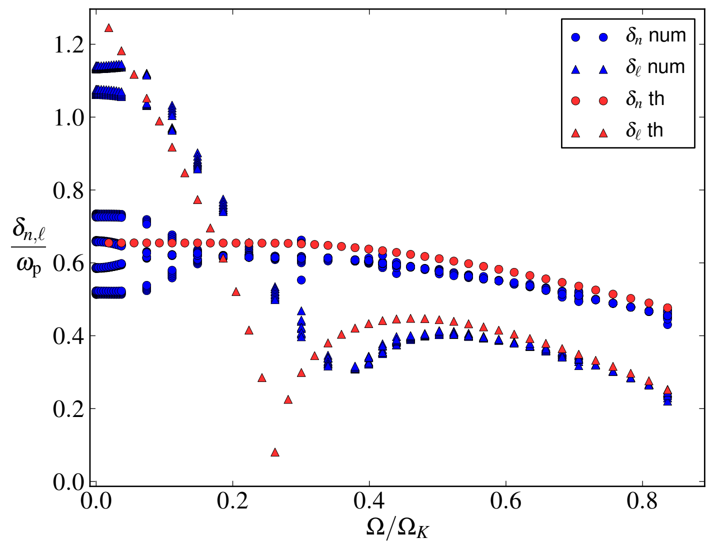
Frequency spacings are computed as follows:

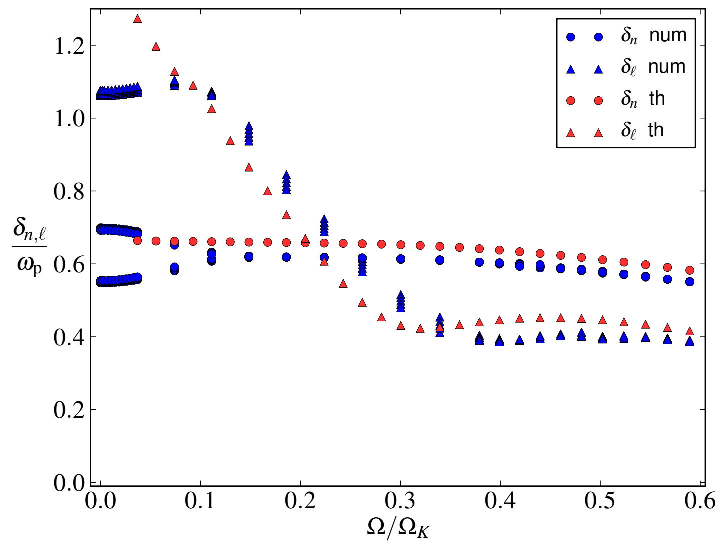
$$\delta_n = \omega_{n+1,\ell,m} - \omega_{n,\ell,m}$$

$$\delta_\ell = \omega_{n,\ell+1,m} - \omega_{n,\ell,m} \ .$$

Theoretical formula vs numerics

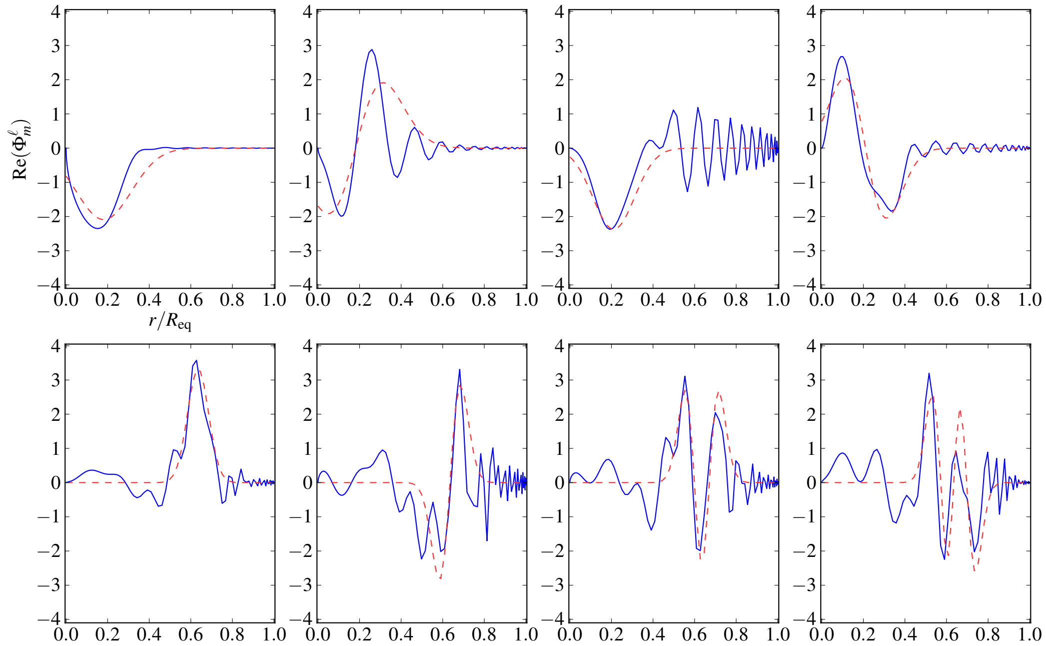
- The semi-analytical regularities and the full computations of high-frequency p-modes are in good agreement (except at bifurcation for $m = 0$).





Amplitude distribution of the modes

- We can also check the agreement between mode amplitude distributions.
- There is a good agreement, except when edge effects or couplings are present.



Astrophysical applications

- The formula for frequencies relates potential seismic observables to internal properties of the star (good for asteroseismology).
  - δ_n depends only on the acoustic time along the periodic orbit.
  - δ_ℓ depends on the acoustic time and the second derivatives of the celerity of sound transverse to the trajectory (i.e. stability).
- For a given stellar model, the method allows a rapid computation of asymptotic regularities to help one search for patterns in numerical or observed spectra.
- Computation of mode visibilities.
- Predicts avoided crossings at rational values of α/π with rotation as the control parameter.

Regular spectrum: 6-period island modes

In the frequency interval considered, these modes have a similar structure in the direction transverse to the central orbit and should therefore be associated with the same ℓ value.

Theoretical formula is thus:

$$\omega_{n'} = n'\delta'_n + \alpha' \tag{3}$$

with

$$\delta'_n = \frac{\pi}{\int_{a'}^{b'} d\sigma / c_s} \tag{4}$$

this theoretical value of δ'_n differs by only a few percents from the empirical determination of $\delta'_n = 0.186\omega_0$.

Chaotic modes

- These modes should be visible, and give informations on the stellar core which is important to infer the age of a star.

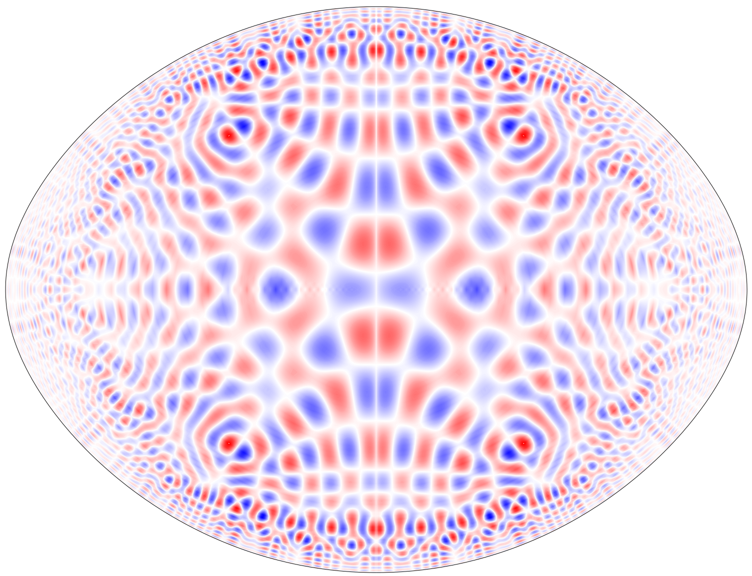


Figure: Chaotic mode for $m = 0$ and $\Omega/\Omega_K \simeq 0.783$.

Irregular spectrum

⇒ No simple asymptotic formula for chaotic modes

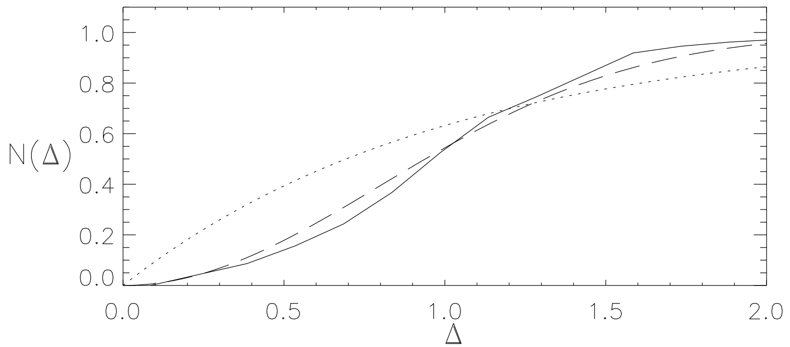
⇒ Conjecture (Bohigas-Giannoni-Schmit): level spacing statistics of chaotic modes should follow Random Matrix Theory

⇒ Verified by the numerical acoustic stellar modes

Integrated spacing distribution $N(\Delta)$ of chaotic modes (full line).

Dashed line: Random Matrix Theory

Dotted line: Poisson distribution typical of integrable systems.



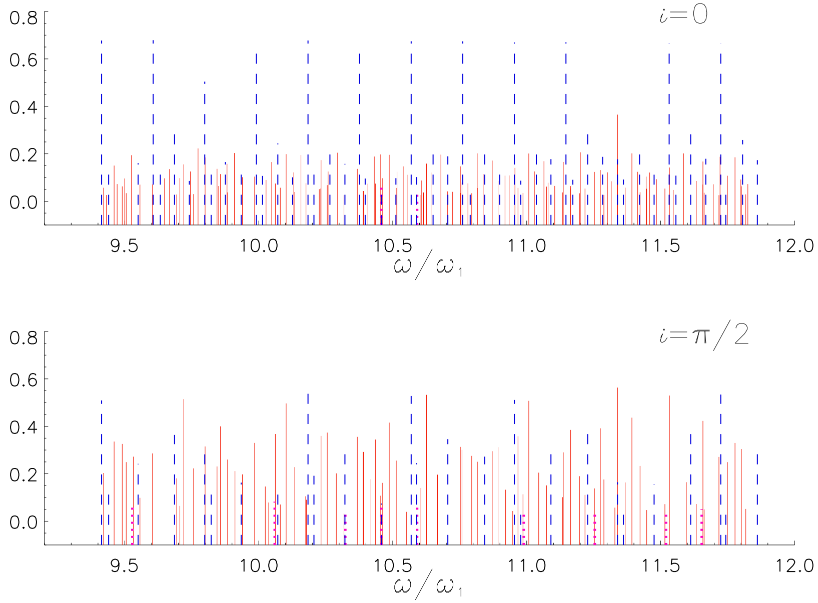
Weyl formula

- Weyl, early XXth century: the number of modes in a range of frequency is proportional to the phase space volume available
- For mixed system: the number of modes in each subspectrum should be proportional to the phase space volume of the associated region
- Monte-Carlo numerical calculation of four-dimensional trajectories: predicts the size of each subspectrum for stellar pulsations
- Results: for $\Omega/\Omega_K = 0.59$, in $[9.42\omega_1, 11.85\omega_1]$, predicts 34 ± 2 modes in the 2-period island chain, 270 ± 8 modes outside the whispering gallery region
- 50 island modes and 276 modes outside the whispering gallery region obtained using the Husimi phase space representation
- Difference can be attributed to next order in the formula, and modes at the frontier between zones
- important to interpret observed spectra

Visibility of the modes

- What is really detected from the earth?
- Only average luminosity over one hemisphere is detected.
- The angle of detection with the rotation axis is not known in general.

Frequency spectra with amplitude given by the visibility for a star seen pole-on $i = 0$ and equator-on $i = \pi/2$: 2-period island modes (blue), chaotic modes (red), 6-period island modes (magenta)



Conclusion

- Dynamics of acoustic rays shows a transition from integrable to mixed system when rotation increases
- For sufficiently large rotation, the spectrum should be divided into well-defined regular or irregular subsets
- This picture holds for numerical modes computed from a realistic star model, in the frequency range where modes are observable.
- The regular and irregular modes have both high visibility
- First results of COROT show hundreds and sometime thousands of frequencies; some regularity seems to be detected in δ scuti stars (very recent) $\rightarrow$ more work to connect to observed spectra (collaboration with E. Michel, Meudon)
- Identification of the spectra should lead to better understanding of the star interior.
- Extensions: more refined numerical models, amplitude of the modes, inertial modes, etc...