Escapers and non-escapers in star clusters

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Outline

Celestial mechanics and stellar dynamics

- Similarities
  - The classical N-body equations
  - M. Hénon
- Differences of emphasis

	Celestial Mechanics	Stellar dynamics
Style	Mathematical	Physical (theoretical/computational)
Focus	***x***(t), ***v***(t) Large mass ratio Resonance	Distribution functions Comparable masses Randomness
Additional effects	Internal tides Radiation pressure Outgassing	External tides Mass loss Stellar evolution
Applications	Planetary systems Comets Asteroids	Galaxies Star clusters

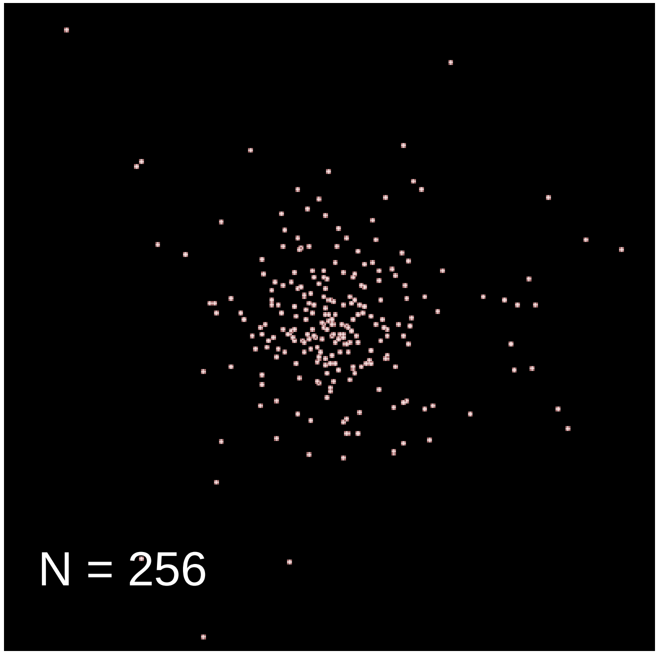
Globular star clusters

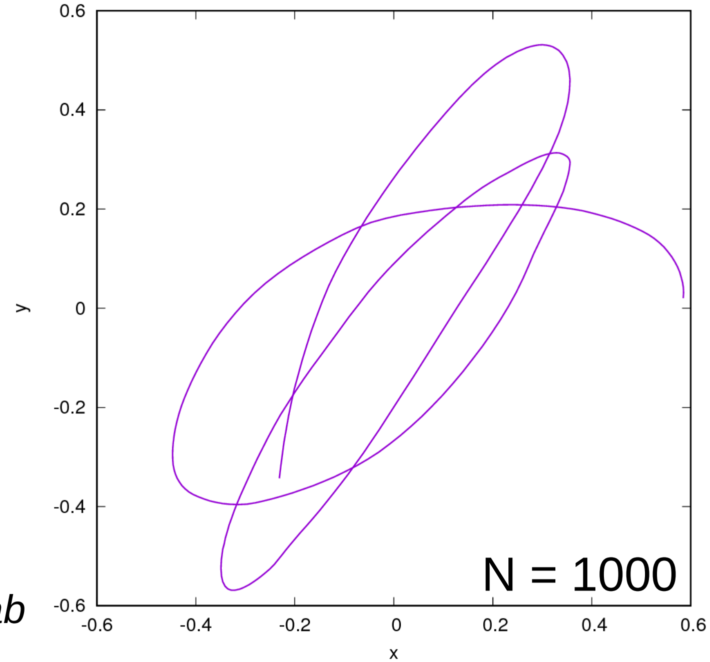
- N ~ 10^5 – 10^6
- Age ~ 10^{10} yr
- Galactic orbital period ~ 10^8 yr

Messier 4 (*ESO*)

Dynamics of globular star clusters

- Potential approximately spherical
- Stellar orbits resemble rosettes
- constant energy E and angular momentum J
- (radial) period (~ "crossing time") ~ 10^6 yr
- E, J evolve by random walk on time scale of the "relaxation time" ~ 10^9 yr
- relaxation time $\simeq \frac{0.1\,N}{\ln N}$ x crossing time
- external potential due to the Galaxy

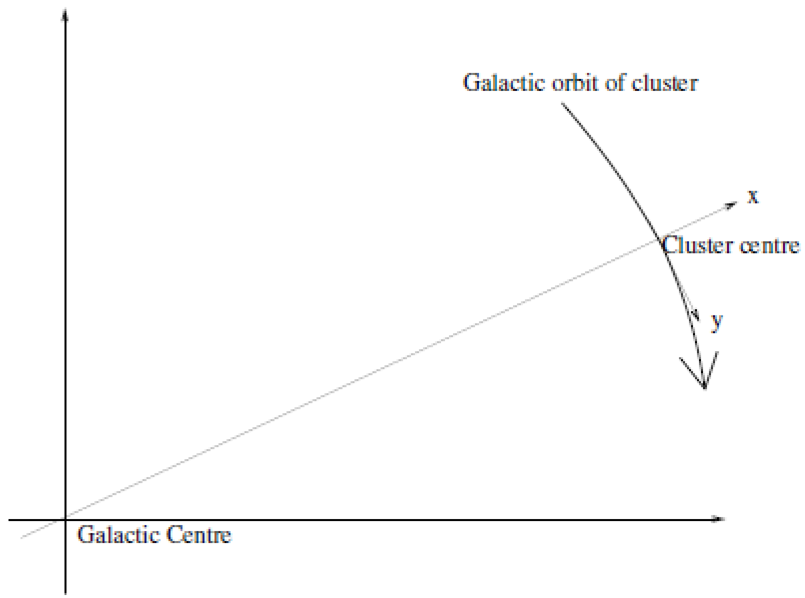




Simulations with starlab

Equations of motion (circular Galactic orbit)

Use a rotating, accelerating frame of reference with origin at the centre of the cluster, and x-axis pointing to or away from the Galactic Centre.



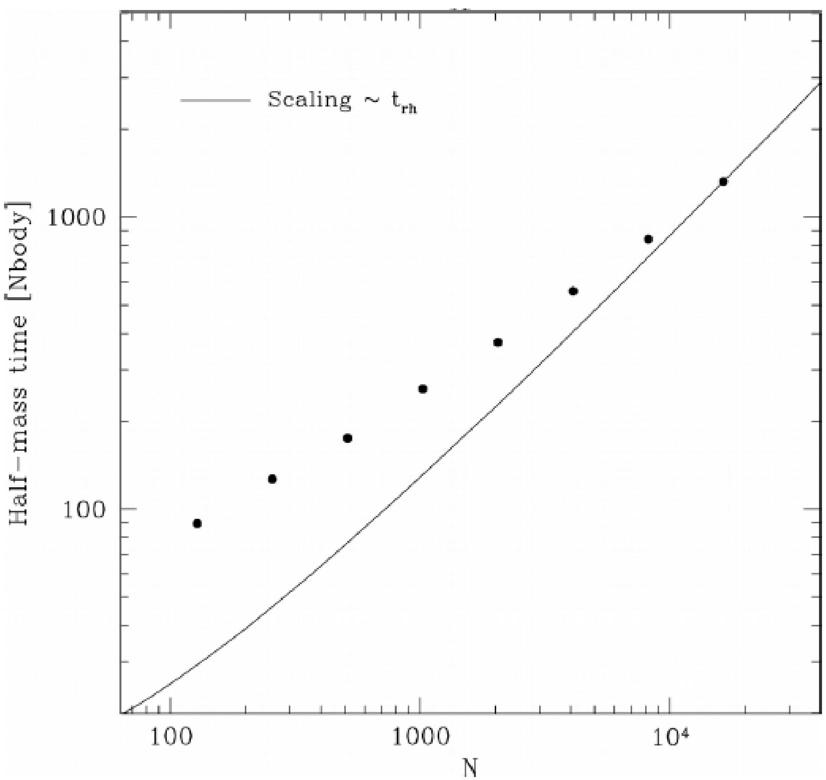
$$\begin{aligned}\ddot{x} - 2\omega\dot{y} + 2(\omega\omega' R)x &= -U_x \\ \ddot{y} + 2\omega\dot{x} &= -U_y \\ \ddot{z} \qquad\qquad + \omega^2 z &= -U_z\end{aligned}$$

where
- ω(R) is angular velocity of the cluster at Galactocentric distance R
- U is potential due to cluster stars

If the Galaxy and cluster are represented by point masses, in appropriate units these are the equations of Hill's problem

Escape from a cluster in the tidal field

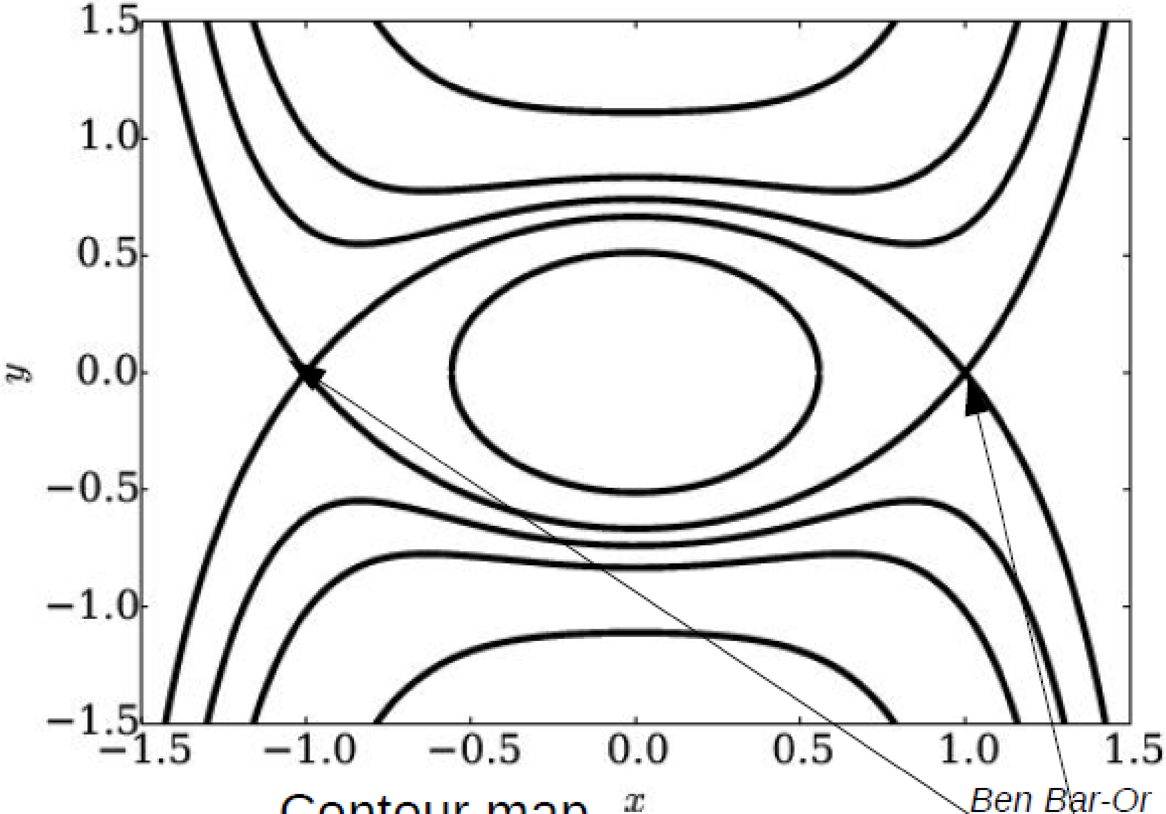
- Relaxation (cumulative effect of gentle two-body encounters) changes energy of stars
- Therefore time scale to achieve escape energy is the relaxation time $t_r \simeq \frac{0.1\,N}{\ln N}$ x crossing time (t_{cr})



Baumgardt 2001

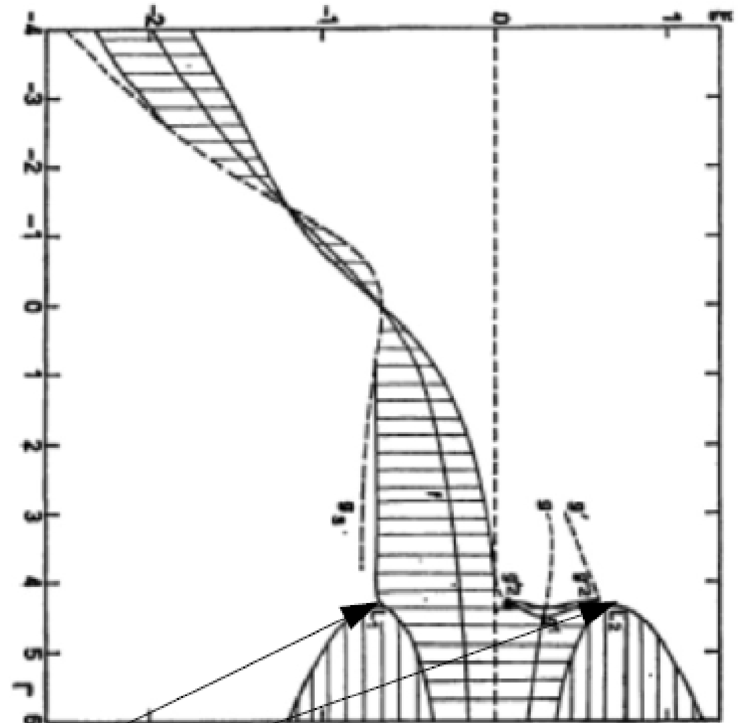
- But *N*-body simulations show that the time scale for half the stars to escape is $\simeq t_r^{3/4}\ t_{cr}^{1/4}$
- Baumgardt attributed this result to a population of "potential escapers", i.e. stars which can remain inside the cluster, above the escape energy, without escaping.

The possibility of potential escapers



Contour map

Ben Bar-Or



Cross section

Hénon 1970

Lagrange points

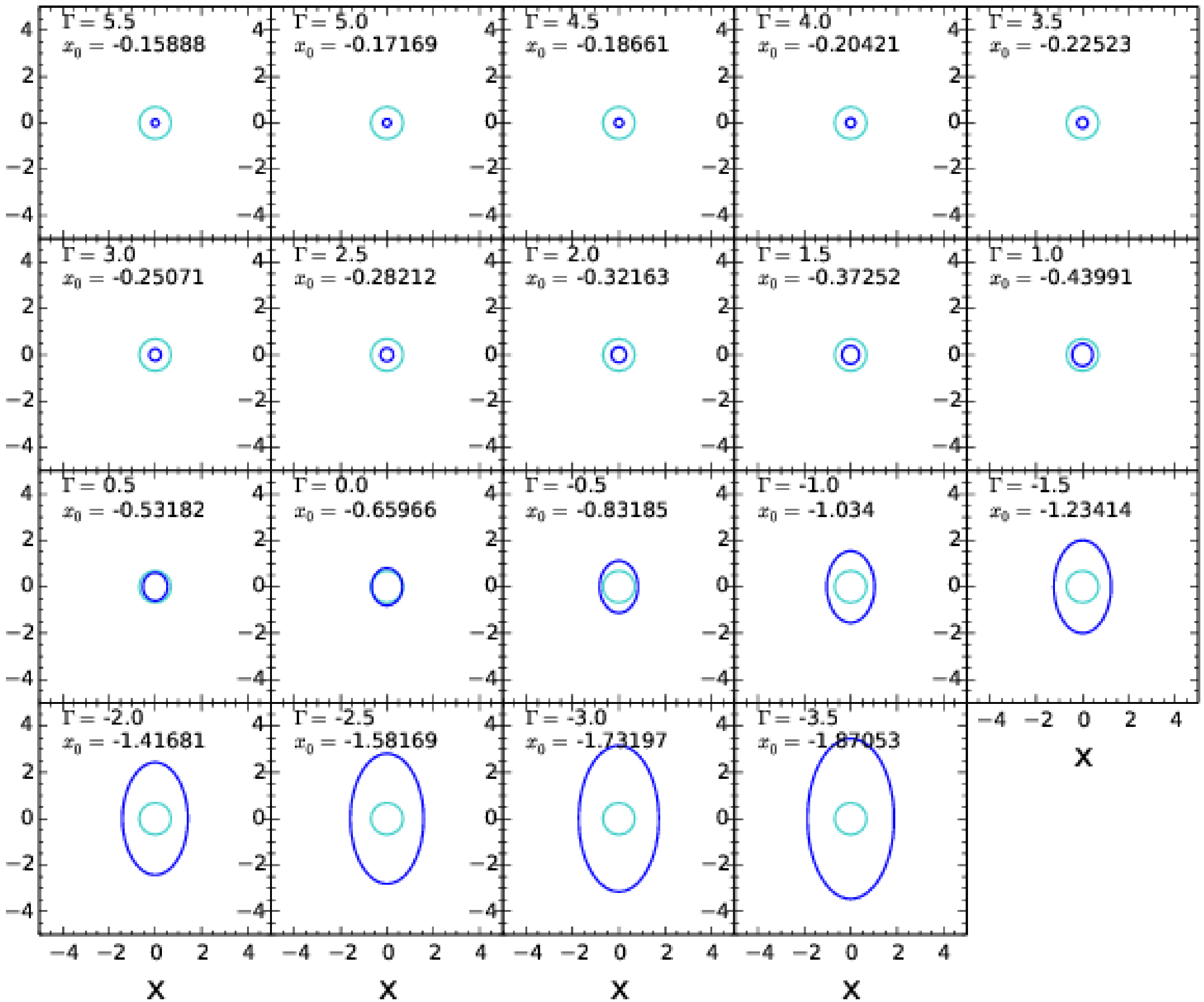
Contours of the effective potential in the *x,y* plane in Hill's problem

Γ is -2 x energy in the rotating frame

Family *f* of stable periodic orbits of the planar Hill's problem, and their associated quasi-periodic orbits

Axes: abscissa *x*
ordinate Γ (Jacobi integral)

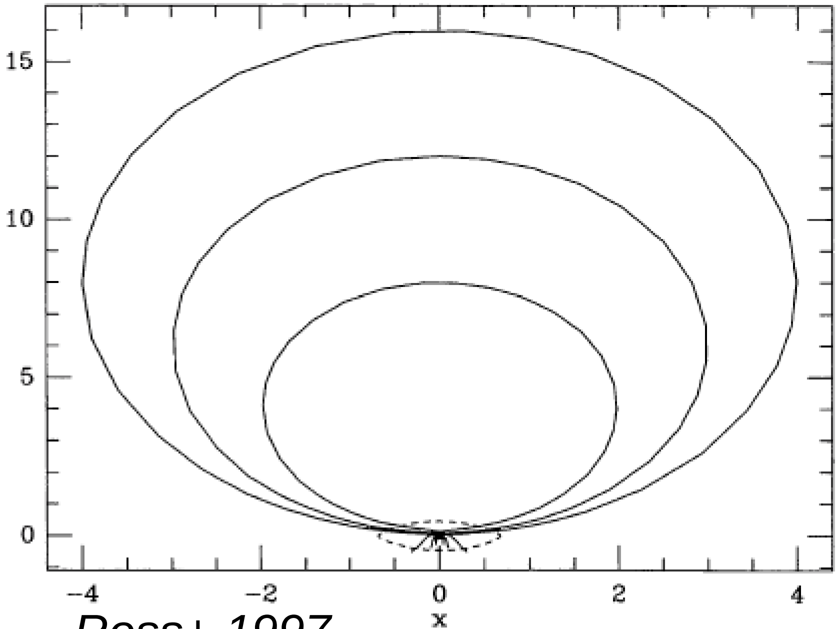
Family *f* of periodic orbits



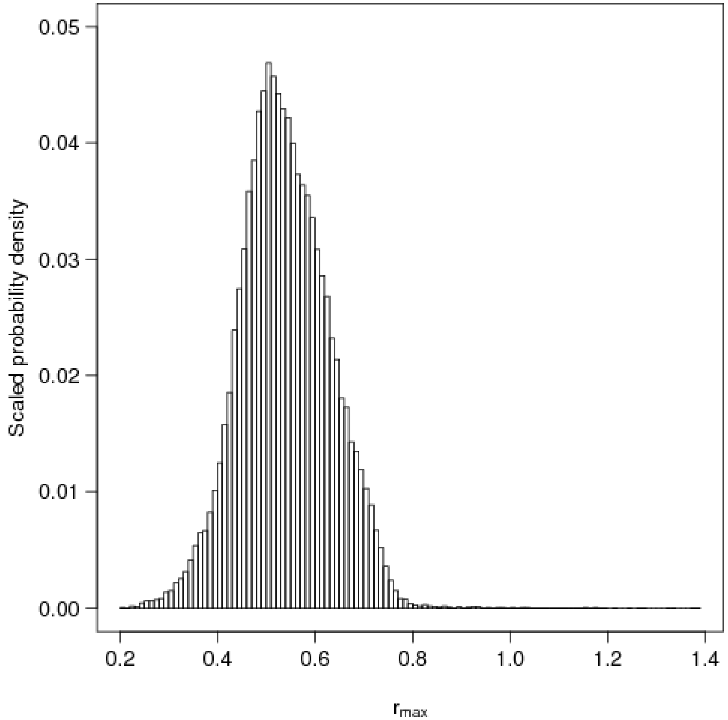
- Lagrange points lie on the pale blue circle ("Jacobi radius")
- Tableau runs from low Γ to high Γ (high energy to low)
- Jacobi energy is $3^{4/3} = 4.32...$
- At high Γ orbits (low energy) are small retrograde Keplerian motions (perturbed by external potential and inertial acceleration)
- These move outside the Jacobi radius at about Γ = 0
- Below Γ = 0 (high energy) the orbits are epicycles in the field of the Galaxy, mildly perturbed (and stabilised) by the cluster potential

Kate Daniel

What is an escaper?

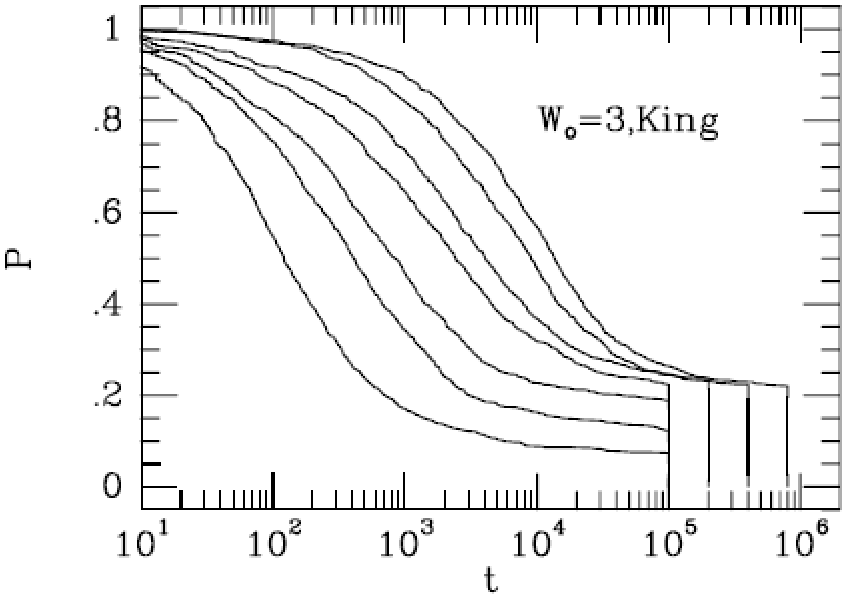


Ross+ 1997



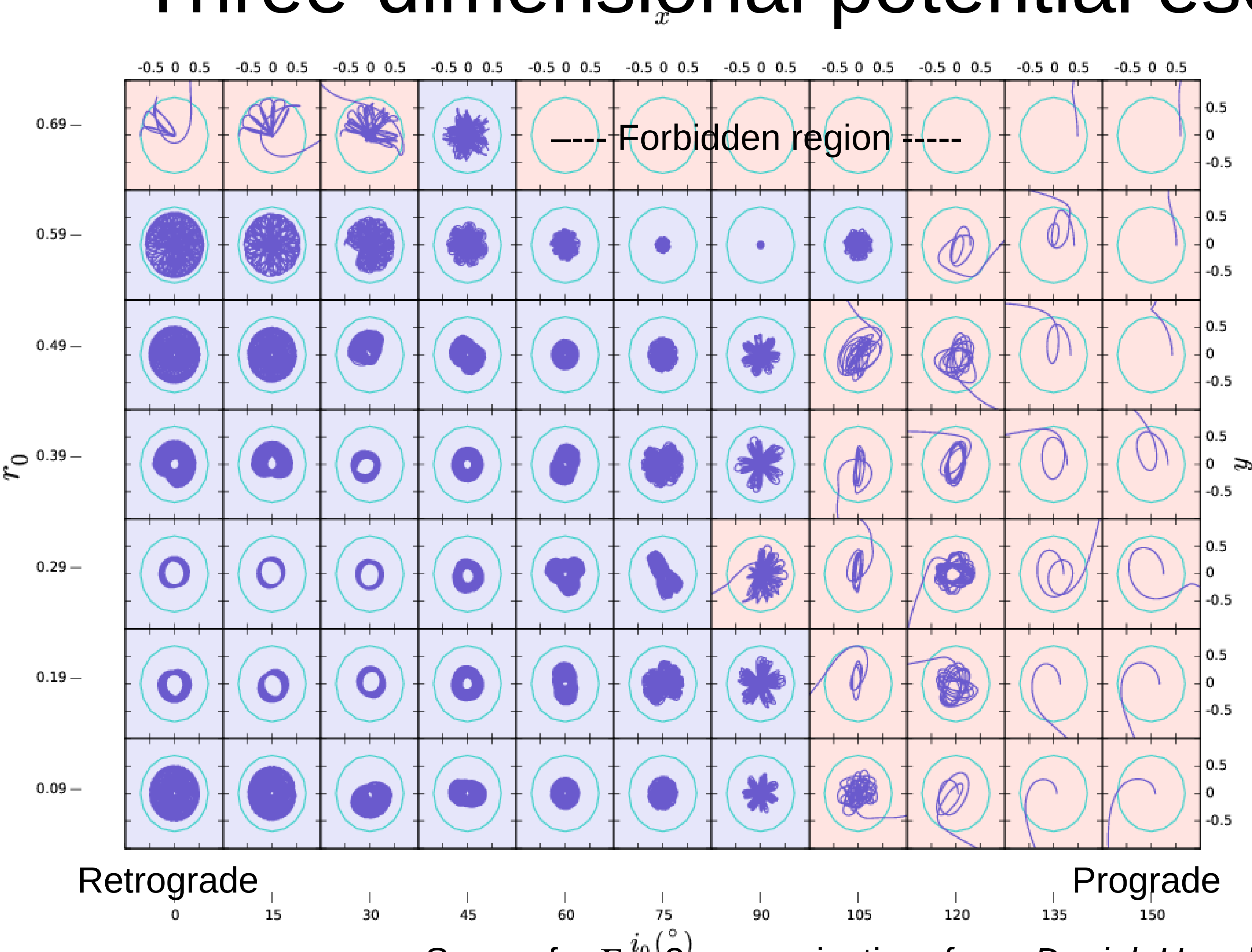
Daniel+ 2016

Fukushige & H 2000



- Stable non-escapers can lie at arbitrary distances (e.g. family *f*)
- “Escapers” can recede to arbitrary distance and return
- The time scale on which escapers leave the Jacobi radius depends strongly on energy
- Between the escape energy and zero energy the maximum radius of non-escapers exceeds the Jacobi radius only slightly

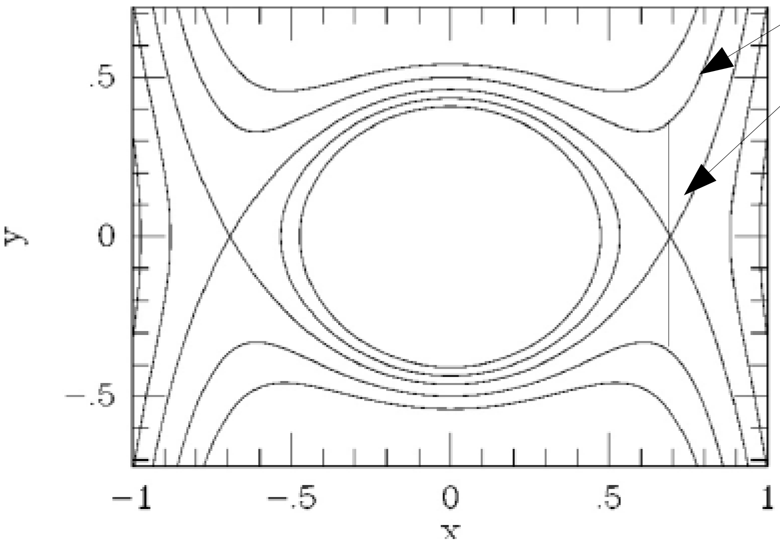
Three-dimensional potential escapers

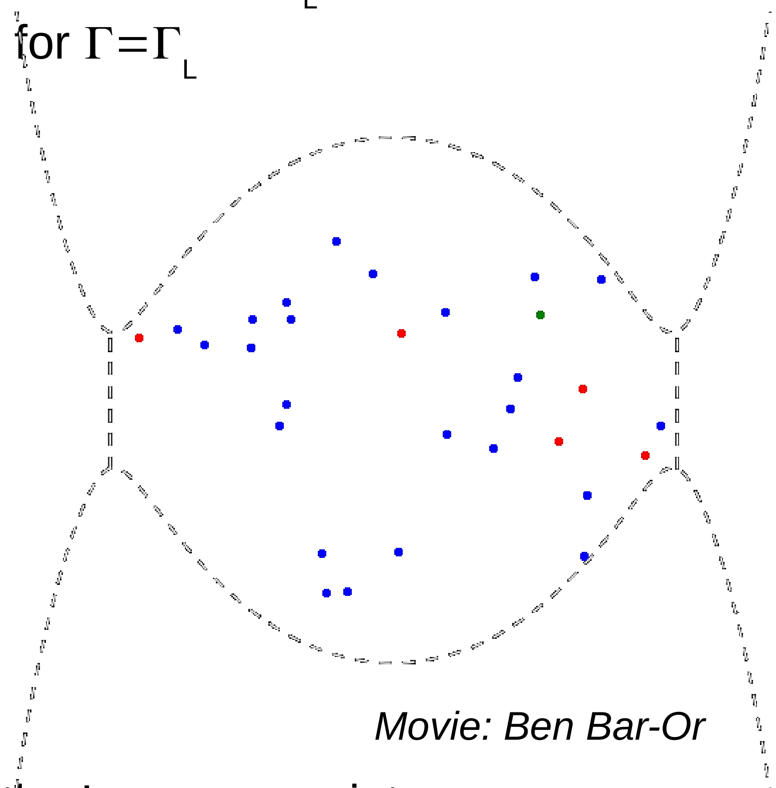


Survey for $\Gamma = 3$, *x-y* projection, *from Daniel, H and Varri, submitted*

- Note Lidov-Kozai behaviour for high-inclination motions
- Suggests approximate invariants with which to describe domain of potential escapers

Rate of escape of phase volume





Movie: Ben Bar-Or

- Aim to compute flux of phase-space volume across $x = r_J$ on Γ-hypersurface
- In three dimensions this is

$$F=\int \dot{x}\,\delta\left(\Gamma+v^2-\frac{2}{r}-3x^2+z^2\right)dy\,dz\,d\dot{x}\,d\dot{y}\,d\dot{z}$$

- This can be evaluated for $\Gamma \simeq \Gamma_J$ after linearisation near the Lagrange point
- Hence $F \propto (\Gamma_J - \Gamma)^2$ for $\Gamma < \Gamma_J$ (with explicit constant)
- Similarly evaluate the phase space volume V inside r_J per unit Γ (numerical integration required)
- Hence time scale on which phase volume escapes is $V/F \sim t_{cr} (\Gamma_J - \Gamma)^{-2}$ (Fukushige & H 2000)

Derivation of the escape rate

- Evolution of the distribution function of potential escapers evolves by two processes
  - Relaxation, which is diffusive, on time scale t_r
  - Escape, on time scale t_{cr} $(\Gamma_J - \Gamma)^{-2}$
- Describe by a toy model (Baumgardt 2001)

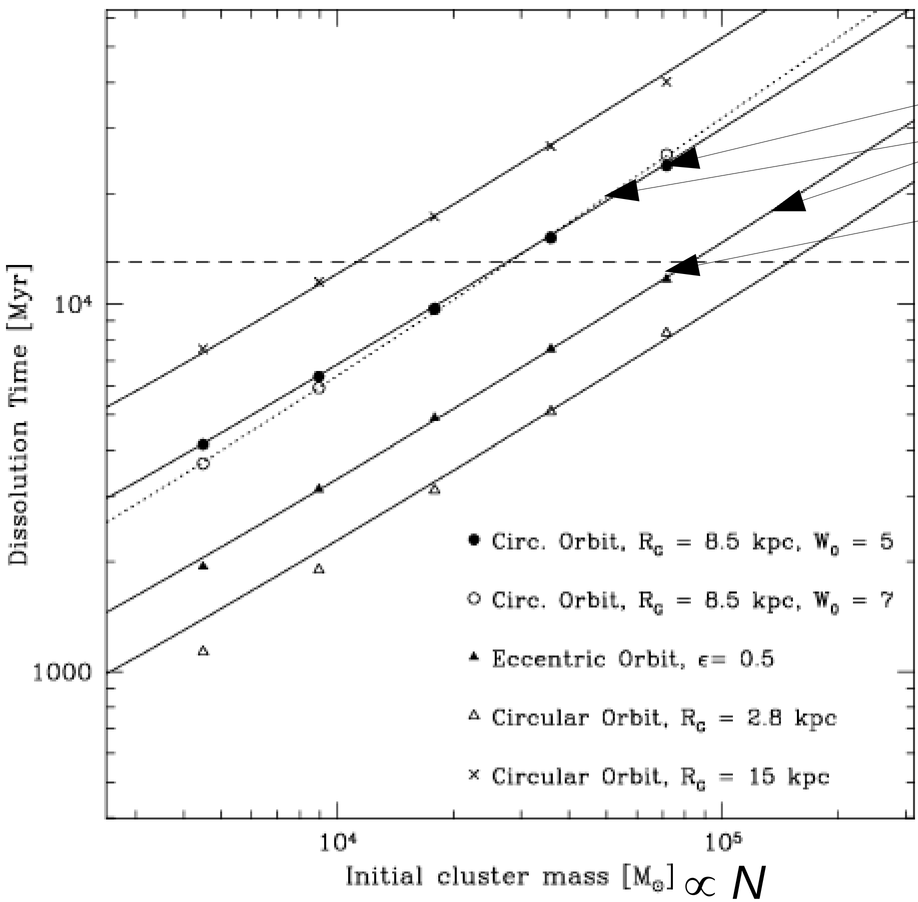
$$\frac{\partial f}{\partial t} = \frac{1}{t_r}\frac{\partial^2 f}{\partial \Gamma^2} - \frac{(\Gamma_J - \Gamma)^2}{t_{cr}} f$$

(derivable from approximating the Fokker-Planck equation of collisional stellar dynamics)

- Obvous scaling $(\Gamma_J - \Gamma)^4 \sim t_{cr}/t_r$. Hence
  - Width of distribution of Γ in the potential escapers scales as $(t_{cr}/t_r)^{1/4}$
  - Number of potential escapers scales as $N(t_{cr}/t_r)^{1/4}$
  - Escape time scale in this range of $\Gamma_J - \Gamma$ scales as $t_{cr}(t_{cr}/t_r)^{-1/2}$, $= (t_{cr}t_r)^{1/2}$
  - Time for cluster to lose half its stars ~ $(t_{cr}t_r)^{1/2}(t_{cr}/t_r)^{-1/4}$, $= t_r(t_{cr}/t_r)^{1/4}$
  - Same scaling as found by Baumgardt (2001) with N-body simulations

- *All these results depend on the assumption that the orbit of the cluster about the Galaxy is circular*

The case of an elliptic Galactic orbit



$\propto N$

Baumgardt & Makino 2003

Circular orbit

$\propto t_r^{3/4}$

Elliptic orbit, e = 0.5

The scaling of the lifetime with *N* is the same for an eccentric orbit as for a circular orbit.

But there is no known frame in which the equations of motion are autonomous (no explicit *t*-dependence), and no known integral of motion analogous to the Jacobi integral. Hence no escape energy.

What about Lagrange points? In the circular case these are

- equilibria
- critical points of the effective potential
- orbits

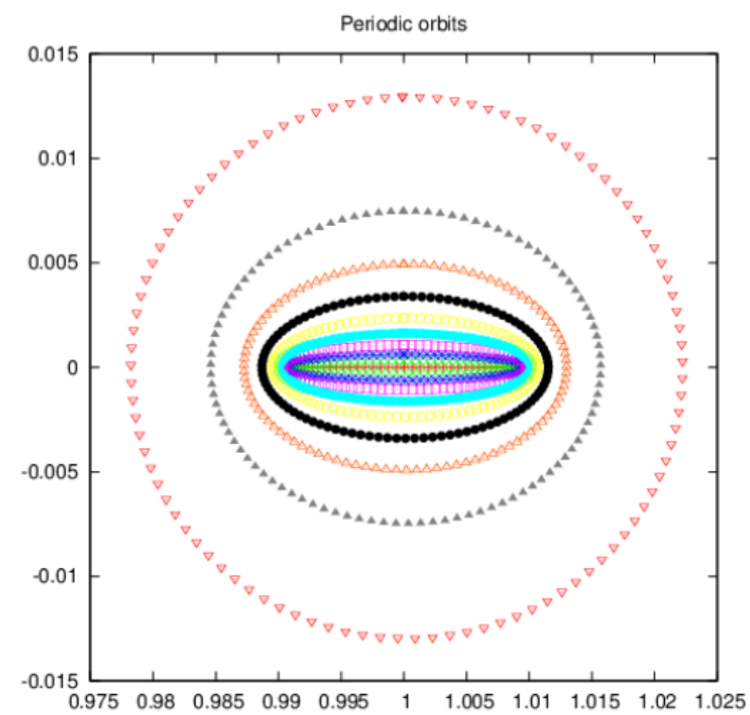
Elliptical case: Lagrange points

- Use rotating coordinates, *x*-axis always points to centre of Galaxy
- Equations of motion

$$\begin{aligned}
\ddot{x} &= -2\frac{\dot{R}}{R}\Omega y - \left(\Phi_g''(R) - \Omega^2\right)x + 2\Omega\dot{y} - \Phi_c'(r)\frac{x}{r} \\
\ddot{y} &= 2\frac{\dot{R}}{R}\Omega x - 2\Omega\dot{x} - \left(\frac{\Phi_g'(R)}{R} - \Omega^2\right)y - \Phi_c'(r)\frac{y}{r} \\
\ddot{z} &= -\frac{\Phi_g'(R)}{R}z - \Phi_c'(r)\frac{z}{r}
\end{aligned}$$

- where
  - *R* is distance to Galactic centre
  - Ω is angular velocity of motion about Galactic centre
  - Φ_c, Φ_g are cluster and Galactic potential, respectively

- There are no equilibria, but
- There are periodic solutions which reduce to the Lagrange points when eccentricity of Galactic motion → 0



Periodic motions for power-law Galactic potentials with $\Phi_g \propto R^{-\alpha}$, $\alpha = 1$ *(innermost) to* $\alpha = 0.1$ *(outermost)*
For $\alpha = 1$ *(Keplerian) motion is rectilinear*

Lagrange points in the elliptic Hill problem

- Rectilinear homothetic solutions of any eccentricity exist for the elliptic three-body problem
- For the elliptic Hill problem two of these correspond to the two Lagrange points
- Their existence becomes obvious with the use of rotating, pulsating coordinates and a change of independent variable (cf. Szebehely 1967 §10.3 for the restricted problem)

Equations of motion

- Equations of motion (planar case)

$$\rho'' = -2\boldsymbol{\Omega} \times \rho' - \frac{1}{1 + e\cos\phi}\nabla_\rho U$$

where

- $\rho = (\xi,\eta)$ – pulsating, rotating coordinates
- independent variable is ϕ, the true anomaly
- $U = -1/r - 3\xi^2/2$
- $\boldsymbol{\Omega}$ is unit *z*-vector
- Lagrange points at $(\pm 3^{-1/3}, 0)$

STRATEGY:

- Flux of phase space per unit “energy” at the Lagrange points
- Phase-space volume per unit “energy”
- Hence time scale of escape (as a function of “energy”)
- Model effect of time-dependent external field as an additional kind of relaxation
- Combine with two-body relaxation – effective time of relaxation
- Combine with escape time to estimate escape rate of stars

Flux at the Lagrange points

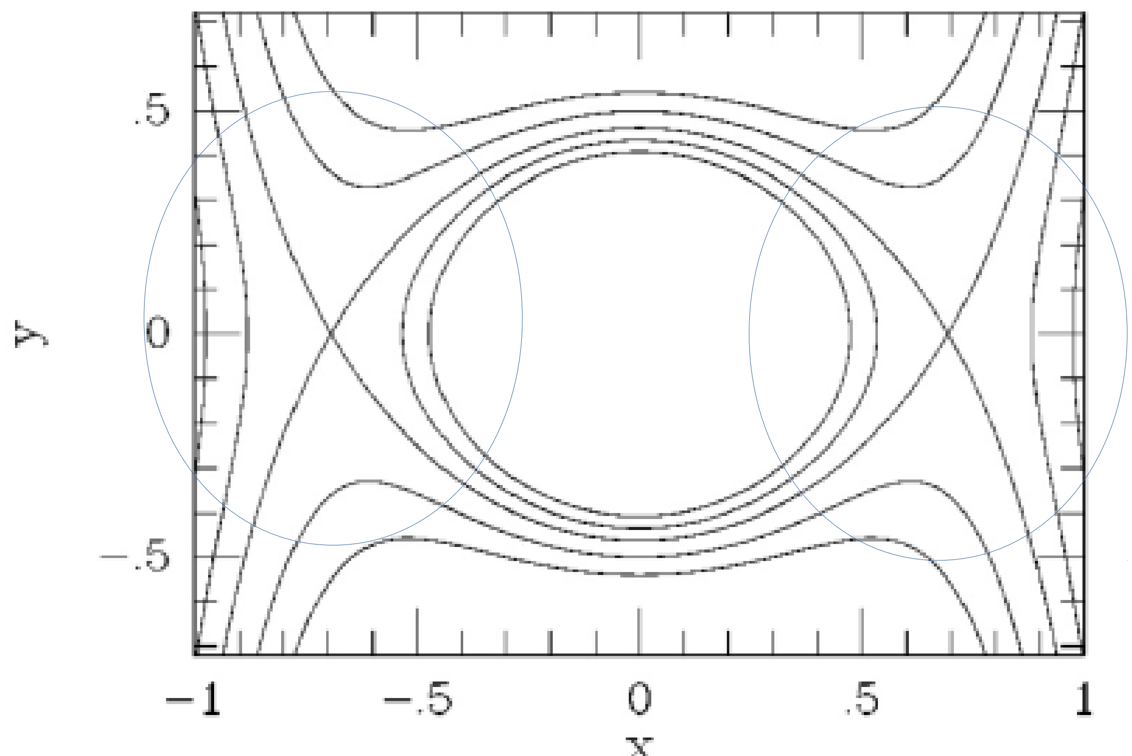
- Shifting origin to the Lagrange point and linearising there, equations have Hamiltonian form, with Hamiltonian

$$
\begin{aligned}
H &= \frac{1}{2}\left(p_1^2 + p_2^2\right) + p_1 q_2 - p_2 q_1 + \\
&\quad \frac{1}{2}\left(q_1^2 + q_2^2\right) + \frac{1}{2(1 + e\cos\phi)}\left(-9q_1^2 + 3q_2^2\right)
\end{aligned}
$$

- To first order in *e,* the single ϕ-dependent term can be removed by canonical transformation, *H* → *H'*
- Flux per unit energy exactly as in circular case (to first order in *e*), but
  - different independent variable
  - different (local) definition of energy *H'*
- Now attempt to calculate the time scale of escape as t_e = V/F, where F is the flux per unit energy, and V is the volume of phase space, inside the cluster, per unit energy

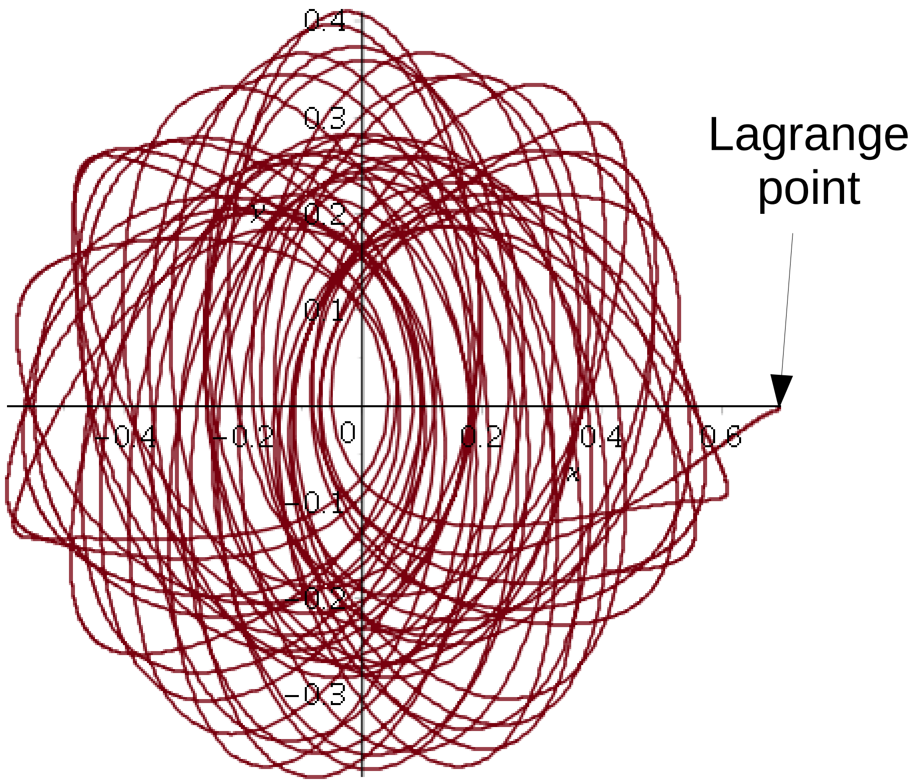
Phase space volume per unit "energy"

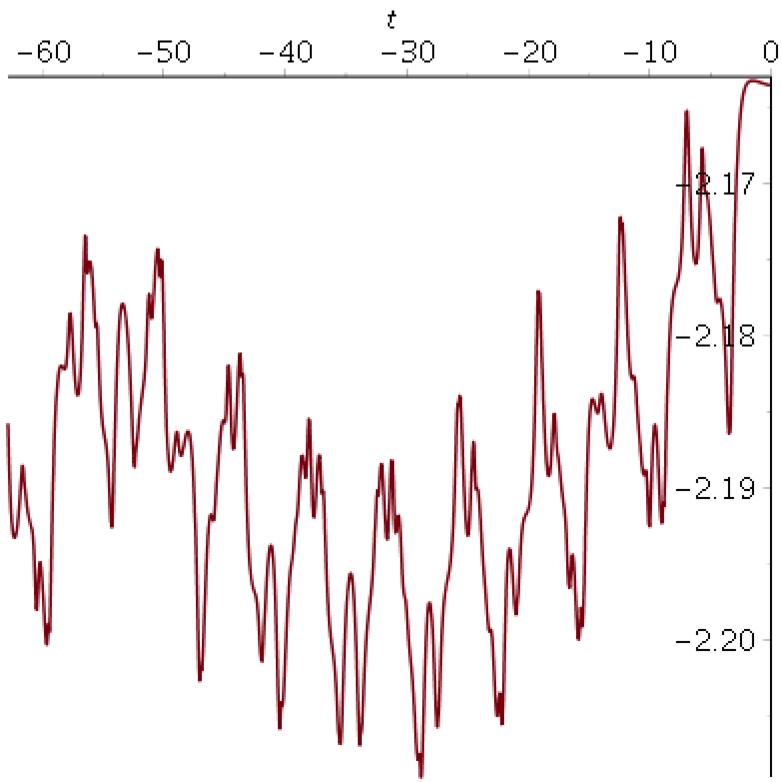
- In the circular case it is straightforward to calculate V
- Use Hamiltonian *H'* to calculate the phase volume per unit *H'* in the vicinity of the Lagrange points



Behaviour before escape

- Numerically computed orbit in the stable manifold of one Lagrange point (rotating, non-pulsating frame, $e = 0.01$)
- Integration time = 10 Galactic orbits
- Jacobi “integral” J varies because of time-dependent external field, in addition to two-body relaxation
- $\mathrm{d}J/\mathrm{d}t = O(e)$



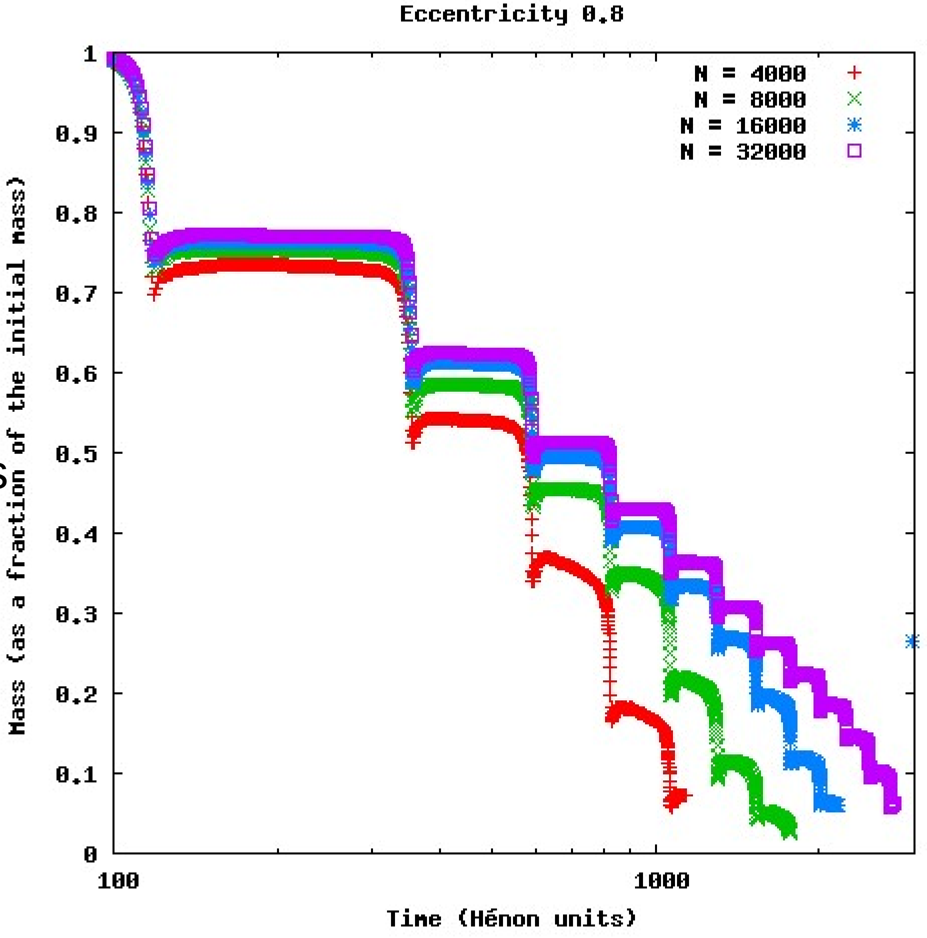


- During time 2π, we assume mean square change in J is given by
  $<(\Delta J)^2>/J^2 \sim 2\pi/t_{\mathbf{r}} + 2\pi e^2/\alpha$ (α = constant)
- effective relaxation time given by $1/t_{eff} \sim 1/t_r + e^2/\alpha$
- transition from relaxation-dominated evolution to tidally-dominated evolution as t_r increases (i.e. as N increases) or as e increases

N-body results on *N*-dependence of lifetime on an eccentric Galactic orbit

Interpretation of figure

- figure plots remaining mass *f* against time
- sharp drops caused by pericentre passage
- at early times mass loss dominated by tidal effect
- by *f* = 0.6 the two largest models still evolve similarly (tide dominant), two smaller models strongly affected by two-body relaxation
- by *f* = 0.1 all models strongly affected by relaxation



My problem

Circular case

- Escape requires $E > E_{crit}$
- Time scale of escape

  (= volume of phase space per unit E/flux of phase space per unit E past the Lagrange points)

  $\propto (E - E_{crit})^{-2}$

Elliptical case

- What is the successful route to an analogous result?

My goals

Assume small orbital eccentricity *e*

- At first order the rate of mass loss is independent of *e*; it equals the rate of mass loss on a circular orbit of radius equal to the average of apogalactic and perigalactic distance
- At second order the rate of mass loss depends on
  - the relative effect of tides and two-body relaxation
  - the Galactic gravitational field

Thanks to collaborators Ben Bar-Or, Kate Daniel, Anna Lisa Varri