Chaos and fractal structures in the planar restricted three-body problem

Guillaume Rollin, José Lages, Dima Shepelyansky

School for advanced sciences of Luchon

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References :
G. Rollin, *Dynamical chaos in the restricted three body problem*, Ph.D. thesis, 2015.
G. Rollin, J. Lages, D. L. Shepelyansky, *Fractal structures for the Jacobi Hamiltonian of restricted three-body problem*, New Astron., 47: 97–104, 2016.
G. Rollin, J. Lages, *Chaos border for particle captured by binaries*, in prep.

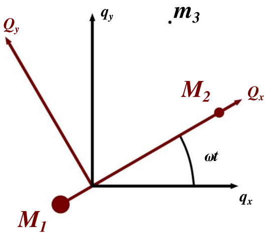
Outline

Outline

The planar restricted three body problem

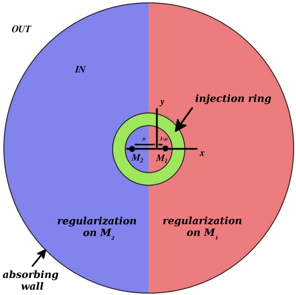
- In the 19th century : Poincaré studied the restricted three body problem.
- The primaries M_1 and M_2 which move in the q_x, q_y plan.
- An object m_3 of very low mass compared to M_1 and M_2.
- The aim of our study was to explore the proprieties of the trajectory of m_3.
- We put our system in the sydodic reference frame Q_x, Q_y.
- Due to this transformation the hamiltonian of the system becomes autonomous.

H. Poincaré (1854-1912)



The planar restricted three body problem

The system



The system

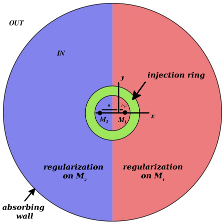
- The hamiltonian of the system in the synodic reference frame is the following :

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) - yp_x - p_y x - \left(\frac{1-\mu}{R_1} + \frac{\mu}{R_2}\right)$$

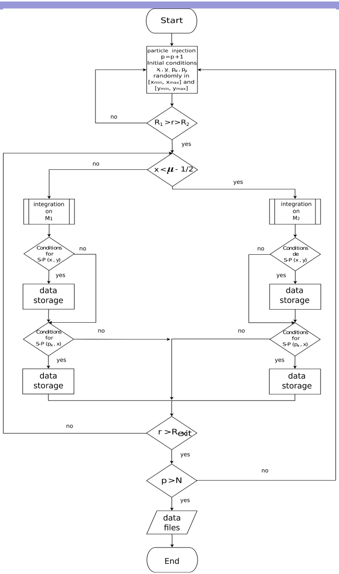
- To avoid the numerical problem of the close encounter we have used the Levi-Civita regularization.
- We add an absorbing wall around the system to simulate the ejection of the particles ⇒ the system is open !

The algorithm

- Coded in Fortran.
- The regularization is made on the one or other one of both masses.
- Poincaré section was made in the x, y space and p_x, x space.
- When $t = 0$, $\dot{r} = 0$.
- Poincaré section are taken when $\dot{r} = 0$ and $\dot{\Phi} < 0$ (angular velocity).



Synodic frame

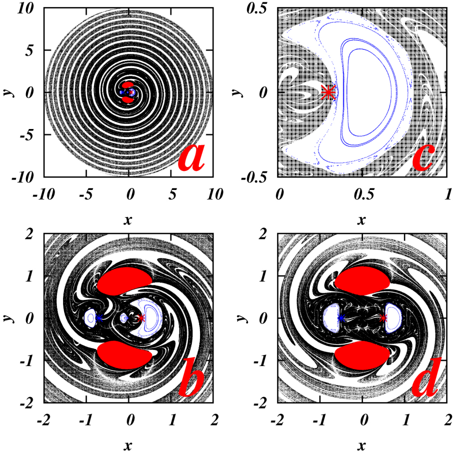


Algorithm

Outline

Results

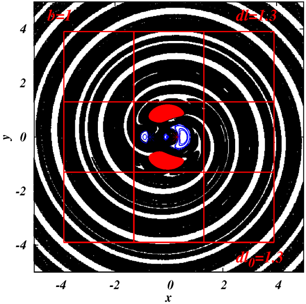
- Poincaré section in x, y plan.
- Black points are the chaotic trajectories.
- Red star are $M_1 = 1 - \mu$, blue star are $M_2 = \mu$.
- Red areas are the forbidden zones.
- Blue lines are the invariant KAM curves.
- We see the particles which remain in the system after $t = 10$.
- Open system $\Rightarrow$ we clearly see the fractal structures of the strange repeller.



a) $\mu = 0.3$, $E = -1.5$, b) idem with zoom, c) idem with zoom, d) $\mu = 0.5$, $E = -1.5$.

References :
G. Rollin, J. Lages, D. L. Shepelyansky, New Astron., 47: 97–104, 2016
J. Nagler, Phys. Rev. E, 69(066218), 2004. *ibid* 71(026227), 2005.

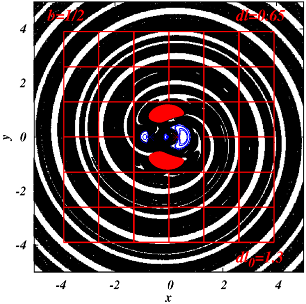
Fractal structures



Fractal analysis for $b = 1$, $dl = 1.3$, $dl_0 = 1.3$.

- We have used the "Box-Counting" method.
- We have used this method on a square ring with a square hole at its center.
- Algorithm : We split the 8 squares with different scale $b = 1, 1/2, 1/4 \ldots$. At each step we obtain $1/b^2$ square with the width $dl = bdl_0$.
- The dimension is given by : $D = \frac{\ln(N_b)}{\ln(1/b)}$.

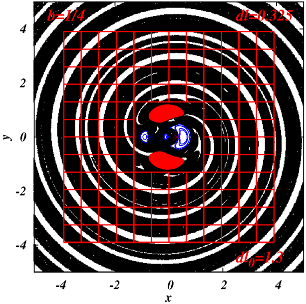
Fractal structures



Fractal analysis for $b = 1/2$, $dl = 0.65$, $dl_0 = 1.3$.

- We have used the "Box-Counting" method.
- We have used this method on a square ring with a square hole at its center.
- Algorithm : We split the 8 squares with different scale $b = 1, 1/2, 1/4 \ldots$. At each step we obtain $1/b^2$ square with the width $dl = b dl_0$.
- The dimension is given by : $D = \frac{\ln(N_b)}{\ln(1/b)}$.

Fractal structures

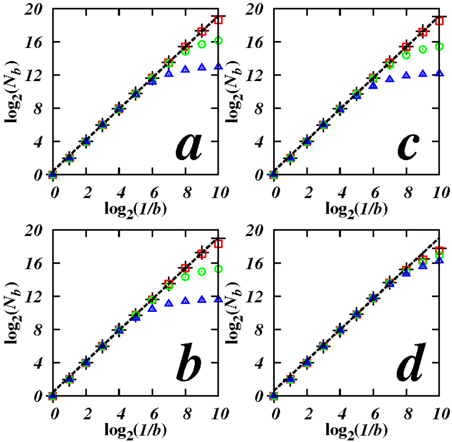


Fractal analysis for $b = 1/4$, $dl = 0.325$, $dl_0 = 1.3$.

- We have used the "Box-Counting" method.
- We have used this method on a square ring with a square hole at its center.
- Algorithm : We split the 8 squares with different scale $b = 1, 1/2, 1/4 \ldots$. At each step we obtain $1/b^2$ square with the width $dl = bdl_0$.
- The dimension is given by : $D = \frac{\ln(N_b)}{\ln(1/b)}$.

Fractal structures

- The fractal dimension is computed for different parameters : μ and E.
- For each panel the fractal dimension is computed for the remaining particles when $t > 3$, $t > 10$, $t > 30$, $t > 50$.
- The results are substantially the same and we have found : $D \sim 1.87$.
  CAUTION : for $t > 30$ and $t > 50$ the curve seem to have an "inflection" $\Rightarrow$ low number of available points for the analysis.
- Conclusion : The fractal pattern of the strange repeller is done after $t = 3$ and will not change any more.
- The fractal dimension of the invariant strange attractor formed with the particles which neither leave the system in the past nor in the future is given by $D_0 = 2(D-1)$, here : $D_0 = 1.74$



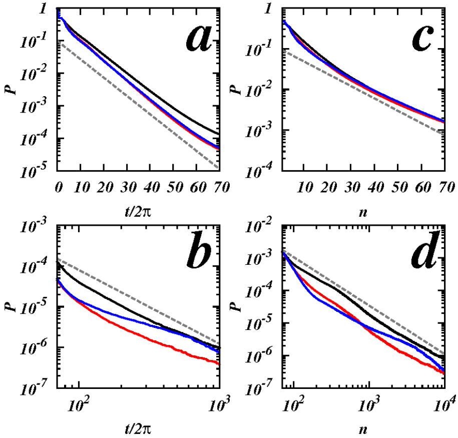
Dimension analysis for $R_{exit} = 10$. a) $\mu = 0.3$, $E = -1.5$, $D \simeq 1.87$ ($t > 10$). b) $\mu = 0.4$, $E = -1.5$, $D \simeq= 1.87$ ($t > 10$). c) $\mu = 0.3$, $E = -1.3$, $D \simeq= 1.87$ ($t > 10$). d) $\mu = 0.3$, $E = -1.7$, $D \simeq= 1.83$ ($t > 10$).

References :
G. Rollin, J. Lages, D. L. Shepelyansky, New Astron., 47: 97–104, 2016

Outline

Survival probability

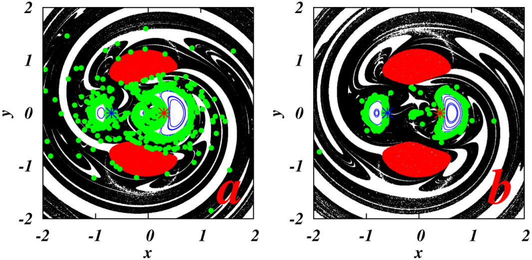
- Survival probability is computed for different μ.
- The time is counted in number of binary rotation and in number of appearance on the Poincaré section.
- We note two behaviour.
  - For the short times $\Rightarrow$ exponential decrease $P(t) \propto e^{-t/t_s}$ with $1/t_s = 0.13$ (for P vs n : $1/\tau_s = 0.07$).
  - For the long times $\Rightarrow$ algebric decrease $P(t) \propto 1/t^{\beta}$ with $\beta = 1.82$ (for P vs n : $\beta = 1.49$).
- Exponential : typical behaviour of strange attractor.
- Algebric : typical behavior due to the decrease of the Poincaré recurences probability. $\beta = 1.49$ is close to the the value $\beta = 1.5$ found usually in symplectic map.



Survival probability for $E = -1.5$, black curve is for $\mu = 0.3$, red curve for $\mu = 0.4$ and blue curve for $\mu = 0.5$.

Survival probability

An algebric decrease :

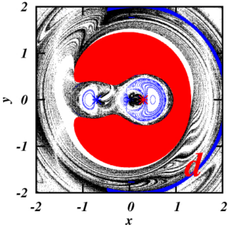


In green particles positions for $t > 100$ and $E = -1.5$. a) $\mu = 0.3$ b) $\mu = 0.4$.

- For $t > 100$ particles are sticked around the KAM island.
- $\Rightarrow$ We understand that the probability decrease is no longer exponential.

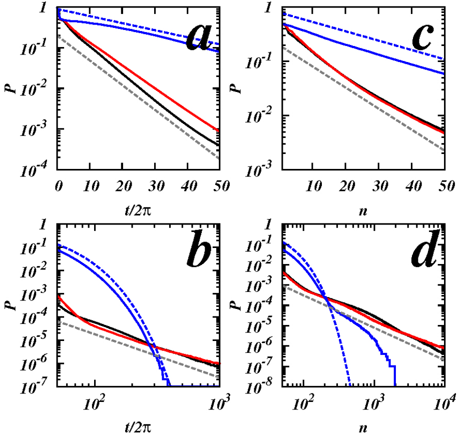
Survival probability

- Same behaviour as before for the black and red curve.
- New behaviour for the blue curve which is almost only exponential.



Poincaré section for $\mu = 0.3$ and $E = 1.7$.

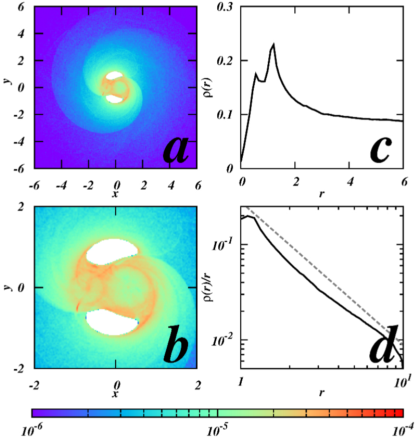
- Being close to islands is almost impossible $\Rightarrow$ survival probability is exponential.



Survival probability for $\mu = 0.3$, black curve is for $E = -1.3$, red curve for $E = -1.5$ and blue curve for $E = -1.7$.

In real space...

- A part of the spiral structure is preserved in the real space !
- The surface density shape is in agreement with previous results.
- It would be interesting to explore further this kind of work $\Rightarrow$ the structure in spiral resembles that observed in the galaxies.



Density of presence of particles in real space for $t = 10$.

References :
G. Rollin, J. Lages, D. L. Shepelyansky, New Astron., 47: 97–104, 2016
G. Rollin, J. Lages, D. L. Shepelyansky, A&A, 576:A 40, 2015
J. Lages and D. L. Shepelyansky, MNRAS, 430(L25), 2013

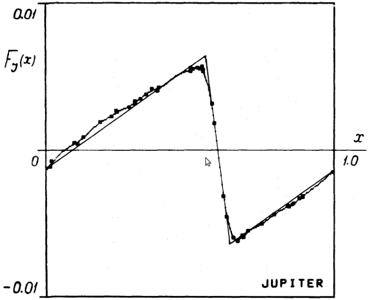
Outline

The Kepler map

- To describe the dynamics of the particles we have used a symplectic map description : the "Kepler Map."
- The Kepler map is described by the following equations :

$$w_{n+1} = w_n + F(x_n)$$
$$x_{n+1} = x_n + w_{n+1}^{-3/2}$$

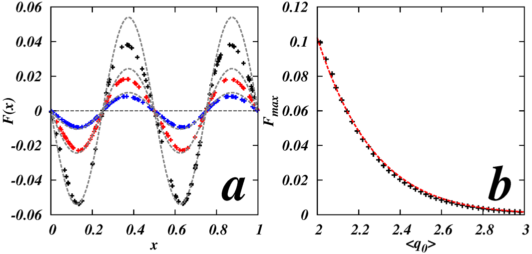
- w_n is the energy of the particle at its perihelion, x_n the phase of the binary at the perihelion of the particle, $F(x_n)$ is called the "kick function".
- Originally used to study the quasi-parabolic dynamics of comets.
- The kick function for a binary with identical masses is $F(x) = A\sin(4\pi x)$.



Original kick function found by Chirikov and Vesheslavov for the Halley comet and Sun-Jupiter binary.

References :
T. Y. Petrosky, Phys. Letters A, 117(328), 1986.
T. Y. Petrosky and R. Broucke, Celestial Mechanics, 42 :53–79, 1988.
B. V. Chirikov and V. V. Vecheslavov, Astron. Astrophys., 221 :146–154, 1989.
I. I. Shevchenko, The Astrophysical Journal , 799(1):8, 2015.

The kick function



a) Kick function for $\mu = 0.5$, in black $q = 2.2$, in red $q = 2.4$, in blue $q = 2.6$, b) Evolution of F_{max} with q

- We have used the same software as before to compute the amplitude evolution of the kick function with to the periatron distance q.
- The theorical evolution is given by (Shevchenko '15) :

$$F_{max} = Aq^{3/4} \exp\left\{\frac{-2^{5/2}q^{3/2}}{3}\right\}$$

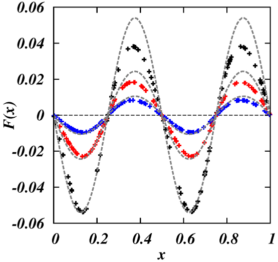
- The theorical value of A is : $A = 2^{11/4}\pi^{1/2} \simeq 11.9236$
- We have found $A \simeq 12.5583 \pm 0.04407$

References :
I. I. Shevchenko, The Astrophysical Journal , 799(1):8, 2015.

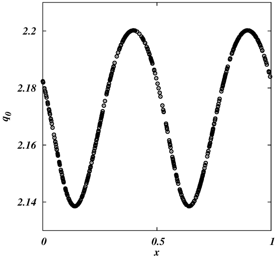
Non-symetric kick function

- The kick is non-symetric, how can we explain this fact ?
- Unlike the two body problem with a fixed q,

  particles which come from the infinity are
  - "more attracted" at small q when they feel an decrease of w.
  - "less attracted" at small q when they feel an increase of w.
- In conclusion :
  - Positive part of the kick function is given by particles with a large q.
  - Negative part of the kick function is given by particles with a small q.
- Consequently the kick function is non-symetric.



Non-symetric kick function



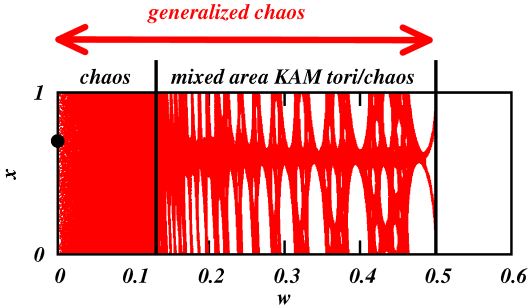
q vs x for particles which come from the infinity and which should have a $q_0 = 2.2$ in two body approximation.

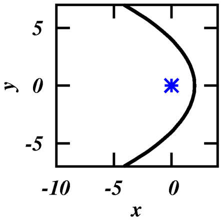
Space phase diffusion and chaos border

- The Kepler map mimics the dynamics by an iterative process.

$$
\begin{aligned}
w_{n+1} &= w_n + F(x_n) \\
x_{n+1} &= x_n + w_{n+1}^{-3/2}
\end{aligned}
$$

- Poincaré Section in phase space :





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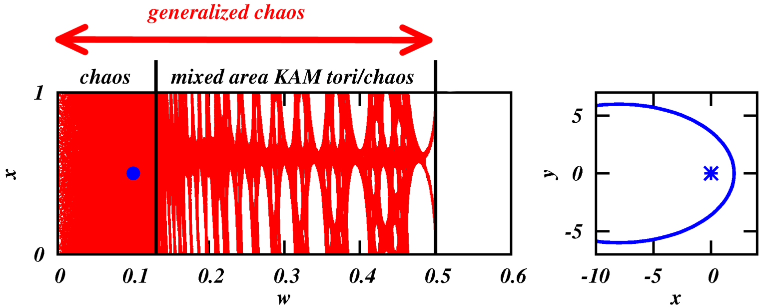

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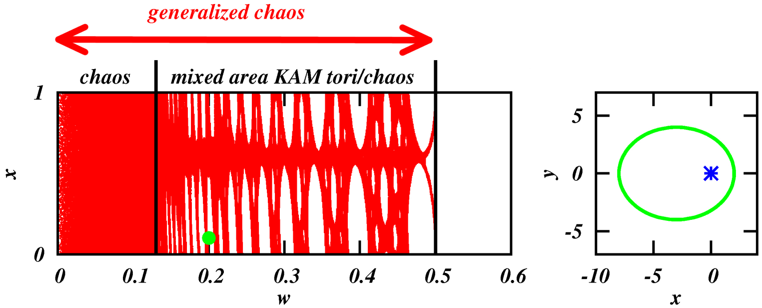

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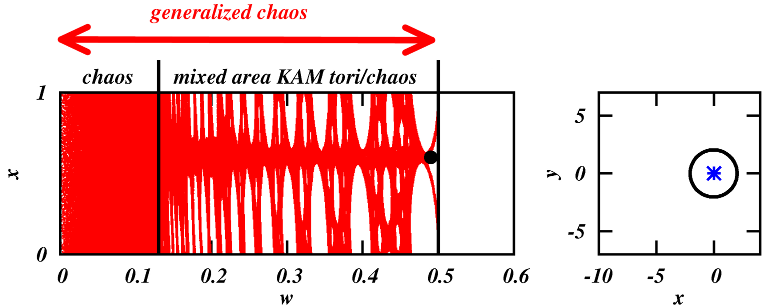

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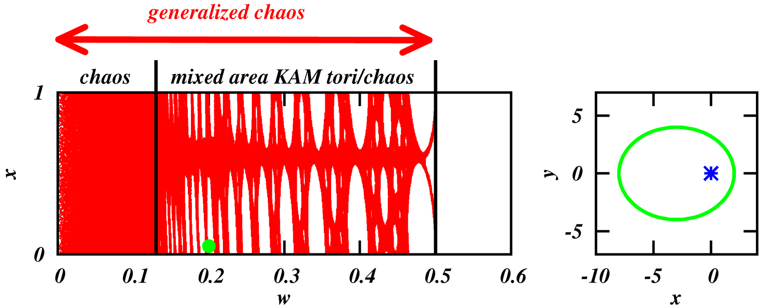

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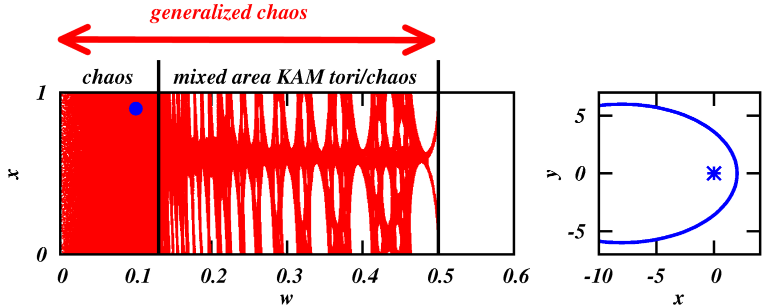

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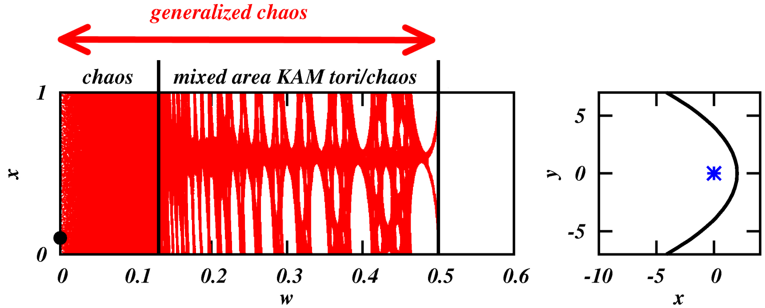
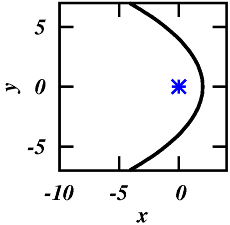
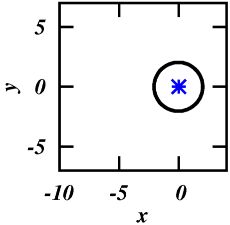


What about the kick function during the diffusion ?

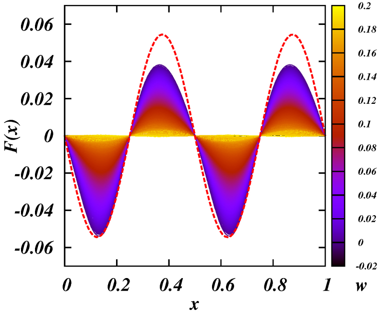


Dynamics close to $w = 0$.



Dynamics close to $w = w_{ch}$.

- Can we think that the kick function is the same during all the studied dynamics ?
- The answer is
  - YES if μ is small (comets in SS) $\Rightarrow$ kick function remains the same during a long time.
  - NO if μ is close to 0.5.



Kick function evolution with the energetic position of the particles.

Periastron distance evolution

- In the planar restricted three body problem, the Jacobi constant is given by $C = w + 2\ell$ where C is the Jacobi constant, w is the energy, ℓ is the angular momentum.
- Due to the constant C : when one particle undergoes an increment Δw in energy it feels a decrease $\Delta \ell = -\Delta w/2$ of its angular momentum.
- After n iteration, we have :

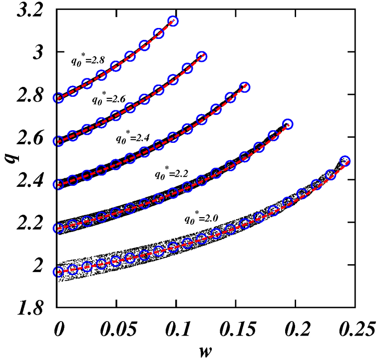
$$\begin{aligned} w_n &= w_0 + \Delta w \simeq \Delta w \\ \ell_n &= \ell_0 + \Delta \ell = \ell_0 - \frac{\Delta w}{2} \end{aligned}$$

where w_0 is the first energy of the quasi-parabolic particles (only ones who can be captured) and ℓ_0 is the first angular momentum.

- In two body approximation we have $q_n = \ell_n^2/(1+e_n) = a_n(1-e_n)$ here $a_n = 1/w_n$ is the semi-major axis of the resulting ellipse and $e_n = \sqrt{1 - w_n \ell_n^2}$ is its eccentricity.
- So we can write :

$$q_n = \frac{\left(\sqrt{2q_0} - \frac{w_n}{2}\right)^2}{1 + \sqrt{1 - w_n\left(\sqrt{2q_0} - \frac{w_n}{2}\right)^2}}$$

Periastron distance evolution



Evolution of the periastron distance during the diffusion in w

- Black points are the diffusion in w of the periastron distance q for captured particles.
- Blue circles are the average of the w position in a small windows Δw.
- Red dashed line is the theorical evolution q_n seen before.

Chaos border and modified Kepler map

- The position of the chaos border can be found with the Chirikov criterion (Chi 79'). For a kick funtion $F(x) = F_{max}\sin(4\pi x)$ the chaos border is given by :

$$w_{ch} = (6\pi F_{max})^{2/5}$$

- When the particle rises in energy ⇒ the kick decrease ⇒ the chaos border decrease.
- The encounter between the particle and the chaos border occurs when w respects the following equation :

$$Aq(w)^{3/4}\exp\left\{\frac{-2^{5/2}q(w)^{3/2}}{3}\right\} - \frac{w^{5/2}}{6\pi} = 0$$

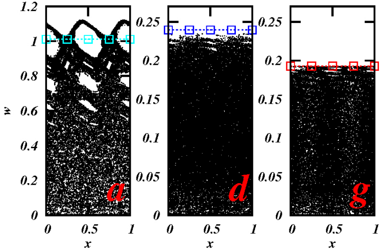
- To mimic the dynamics of one particles when $\mu = 0.5$, we propose a modified Kepler map with an other kick function $F(x, w)$:

$$F(x,w) = A\,\frac{\left|\frac{w}{2} - \sqrt{2q_0}\right|^{3/2}}{\left(1+\sqrt{1-w\left(\sqrt{2q_0}-\frac{w}{2}\right)^2}\right)^{3/4}}\,\exp\left\{-\frac{2^{5/2}\left|\frac{w}{2}-\sqrt{2q_0}\right|\left(\frac{w}{2}-\sqrt{2q_0}\right)}{3\left(1+\sqrt{1-w\left(\sqrt{2q_0}-\frac{w}{2}\right)^2}\right)^{3/2}}\right\}\sin(4\pi x)$$

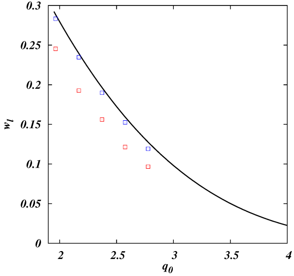
References :
B. V. Chirikov Physics Letters, 52(5) :263-379, 1979

Modified Kepler map results

- Our Kepler map is in agreement with the integration of motion equations.
- Chaos border is lower with our kick function than with a fixed kick function.
- The explanation of the small difference of the chaos border level with our Kepler map and the direct simulation is not still totally solved.
  - Additional component in the decrease of the kick function ?
  - The asymmetry of the kick function ?



a) “traditional” Kepler map. d) Our Kepler map. g) integration of motion equations.

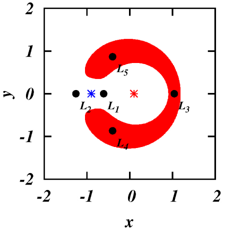


Chaos border evolution with q_0 the first periastron

Outline

Conclusion

- **We have seen that trapped particles in the vicinity of a binary system describe a strange repeller in phase space before their ejection.**
- **The dimension of the strange repeller is $D_0 = 1.74$.**
- **The survival probability follow two laws : exponential decrease and algebric decrease.**
- **The spiral structure leaves traces in the real space.**
- **Perspective : A comprehensive study of "gravitational billard"**



- **We have shown that the chaos border in the phase space (w, x) when μ is close to 0.5 moves during the particles dynamics.**
- **Consequently the diffusive process in phase space is affected.**
- **Perspective : The trajectories of a large number of particles can be computed in a very simple way for statistical studies.**

Thank you for your attention !