A secular representation for the long-term resonant dynamics beyond Neptune

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Plan

Introduction

A well-known but rather unexplored mechanism

Mean-motion resonances with Neptune :

- origin of large orbital variations beyond the planetary region
- strong captures are relatively rare
- variety of possible behaviours yet to be explored

Goal : ***general*** *analysis of the resonant dynamics beyond Neptune*
(extensive exploration of *what can be done* by the known planets)

A quasi-integrable dynamics :

- smooth long-term behaviour
- typical time-scales > 1 Gyr

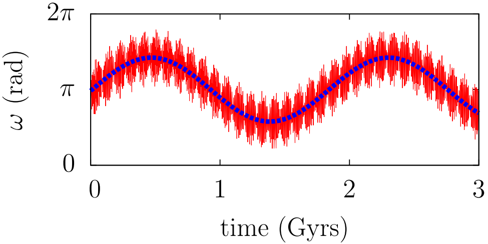
$\Longrightarrow$ *can be described by a* ***secular theory***

What is a secular theory ?

The two-body problem is degenerate :
Only one varying angle in a 3D space

$$\begin{cases} \dot{M} = 2\pi/T \\ \dot{\omega} = 0 \\ \dot{\Omega} = 0 \end{cases}$$

Effect of a small perturbation : ω and Ω become *slow angles*



$$\begin{cases} \dot{M} = 2\pi/T + \mathcal{O}(\varepsilon) \\ \dot{\omega} = \mathcal{O}(\varepsilon) \\ \dot{\Omega} = \mathcal{O}(\varepsilon) \end{cases}$$

Secular theory : study of the slow (dominant) motion

Secular non-resonant theory

Hamiltonian of the problem

Hamiltonian in heliocentric coordinates : $\mu = \mathcal{G}M_\odot$

$$\mathcal{H} = -\frac{\mu^2}{2L^2} + \sum_{i=1}^{N} n_i\,\Lambda_i - \sum_{i=1}^{N} \mathcal{G}m_i \left(\frac{1}{|\mathbf{r} - \mathbf{r}_i|} - \mathbf{r} \cdot \frac{\mathbf{r}_i}{|\mathbf{r}_i|^3} \right)$$

Canonic coordinates (Delaunay elements) :

$$\begin{cases} \ell = M \\ g = \omega \\ h = \Omega \\ \lambda_1, \lambda_2 \ldots \lambda_N \end{cases} \quad \text{et} \quad \begin{cases} L = \sqrt{\mu a} \\ G = \sqrt{\mu a\,(1 - e^2)} \\ H = \sqrt{\mu a\,(1 - e^2)}\,\cos I \\ \Lambda_1, \Lambda_2 \ldots \Lambda_N \end{cases}$$

Secular Hamiltonian $\mathcal{F}$ (1st order of the masses) : average of $\mathcal{H}$ with respect to the fast independent angles ℓ and $\lambda_1, \lambda_2 \ldots \lambda_N$.

Secular non-resonant Hamiltonian

General form of the secular Hamiltonian : $\mathcal{F} = \mathcal{F}(L, G, H, g)$

- conservation of secular momenta L and H
- with L and H as parameters, the dynamics is described by the level curves of $\mathcal{F}$ in the (g, G) plane

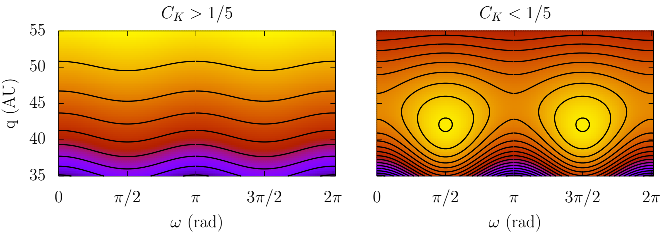
Simpler version of the parameters :

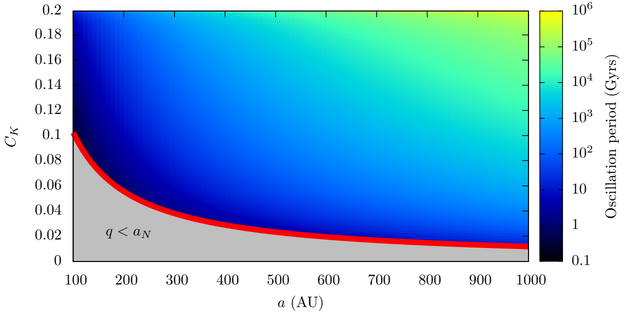
$$\begin{cases} a = L^2/\mu \\ C_K = (H/L)^2 = (1-e^2)\cos^2 I \end{cases}$$

...and of the variables :

$$\begin{cases} \omega = g \\ q = a\left(1 - \sqrt{1-(G/L)^2}\right) \end{cases}$$

Study of the lowest order terms

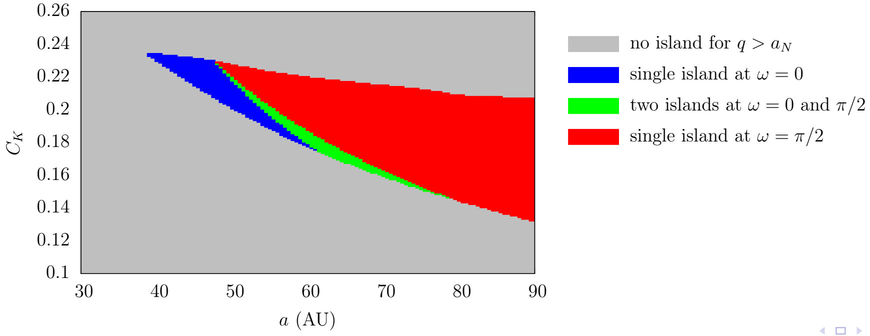




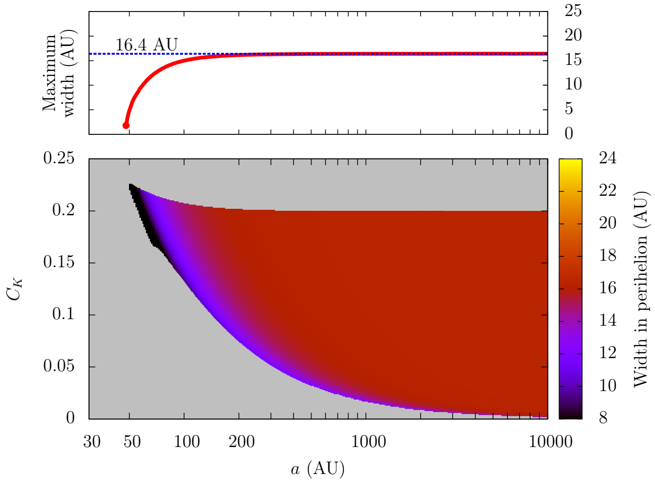
Semi-analytical model

Lowest order terms : only accurate for large semi-major axis and small eccentricity $\Longrightarrow$ *Numerical "exact" secular Hamiltonian :*

$$\mathcal{F}(L,G,H,g) = -\sum_{i=1}^{N}\frac{1}{4\pi^2}\int_0^{2\pi}\!\!\int_0^{2\pi}\frac{\mathcal{G}m_i}{\left\|\mathbf{r}(L,G,H,\ell,g,h)-\mathbf{r}_i(\lambda_i)\right\|}\,\mathrm{d}\lambda_i\,\mathrm{d}\ell$$



Numerical exploration of the parameter space



$\Longrightarrow$ *a non-resonant secular evolution allows a maximum perihelion excursion of about* 16.4 *AU*

Secular theory for a single resonance

Coordinate change

Principal resonant angle : $\sigma = k\,\lambda - k_p\,\lambda_p - (k - k_p)\,\varpi$

with $k, k_p \in \mathbb{N}$ and $k > k_p$

New canonical coordinates : matrices $\mathbf{A}$ and $(\mathbf{A}^T)^{-1}$

$$\mathbf{A}\begin{pmatrix} M \\ \lambda_p \\ \omega \\ \Omega \\ \{\lambda_{i\neq p}\} \end{pmatrix} = \begin{pmatrix} \sigma \\ \gamma \\ u \\ v \\ \{\lambda_{i\neq p}\} \end{pmatrix}$$

- σ ⟵ resonant angle
- γ ⟵ fast angle
- u, v, $\{\lambda_{i\neq p}\}$ ⟵ no change

Three time-scales :

- short periods $< 10^3$ years — $\{\lambda_{i\neq p}\}$ and γ
- semi-secular periods $\sim 10^5$ years — σ
- secular periods $> 10^9$ years — ω and Ω

Semi-secular Hamiltonian

Semi-secular Hamiltonian $\mathcal{K}$ (1st order of the masses) :

average of $\mathcal{H}$ with respect to the fast angles γ and $\{\lambda_{i\neq p}\}$.

general form : $\mathcal{K} = \mathcal{K}(\Sigma, U, V, \sigma, u)$

$\Rightarrow$ two degrees of freedom

$\Rightarrow$ semi-secular momentum conserved : $V = \sqrt{\mu a}\left(\sqrt{1-e^2}\cos I - \frac{k_p}{k}\right)$

Methods to describe the secular dynamics :

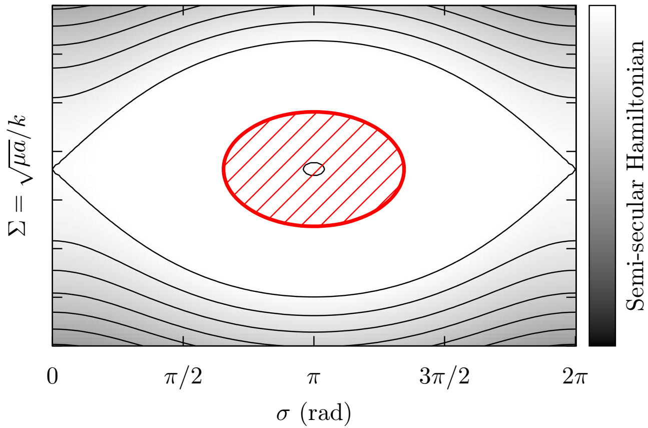
1) Poincaré sections
2) reduction to a one-degree-of-freedom system :

$\Longrightarrow$ **use the adiabatic hypothesis with $T_u \gg T_\sigma$**

The adiabatic invariance

Action-angle coordinates of $\mathcal{K}$ with (U,u) fixed :

- $\theta =$ mean angle along the trajectory
- $J \propto$ enclosed area $=$ *adiabatic invariant*



Resonant secular Hamiltonian

Secular Hamiltonian : $\mathcal{F} = \mathcal{F}_0(J, U, V, u) + \mathcal{O}(\xi)$
where ξ is related to the ratio of frequencies secular/semi-secular.

- conservation of secular momenta J and V
- with J and V as parameters, the dynamics is described by the level curves of $\mathcal{F}$ in the (u, U) plane

Simpler version of the parameters (for a chosen a_0) :

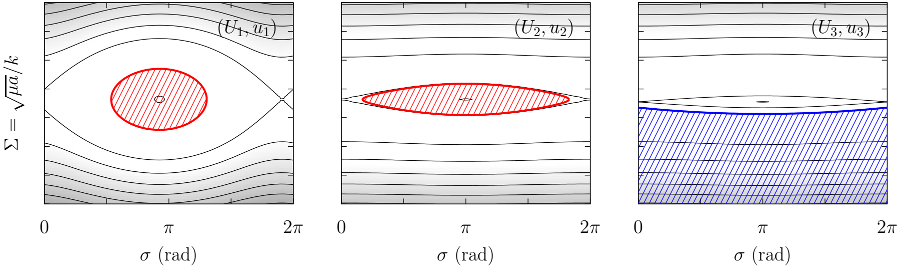
$$\begin{cases} \eta_0 = V/\sqrt{\mu a_0} + k_p/k = \sqrt{1-\tilde{e}^2}\cos\tilde{I} \\ J \text{ (inchanged)} \end{cases}$$

...and of the variables :

$$\begin{cases} \omega = u \\ \tilde{q} = a_0\left(1 - \sqrt{1 - (U/\sqrt{\mu a_0} + k_p/k)^2}\right) \end{cases}$$

Separatrix crossing

Stretching of the island during the secular evolution :



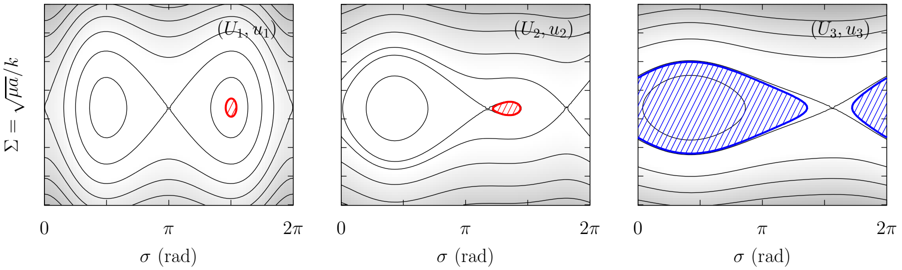
Corresponding secular model :

- trajectory integrable by parts (piecewise model)
- slow chaotic diffusion

Double resonance island

For resonances of type $1 : k$, a range of parameters allows two resonance islands $\Longrightarrow$ *three possible oscillating types for* (Σ, σ)

Disappearance of one island during the secular evolution :

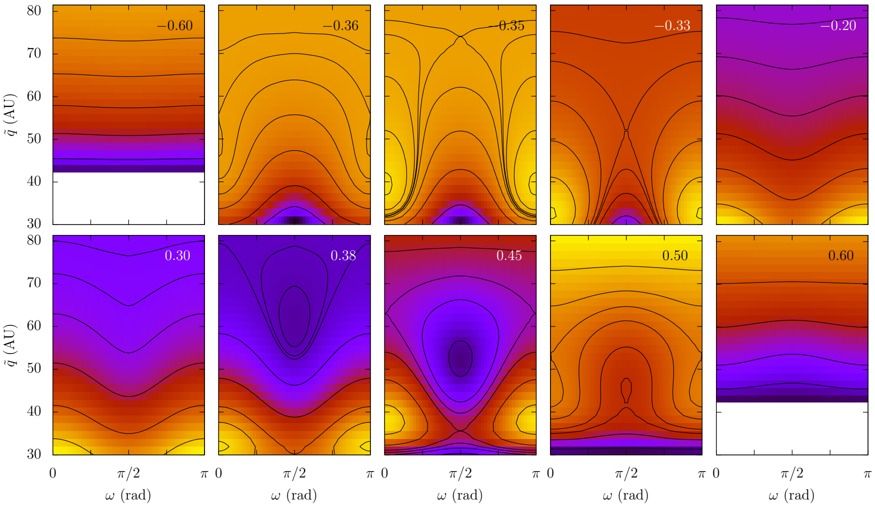


Corresponding secular model :

- frequent separatrix crossings
- long time-scale chaos

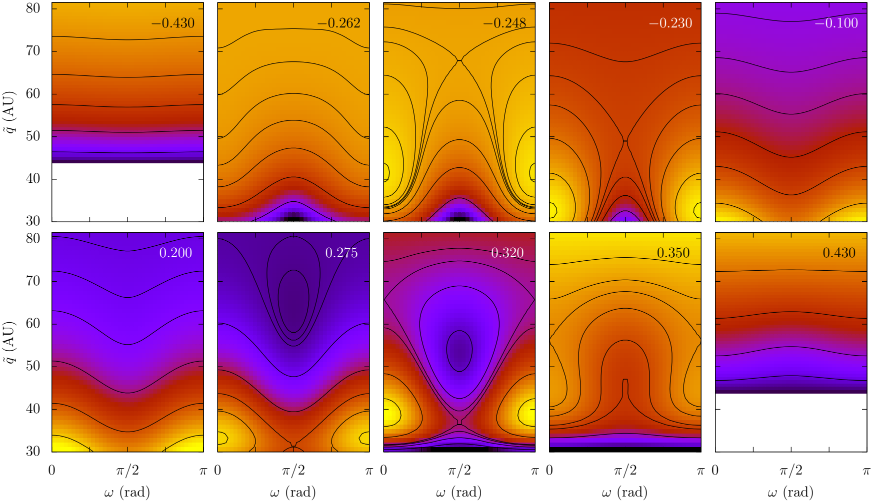
Exploration of the parameter space

Resonances other than $1:k$ (single island)



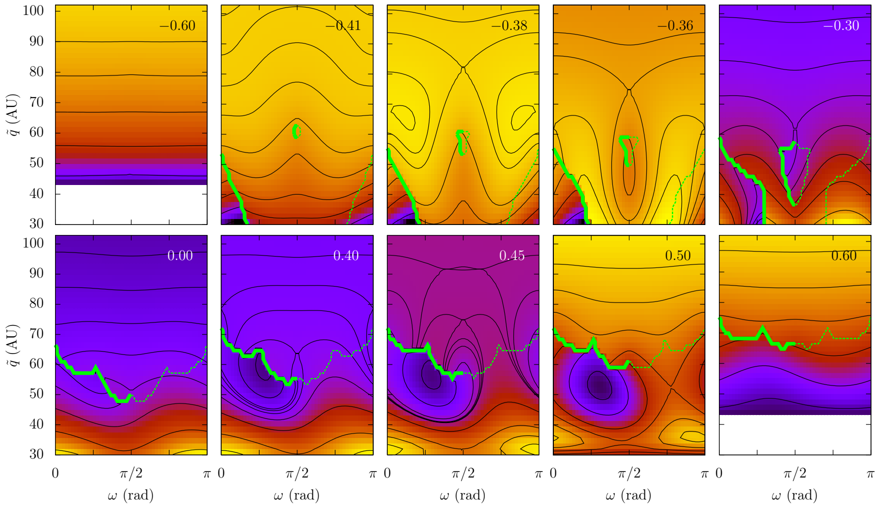
Example : resonance N2 : 37 ($a_0 = 210.99$ AU)

Resonances other than $1\!:\!k$ (single island)



Example : resonance N2 : 115 ($a_0 = 449.36$ AU)

Resonances of type $1:k$ (double island)



Example : resonance N1 : 19 ($a_0 = 214.78$ AU)

Observations about the resonant secular dynamics

General geometry of the phase space :

- an equilibrium point at $\omega = 0 \mod \pi$
- an additional equilibrium at $\omega = \pi/2$ for prograde orbits
- large oscillations of q near the equilibrium points

Resonances of type $1\!:\!k$ are specific :

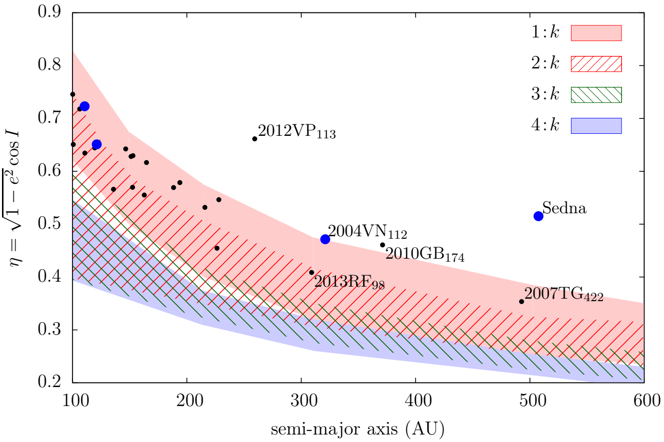
- same features but distorted (asymmetric islands)
- truncation of the secular trajectories by the "green line"

$\Longrightarrow$ *possible segregation of ω and/or wide excursions of q* ***whatever the resonance****, provided that η_0 is in the required range.*

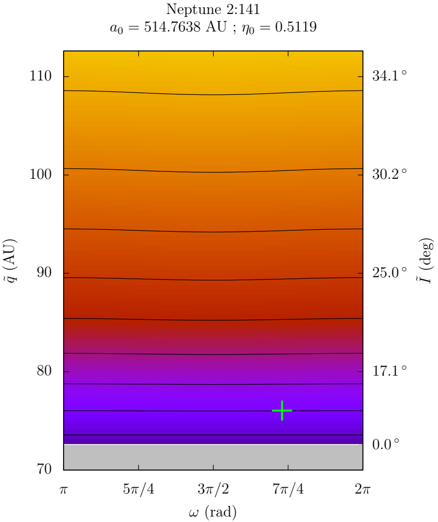
Application to known objects

The "zone of interest"

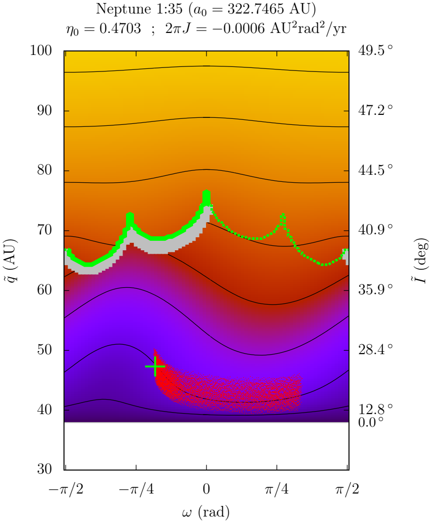
Zone of interest : interval of parameters allowing stable equilibrium points



Out of the zone of interest : Sedna



$2004VN_{112}$ and 303775



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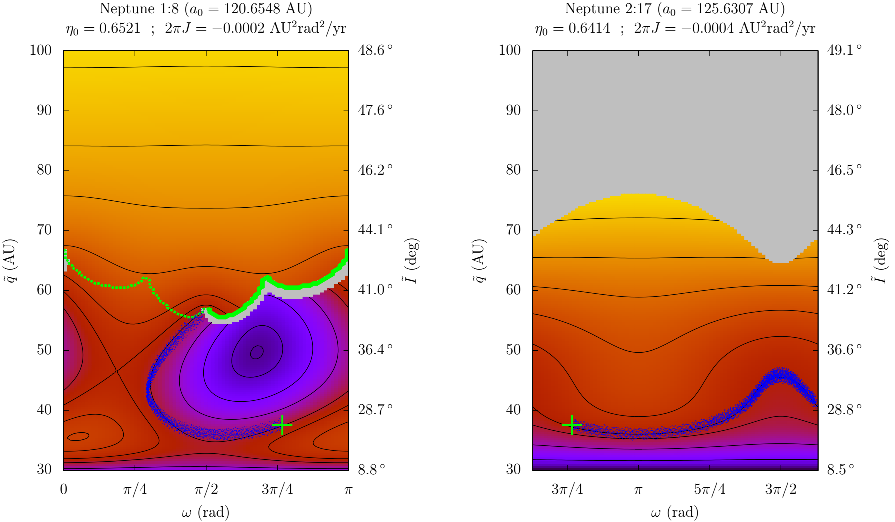
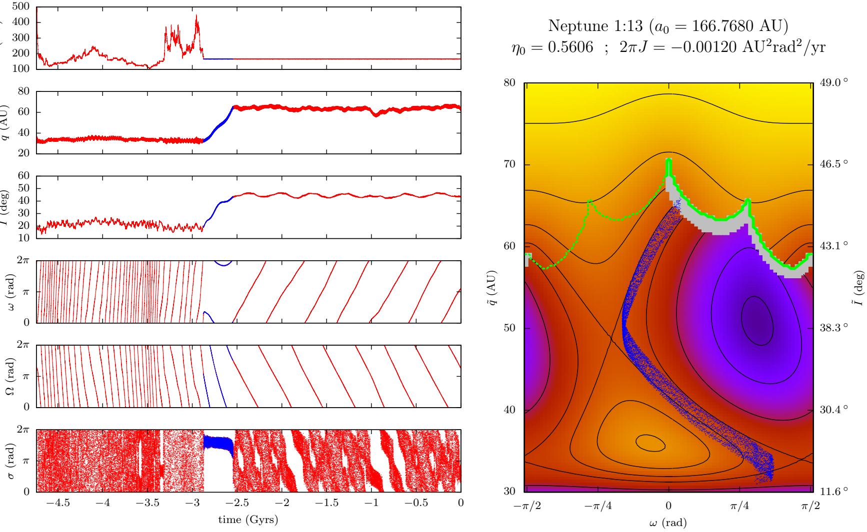
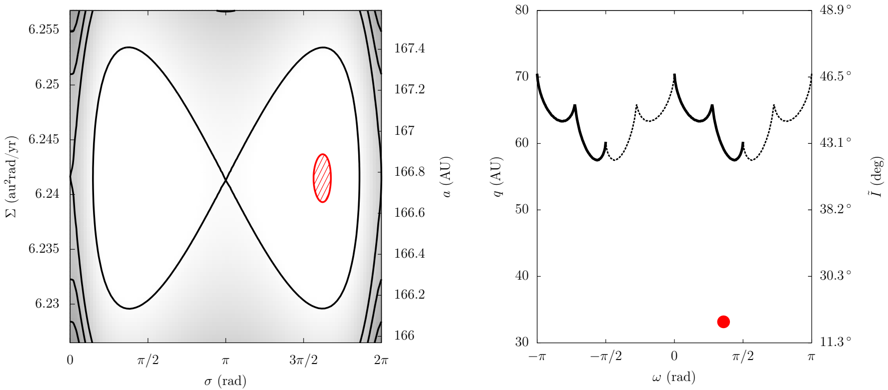

Neptune 1:8 ($a_0 = 120.6548$ AU)
$\eta_0 = 0.6521$; $2\pi J = -0.0002$ AU^{2}rad^2/yr
$\tilde{q}$ (AU)
$\tilde{I}$ (deg)
ω (rad)
0
$\pi/4$
$\pi/2$
$3\pi/4$
π
100
90
80
70
60
50
40
30
48.6 °
47.6 °
46.2 °
44.1 °
41.0 °
36.4 °
28.7 °
8.8 °
Neptune 2:17 ($a_0 = 125.6307$ AU)
$\eta_0 = 0.6414$; $2\pi J = -0.0004$ AU^{2}rad^2/yr
$3\pi/4$
π
$5\pi/4$
$3\pi/2$
49.1 °
48.0 °
46.5 °
44.3 °
41.2 °
36.6 °
28.8 °
8.5 °

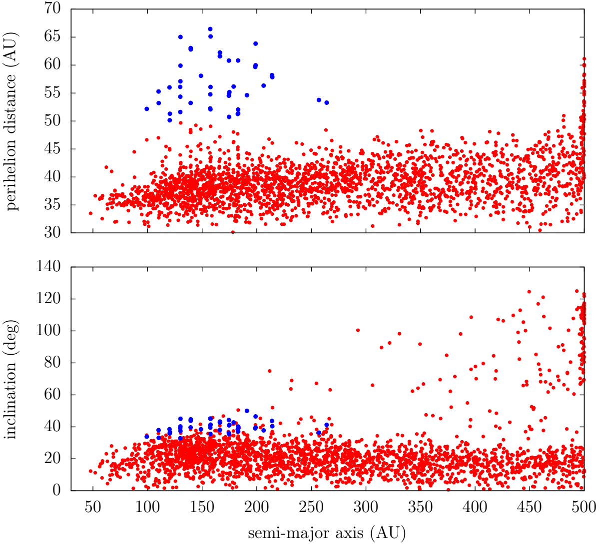
A high-perihelion trapping mechanism



A high-perihelion trapping mechanism



A high-perihelion reservoir



Conclusion

Conclusion

Secular resonant dynamics beyond Neptune :

- one-degree-of-freedom system with two parameters
- straightforward way to explore the dynamics

Observed geometries :

- wide variations of q for some ranges of η_0
- dynamical paths from low to high q and I
- high-perihelion trapping mechanism for $1\!:\!k$ resonances

Implication for the known high-perihelion objects :

- confinement of ω at 0 or π (*but no way to favour 0 against π*)
- possible accumulation of resonant high-perihelion objects (with circulating ω)
- 4×10^6 trapped objects from the Oort Cloud $(1/250000)$